Realization of Relativistic Quantum Physics

Glenn Eric Johnson
Corolla, NC

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Contents

1 Introduction

> It seems clear that the present quantum mechanics is not in its final form. Some further changes will be needed, just about as drastic as the changes made in passing from Bohr's orbit theory to quantum mechanics. Some day a new quantum mechanics, a relativistic one, will be discovered, in which we will not have these infinities occurring at all. – Paul Dirac, in *Albert Einstein : Historical and Cultural Perspectives: The Centennial Symposium in Jerusalem*, edited by Gerald James Holton and Yehuda Elkana, 1979, p. 85.

That relativistic quantum field theory (RQFT) [8, 24, 25, 52, 61] does not realize physically nontrivial quantum mechanics [9, 10, 56] suggests either that quantum mechanics is insufficient to describe nature, or there are unnecessary considerations within RQFT that preclude quantum mechanical realization. Quantum mechanics does suffice if revisions necessary for relativistic position are also applied to quantum fields. Canonical quantization[a] of location is inconsistent with relativistic quantum physics [43, 66]. Similarly, without imposition of canonical quantization for fields, relativistic quantum physics (RQP) is realizable, section 3 and [31, 33, 35, 37]. Constructed realizations of RQP satisfy physical requirements, in particular, the Wightman axioms [56, 62] with relaxation of technical assertions that implement canonical quantization. These notes present a construction and characterization of these realizations.

In this note, constructions demonstrate physically nontrivial realizations of relativistic quantum physics. The established properties of quantum mechanics and relativity [10, 14, 60, 61] are satisfied without Hermiticity of the fields that appear in the Hilbert space scalar product. Realization of quantum mechanics is prioritized over canonical quantization. Interaction is implemented in the scalar product with physically nontrivial vacuum expectations values (VEV) of non-Hermitian field operators. The Hamiltonian is a "trivial" free field-like form that achieves Poincaré covariance: the constructions provide relativistically covariant, general state transition likelihoods $|\langle \underline{f}|\underline{h}\rangle|^2$. Infinite transition interval limits of these state transition likelihoods include low order contributions from Feynman series scattering likelihoods. Scattering likelihoods from the constructions approximate RQFT scattering likelihoods. Although the Hamiltonians are "trivial," interaction follows from the VEV. In the example of the Feynman-Dyson series for scattering amplitudes,

$$\lim_{\lambda\to\infty} |\langle \underline{f}|e^{iH\lambda}e^{-i(H+gH_{eff})2\lambda}e^{iH\lambda}\underline{h}\rangle_o| \approx |\langle \underline{f}|\underline{h}\rangle| \tag{1}$$

[a] A quantization of classical physics includes many considerations, [4] and section 7. Here, the term canonical quantization is applied if a classical dynamical variable q pairs with a densely defined Hermitian operator Q implemented $Q\psi(q) = q\psi(q)$ with quantum state descriptions that are functions $\psi(q)$ over a domain determined by the type of classical dynamical variable, appendix 8.2. Then the (generalized) eigenfunctions of Q, $Q\psi_{q_o}(q) = q_o\psi_{q_o}(q)$ for a constant q_o, are Dirac delta functions for observables with continuous values, or Kronecker delta functions for observables with discrete values.

with: selected particle-like state describing function sequences $\underline{f}, \underline{h}$; "trivial" free field scalar product $\langle\cdot|\cdot\rangle_o$; Hamiltonian H_{eff} expressed in free fields and derived from a classical action; "trivial" free field Hamiltonian H that is a generator of a representation of the Poincaré group; transition interval λ; $g^2 \ll 1$ to approximate the Feynman-Dyson series with a limited number of terms; and $\langle\cdot|\cdot\rangle$ is a scalar product constructed to approximate H_{eff}, appendix 8.9 and [35]. Particle-like state describing function sequences provide $\langle \underline{f}|\underline{h}\rangle \approx \langle \underline{f}|\underline{h}\rangle_o$ at either $g = 0$ or $\lambda = 0$, section 4.3. State describing functions are classical particle-like if the spatial support of each argument is isolated and well-represented by a single location, the support of the Fourier transform of each argument is well-represented by a single momentum, and supports are not entangled. The Feynman-Dyson series are not relativistically covariant for finite transition times λ but the $\lambda \to \infty$ limits, the S-matrices, are covariant [9, 61]. Selection of a Hamiltonian distinct from the generator of time translation in a representation of the Poincaré group precludes relativistic covariance in RQFT.

At issue is whether canonical quantization is a characteristic of nature or a peculiarity of non-relativistic quantum mechanics. One rationale for canonical quantization is if, upon measurement, a general description for an observable "collapses" to an eigenfunction that describes a classical state of the observable, then canonically quantized operators corresponding to observables must be Hermitian with eigenfunctions that describe classical states.[b] Canonical quantization elevates[c] classical dynamical variables to quantum mechanical operators. In a non-relativistic canonical quantization, position and momentum are canonically commuting, Hermitian operators with eigenfunctions that provide classical descriptions of the observables. However, the eigenfunctions of both position and momentum are generalized eigenfunctions, not elements of the Hilbert space of states. Quantum fields also generally do not have eigenfunctions within their Hilbert spaces, section 3.1.3. The "collapse" argument for canonical quantization, classical dynamical variables elevating to Hermitian operators with eigenfunctions precisely corresponding with the classical dynamical variables, is disputed for its technical discrepancies as well as consequent measurement paradoxes [17, 51, 65] and the inconsistency of an instantaneous collapse with relativity.[d] A physical understanding of the formalism of

[b] Classical description for the classical dynamical variable that corresponds to the operator. Quantum state descriptions include no complete classical state descriptions: the Heisenberg uncertainty principle depicts that both location and momentum can not be specified exactly in a quantum description. Generalized functions of point support do not have Fourier transforms of point support. And, no Hermitian operator in a separable Hilbert space has an eigenfunction for every real number. A Hermitian operator in a rigged Hilbert space may have a generalized eigenfunction for any real number, but these generalized eigenfunctions are not elements within the Hilbert space of states. There is a countable number of mutually orthogonal elements in a separable Hilbert space, and the eigenfunctions of a Hermitian operator with distinct eigenvalues are necessarily orthogonal [3].

[c] Here, a canonical quantization is also designated an *elevation*. E.g., the elevation of location is $X\psi(x) = x\psi(x)$ and of momentum is $P\tilde{\psi}(p) = p\tilde{\psi}(p)$ in the Fourier transform domain.

[d] The Einstein, Podolsky, Rosen argument emphasizes the inconsistency of a classical concept for the description of objects, e.g., classical dynamical variables that characterize distinguishable points, with the state describing functions of quantum mechanics.

quantum mechanics remains controversial, sections 2 and 5, and appendix 8.2.8. Relativistic location [43, 66] demonstrates that the canonical quantization of a classical dynamical variable is not necessarily possible. Eigenvectors of the Hermitian operator that best corresponds with location [43] only correspond approximately with points in space. Hermiticity of operators representing observables is necessary for real eigenvalues [3]. Hermiticity applies to complex and multiple component fields in a notation discussed in sections 3.1.3 and 3.1.4.

Classical dynamics must approximate "macroscopic" quantum state descriptions, those perceived as classically described, but the approximation need not be arbitrarily precise and need not apply to any state. States that exhibit indistinguishablity, entanglement or particle production do not necessarily have classical correspondences. These characteristics are not classical. A classical dynamical approximation of quantum mechanics is a quantum-classical correspondence. Relativistic location [43], discussed in section 2 below, is the archetype for a relativistic quantum-classical correspondence. Classical dynamical variables are considered representatives for sets consisting of the dominant supports of quantum state describing functions.[e] For location, this representation is never exact, and the representative is necessarily a good approximation only when the dominant support is localized and does not overlap the dominant support of other state describing function arguments. In the constructions, the quantum-classical correspondence necessary for relativistic position is extended to fields.

The richness of quantum mechanics manifests in the quantum-classical correspondences for the constructions and a significant effort to characterize these correspondences remains. One relativistic realization is perceived with multiple scenario-dependent classical descriptions. No one quantum-classical correspondence necessarily applies universally. Scattering likelihoods, section 4.5 and [31, 33, 35, 37], provide one quantum-classical correspondence. Another, is a $-g/r$ classical interaction potential, demonstrated at the lowest level of non-relativistic, brief interval approximation in section 5. These potentials include approximations to the long-range classical descriptions: electrostatics and gravity. Presumably, classical geometrodynamics and electrodynamics emerge from more precise approximation over greater intervals. The constructions include bound states of the elementary particles, section 6.

Discussion emphasizes foundation principles for relativistic quantum physics, establishment of example realizations, and a physical understanding of the development. Associated phenomenology is relatively unexplored other than to establish connections with Feyman series and classical dynamics. New constructions, matured from [31, 33, 35, 37], are developed in these notes. Primary references are Borchers' [11] and Wightman's [56, 62] developments of quantum field theory, Bogolubov, Logunov and Todorov's review [10], Weinberg's development of RQFT [61], and mathematical background includes [3, 13, 20, 21, 22, 25, 27, 40, 46, 48, 60].

[e]The support of a function is commonly defined as the set of points in the domain where the function is not zero. Here, for position and momentum, the dominant support of a function refers to Lebesgue summations over finite volumes that include a dominant contribution to the mean squared amplitude of the function. The dominant supports of interest are discussed further in section 5.1. Relatively negligible mean squared amplitudes over finite volumes are taken as zero to apply the more standard definition of function support.

2 Realization of relativistic quantum physics

Classical and quantum state descriptions must correspond in appropriate instances but a canonical quantization is not necessary, and for relativistic location, is not possible, [43, 66] and appendix 8.4. The technically relaxed quantum-classical correspondence is realizable. Relativistic location provides an archetype for realizable quantum-classical correspondences. The canonical quantization of classical location is not Hermitian in relativistic physics: delta functions are not eigenfunctions of a Hermitian operator in relativistic physics.[f] The elevations X_1, X_2, X_3 of location $\mathbf{x} \in \mathbb{R}^3$,

$$X_\nu = -i\frac{d}{d\mathbf{p}_\nu},$$

are distinct from the Hermitian relativistic location operators[g] [43]

$$\hat{X}_\nu = -i\,\omega^{1/2}\frac{d}{d\mathbf{p}_\nu}\,\omega^{-1/2}. \tag{2}$$

The operators X_ν and $\hat{X}_\nu$ are expressed in the Fourier transform, momentum domain and $\nu = 1, 2, 3$.[h] That the canonical quantization of location is not a Hermitian operator illustrates a "localization problem" in RQFT. Nevertheless, the elevated operators X_ν conditionally approximate the Hermitian operators $\hat{X}_\nu$ in the sense that the point support of eigenfunctions of X_ν are representatives for the support of the eigenfunctions of the Hermitian location operators $\hat{X}_\nu$, appendices 8.3 and 8.4. The non-Hermitian X_ν approximate the Hermitian $\hat{X}_\nu$ if applied to appropriate state descriptions. That is,

$$X_\nu \approx \hat{X}_\nu$$

[f] Eigenfunctions of a Hermitian operator are necessarily orthogonal if they have distinct eigenvalues [3]. Dirac delta functions are not orthogonal for a relativistic scalar product.

$$\int dxdy\ \Delta^+(x-y)\delta(x-x_o)\delta(y-y_o) = \Delta^+(x_o-y_o) \neq 0$$

for $x_o \neq y_o$ with $\Delta^+(x)$ a Källén-Lehmann form [10, 54]. There are elevations of the momentum operators,

$$\int dxdy\ \Delta^+(x-y)e^{-ip_1x}e^{ip_2y} = (2\pi)^4\delta(p_1-p_2)\theta(E_2)\delta(p_2^2-m^2) = 0$$

if $p_1 \neq p_2$ for the example of the Pauli-Jordan function. The e^{ipx} are inverse Fourier transforms of Dirac delta functions over momenta.

[g] The Newton-Wigner location operators $\hat{X}_\nu$ canonically commute with the Hermitian elevations of momenta and are Hermitian for the relativistic free field scalar product, appendix 8.3. From the Baker-Campbell-Hausdorff relations, momentum generates spatial translations.

[h] Spacetime vectors $x = (x_0, \mathbf{x}) \in \mathbb{R}^4$ with lengths $x_0 = ct$ and spatial vectors $\mathbf{x} = x, y, z \in \mathbb{R}^3$. c is the speed of light and t is time. Energy-momenta are wave numbers designated $p = (p_0, \mathbf{p}) \in \mathbb{R}^4$ with momentum vectors $\mathbf{P} = \hbar\mathbf{p} \in \mathbb{R}^3$, section 3.1.1. Energies are $E = \hbar c p_0$ and $\hbar$ is Planck's constant h divided by 2π. Momentum vectors $\mathbf{p}$ have components p_ν, $\nu = x, y, z$ and $\omega = \omega(\mathbf{p})$ is the wave number proportional to an energy on the mass m shell. An energy $\omega = \sqrt{m^2c^2/\hbar^2 + \mathbf{p}^2}$ provides that $\hbar^2p^2 = m^2c^2$.

when applied to functions dominantly supported on non-relativistic momenta, those with

$$\hbar^2\mathbf{p}^2 \ll (mc)^2$$

within the dominant support.[i] Relativistic location demonstrates that the assumptions of canonical quantization impose unrealizable constraints on relativistic quantum physics.

In relativistic quantum mechanics, nature is described by sequences of complex-valued functions over three-dimensional space and the evolution of these functions is parameterized by time [10, 11]. Sequences of state describing functions over a 3+1 spacetime

$$\underline{f} = (f_0, f_1(x_1)_1, \ldots f_1(x_1)_{\kappa_1}, \ldots f_2(x_1, x_2)_{\kappa_1,\kappa_2}, \ldots) \tag{3}$$

label elements $|\underline{f}\rangle$ of a rigged (equipped) Hilbert space $\mathbf{H}_{\mathcal{P}}$, [10, 11, 31, 33, 37, 56], section 3 and appendix 8.2. These function sequences include descriptions of the vacuum, single particle, multiple particle, wave-like, and composite states. This Hilbert space description fulfills essential properties of nature [14, 60]. Each $x_j \in \mathbb{R}^4$, $j, \kappa_j \in \mathbb{N}$, $\kappa_j \in \{1, N_c\}$ with N_c the number of component fields. $f_0 \in \mathbb{C}$ is the component of the state in the vacuum state characterized by the sequence

$$\Omega = (1, 0, 0 \ldots),$$

the N_c functions $f_1(x)_\kappa$ describe single particle states with masses m_κ, and understanding of the $(N_c)^n$ functions $f_n((x)_n)_{(\kappa)_n}$ in the $n \geq 2$, n-argument subspaces of $\mathbf{H}_{\mathcal{P}}$ distinguishes this approach to RQP from RQFT developments, section 3.1 and appendix 8.2.7 contrasted with [24, 61]. In the constructions, $f_n((x)_n)_{(\kappa)_n}$ describes n particles if classical approximations apply, but $f_n((x)_n)_{(\kappa)_n}$ can also describe $k \neq n$ particles of varied species, section 3.1.1. A convenient representation expands the description of state (3) as a linear combination of products of a field $\Phi(x)_\kappa$ applied to the vacuum with coefficients that are the functions from $\mathbf{H}_{\mathcal{P}}$.

$$|\underline{f}\rangle = \sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \Omega\rangle \tag{4}$$

with

$$(x)_n = x_1, x_2 \ldots x_n \in \mathbb{R}^{4n}$$

[i] Then $\omega \approx mc/\hbar = \lambda_c^{-1}$, a constant. For example, in the $\mathcal{L}^2(\mathbb{R}^3)$ norm,

$$\frac{\|(X_\nu - \hat{X}_\nu)\psi\|}{\|\psi\|} \leq \frac{\lambda_c^2}{\sqrt{8}\,\alpha} \ll \lambda_c$$

if $\lambda_c \ll \alpha$ with α characterizing the extent of the spatial support and $\hbar/\alpha$ characterizing the extent of the momentum support of $\psi(\mathbf{x}) = \exp(-\mathbf{x}^2/(2\alpha^2))$. λ_c is the Compton wavelength (11) for the finite mass m particle. The condition $\lambda_c \ll \alpha$ provides that the dominant support of $\psi(\mathbf{x})$ is non-relativistic. On appropriate states, the expected values of X_ν and $\hat{X}_\nu$ are nearly equal neglecting location differences small with respect to the Compton wavelength.

the designation for a sequence of arguments $x_j \in \mathbb{R}^4$. The conditional particle interpretation of the $n \geq 2$, n-argument functions in (3) follows from physically nontrivial vacuum expectation values of the fields $\Phi(x)_\kappa$, section 3. The selection of basis function space $\underline{\mathcal{P}}$ underlying the Hilbert space $\mathbf{H}_\mathcal{P}$ implements the nonnegativity of energies. The product of quantum fields $\prod_k \Phi(x_k)_{\kappa_k}$ is implemented as a multiplication $\underline{f} \times \underline{g}$ of function sequences, [11] and section 3.1.3.

$$\sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \, f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \underline{g} \rangle = |\underline{f} \times \underline{g}\rangle. \tag{5}$$

The $\times$-multiplication of function sequences is

$$\underline{f} \times \underline{g} = (f_0 g_0, \ldots, \sum_{\ell=0}^{n} f_\ell(x_1, \ldots x_\ell)_{(\kappa)_{1,\ell}} \, g_{n-\ell}(x_{\ell+1}, \ldots x_n)_{(\kappa)_{\ell+1,n}}, \ldots). \tag{6}$$

The vacuum Ω is the identity for the $\times$-multiplication. In (4), the quantum fields map

$$|\Omega\rangle \in \mathbf{H}_\mathcal{P} \quad \mapsto \quad |\underline{f}\rangle \in \mathbf{H}_\mathcal{P}$$

from $\underline{f} \times \Omega = \underline{f}$. Notation is discussed further in section 3.1 and section 4.4 includes a demonstration that the field is a Hilbert space operator.

Vacuum expectation values (VEV) of the quantum fields

$$\langle \Phi(x_k)_{\kappa_k} \ldots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \ldots \Phi(x_n)_{\kappa_n} \Omega \rangle \in \mathcal{S}'(\mathbb{R}^{4n}) \tag{7}$$

are generalized functions (distributions) in the dual to Schwartz tempered test functions $\mathcal{S}(\mathbb{R}^{4n})$. These VEV together with specification of a basis space of function sequences $\underline{\mathcal{P}}$ determine the Hilbert space realization of RQP. From Born's rule, the scalar product in $\mathbf{H}_\mathcal{P}$ determines transition amplitudes. The VEV determine the scalar product

$$\begin{aligned} \langle \underline{f} | \underline{g} \rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, \langle \Phi(x_1)_{\kappa_1} \ldots \Phi(x_n)_{\kappa_n} \Omega | \Phi(x_{n+1})_{\kappa_{n+1}} \ldots \Phi(x_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ \times \overline{f_n}(x_1, \ldots x_n)_{\kappa_1 \ldots \kappa_n} \, g_m(x_{n+1}, \ldots x_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \end{aligned} \tag{8}$$

with formal summation notation [20] for the generalized functions (7). The VEV and $\underline{\mathcal{P}}$ determine the Hamiltonian and determine the scattering amplitudes as infinite interval limits of relativistically invariant transition amplitudes. The state descriptions $|\underline{f}\rangle \in \mathbf{H}_\mathcal{P}$ are equivalence classes within the Hilbert space norm completion of terminating function sequences $\underline{f} \in \underline{\mathcal{P}}$ [10, 11, 13, 56]. The selected basis spaces of functions $\mathcal{P}(\mathbb{R}^{4n})$ are subspaces of the space of Schwartz tempered functions $\mathcal{S}(\mathbb{R}^{4n})$ [21]. Every $f_n \in \mathcal{P}(\mathbb{R}^{4n})$ has the form

$$\tilde{f}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n} (p_{j0} + \omega_j) \, \tilde{h}_n((p)_n)_{(\kappa)_n} \tag{9}$$

with $h_n \in \mathcal{S}(\mathbb{R}^{4n})$ and the wave number

$$\omega_j = \omega(\mathbf{p}_j) = \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}. \tag{10}$$

The reduced Compton wavelength for the mass m_{κ_j} associated with a component field $\Phi(x_j)_{\kappa_j}$ is

$$\lambda_{cj} = \frac{\hbar}{m_{\kappa_j} c}. \tag{11}$$

There is a distinct subspace $\mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$ determined for each

$$(\kappa)_n = \kappa_1, \kappa_2 \ldots \kappa_n \in \mathbb{N}^n$$

that has a distinct sequence of masses $m_{\kappa_1}, m_{\kappa_2}, \ldots m_{\kappa_n}$, section 3.4. $\mathbf{H}_{\mathcal{P}}$ includes elements described by generalized functions with point support over time, functions used by Lehmann, Symanzik and Zimmermann (LSZ) to describe scattering, [10] and section 4.5. Here, VEV (7) of the quantum fields $\Phi(x)_\kappa$ in (5) satisfy the principles of quantum mechanics and relativity described in axioms A.1-7, section 4.1, without imposition of unnecessary constraints to implement canonical quantization. The axioms revise the Wightman axioms [10, 11, 56, 62] and include cluster decomposition from RQFT [61].

The apparently unrealizable constraint in the Wightman axioms is that physically nontrivial quantum fields $\Phi(x)_\kappa$ are densely defined Hermitian operators. In the construction, section 3, Hermiticity of the field operators $\Phi(x)_\kappa$ is inconsistent with interaction. The fields are densely defined Hermitian if

$$\langle \underline{h} | \Phi(\underline{f})^* \underline{g} \rangle = \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle$$

for "real" function sequences $\underline{f} = \underline{f}^*$ with

$$\langle \Phi(\underline{f})^* \underline{h} | \underline{g} \rangle = \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle, \tag{12}$$

with the composition of component fields (5)

$$\Phi(\underline{f}) = \sum_{\kappa=1}^{N_c} \int dx \; \Phi(x)_\kappa f(x)_\kappa$$

and on a dense domain of function sequences $\underline{h}, \underline{g} \in \mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^*} \subset \mathbf{H}_{\mathcal{P}}$. The Wightman axioms [10, 11, 56, 62] assert that the scalar product (8) simplifies to

$$\begin{aligned} \langle \underline{f} | \underline{g} \rangle &= \underline{W}(\underline{f}^* \times \underline{g}) \\ &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \; \langle \Omega | \Phi(x_1)_{\kappa_1} \ldots \Phi(x_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ &\qquad \times f_n^*(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1} \, g_m(x_{n+1}, \ldots x_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \end{aligned} \tag{13}$$

with $\underline{f} \times \underline{g}$ the same product of function sequences (6) as in the definition (5) of field [11], and that the basis space of function sequences is the $*$-involutive[j] algebra of tempered functions $\underline{\mathcal{S}}$. A $*$-dual sequence $\underline{f}^* \in \underline{\mathcal{S}}$ follows for each $\underline{f} \in \underline{\mathcal{S}}$ from an argument order reversal, complex conjugation and linear transformation D determined from a representation of the Lorentz group.

$$\widetilde{f_n^*}((p)_n)_{(\kappa)_n} = (D^T\cdot)_n \overline{\tilde{f}_n}(-p_n, \ldots - p_1)_{\kappa_n \ldots \kappa_1} \tag{14}$$

in a matrix notation with $(D^T)_{ij} = D_{ji} \in \mathbb{C}$, $\tilde{f}(p)$ is the Fourier transform (22) of $f(x)$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$ and

$$(D\cdot)_n V_{(\kappa)_{n+m}} = \sum_{\ell_1=1}^{N_c} \sum_{\ell_2=1}^{N_c} \ldots \sum_{\ell_n=1}^{N_c} D_{\kappa_1\ell_1} D_{\kappa_2\ell_2} \ldots D_{\kappa_n\ell_n} V_{\ell_1 \ldots \ell_n \kappa_{n+1} \ldots \kappa_{n+m}}. \tag{15}$$

The Fourier transform of $\overline{f(x)}$ is $\overline{\tilde{f}(-p)}$. Example D are provided in section 3.2.2. The scalar product (13) follows from the more general scalar product (8) if the VEV (7) satisfy

$$\begin{aligned} \langle\Omega|\Phi(x_1)_{\kappa_1} \ldots \Phi(x_n)_{\kappa_n}\Omega\rangle &= (D^{-1}\cdot)_k \langle\Phi(x_k)_{\kappa_k} \ldots \Phi(x_1)_{\kappa_1}\Omega|\Phi(x_{k+1})_{\kappa_{k+1}} \ldots \Phi(x_n)_{\kappa_n}\Omega\rangle \\ &= \langle\Phi(x_k)^*_{\kappa_k} \ldots \Phi(x_1)^*_{\kappa_1}\Omega|\Phi(x_{k+1})_{\kappa_{k+1}} \ldots \Phi(x_n)_{\kappa_n}\Omega\rangle \end{aligned} \tag{16}$$

for all k, $0 \leq k \leq n$. D is from the $*$-dual (14) and is non-singular, $D^{-1} = \overline{D}$ from (28) in section 3.1.2. (16) is satisfied if the field is Hermitian, $\Phi(\underline{f})^* = \Phi(\underline{f})$, for real function sequences $\underline{f}^* = \underline{f}$. (16) is satisfied if, formally and in matrix notation, $\Phi(x)^*_\kappa = (D\Phi(x))_\kappa$. The simplified scalar product (13) follows for VEV (7) that satisfy (16). A (16) compliant sequence of VEV is designated a Wightman functional $\underline{W}$. Densely defined Hermitian field operators $\Phi(\underline{f})$ follow if the simplified scalar product (13) applies for a $*$-involutive algebra of function sequences like $\underline{\mathcal{S}}$ [10, 11, 56, 62]. The product in the algebra of function sequences is (6) and the $*$-involution is (14). However, despite concerted efforts [2, 5, 10, 30, 39, 41, 56], no physically nontrivial realization for a Wightman functional $\underline{W}$ has been discovered. Physically trivial free field VEV satisfy the Wightman axioms, but here, to construct physically nontrivial fields, the simplification of the scalar product (13) and $*$-involutivity of the basis space of function sequences are not assumed. Below, the condition (16) that implies the simplification (13) of the scalar product (8) is denoted *formal Hermiticity*. One or both of $*$-involutivity and formal Hermiticity are violated by the constructions that exhibit interaction, section 3. Densely defined Hermitian field operators $\Phi(\underline{f})$ do not follow if there is either the support limitation (9) on the function sequences or violation of formal Hermiticity. If the quantum field operator were unitarily similar to a free field, then the demonstrable Hermiticity of a free field would

[j]Here, an algebra of function sequences $\underline{\mathcal{A}}$ is denoted $*$-involutive if $\underline{f}^* \in \underline{\mathcal{A}}$ for every $\underline{f} \in \underline{\mathcal{A}}$, $(\underline{f}^*)^* = \underline{f}$, $(\underline{f} + \underline{g})^* = \underline{f}^* + \underline{g}^*$ and $(\underline{f} \times \underline{g})^* = \underline{g}^* \times \underline{f}^*$ for the mapping (14).

imply Hermiticity of the field. As a consequence, and consistently with the Haag (Haag-Hall-Wightman-Greenberg) theorem [10, 56], unitary similarity of interacting and free field operators is abandoned.

Physically nontrivial VEV include $n \geq 4$, n-argument connected functions that introduce interaction. For example,

$$\langle \tilde{\Phi}(p_1) \dots \tilde{\Phi}(p_k)\Omega | \dots \tilde{\Phi}(p_n)\Omega \rangle = \dots + c_n \delta(p_1 + p_2 + \dots + p_n) \prod_{j=1}^{n} \delta(p_j^2 - m^2c^2/\hbar^2) \tag{17}$$

in the construction with a single neutral scalar field $\Phi(x)$, section 3.3 and [31, 33]. In (17), $N_c = 1$, $n \geq 4$, $n - 2 \geq k \geq 2$, $\tilde{\Phi}(p)$ designates the Fourier transform (23) of $\Phi(x)$, and the contributions of the fewer than n-point connected functions are understood. VEV are cluster expansions of connected functions, section 3.3.4. The selection (9) of basis function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ limits the states in $\mathbf{H}_{\mathcal{P}}$ to positive energies. From (9), the basis function spaces $\underline{\mathcal{P}}$ are limited to functions with Fourier transforms that vanish on appropriate negative energy mass shells: field component κ_j is associated with mass m_{κ_j}, section 3.4.

$$\delta(p_j^2 - m_{\kappa_j}^2 c^2/\hbar^2)\theta(-p_{j0})\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = 0 \tag{18}$$

if $\varphi_n \in \mathcal{P}(\mathbb{R}^{4n})$ with $1 \leq j \leq n$. Because of this limited support property (18) and the inclusion of $n \geq 4$, n-argument connected functions in the VEV (7), interpretation of state describing functions $f_n((x)_n)_{(\kappa)_n}$ differs from RQFT: the negative energy component of fields $\Phi(x)_\kappa$ are eliminated. In the example of a free field, section 3.2, the annihilation component of a free field $\Phi_o(x)_\kappa$ is eliminated for functions from $\mathcal{P}_\kappa(\mathbb{R}^4)$. Then $\Phi_o(x_1)_{\kappa_1} \dots \Phi_o(x_n)_{\kappa_n}\Omega$ equals n factors of the creation component of the free field acting on the vacuum Ω, and consequently describes exactly n particles when the VEV are free field VEV [31]. And, because physically nontrivial VEV (7) include n-argument connected functions with $n \geq 4$, states with different numbers and species of particles are generally not orthogonal. Since scalar products (8) of state describing functions with different numbers of arguments do not vanish, there are nonzero likelihoods of observing $k \neq n$ particles and distinct particle species for a state described by an n-argument function. But even with interaction, there are n-argument states well-represented as n classical particles. For example, n-argument functions $f_n \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$ with widely space-like separated supports describe n particle states [31]. Except for an unobservable[k] phase difference between forward and scattered contributions, the plane wave scattering amplitudes resulting from VEV such as (17) coincide with first order terms from a Feynman-Dyson series [31, 33] with, in examples with nonzero spin, corrections at very relativistic exchange momenta (small distances) [35]. For the neutral scalar field example (17), the scattering cross sections coincide with the first order contributions from $:P(\Phi)_4:$ interactions. The phase difference between (17)

[k] Unobservable in the scattering limit, and RQFT methods do not provide estimates for finite interval transition likelihoods.

and the Feynman-Dyson series is necessary for nonnegativity of the scalar product (8). The phase does not affect scattering likelihoods. Connected functions (17) introduce interaction to the constructions, but preclude Hermitian field operators. Even though (17) satisfies formal Hermiticity, the limitation (18) on the support of the functions in $\underline{\mathcal{P}}$ eliminates Hermiticity of the field operators (5), sections 3.1.3 and 3.1.4. The support constraint (18) also implies that there are no state describing functions with support strictly limited to bounded spatial volumes within $\mathbf{H}_{\mathcal{P}}$ [33]. Considered as functions over spacetime, state describing functions within $\underline{\mathcal{P}}$ do not vanish within any finite spatial volume unless the function is identically zero: such functions are designated *anti-local* [53]. Nevertheless, comparing summations over equal finite volumes, anti-local functions include functions arbitrarily dominantly supported within one finite spatial volume: such functions are designated *essentially localized* here, appendix 8.17. The physically significant support of functions in $\underline{\mathcal{P}}$ may be local, section 5.2. For the VEV (17) to be generalized functions and include massless particles, at least 3+1 dimensional spacetime is necessary, section 4.2.5.

Equivalently, the constructions may be considered to have unconstrained basis function spaces with VEV modified from symmetric forms such as (17), section 3.4.2. In this alternative development, the basis function space is the $*$-involutive $\underline{\mathcal{S}}$ and the VEV (17) are modified to implement positive energies. The tempered functions $\underline{\mathcal{S}}$ include dense sets of real functions with bounded support. In the neutral scalar field example, the VEV equal, up to a phase, the first order contributions from the Feynman-Dyson series, appendix 8.9 and [31, 35],

$$\begin{aligned}\langle\tilde{\Phi}(p_1)\dots\tilde{\Phi}(p_k)\Omega|\dots\tilde{\Phi}(p_n)\Omega\rangle = \dots + c_n\delta(p_1+p_2+\dots+p_n)\\ \times\prod_{j=1}^{k}\theta(-p_{j0})\delta(p_j^2-m^2c^2/\hbar^2)\prod_{\ell=k+1}^{n}\theta(E_\ell)\delta(p_\ell^2-m^2c^2/\hbar^2).\end{aligned} \tag{19}$$

Truncated at first order, the scalar field Feynman-Dyson series VEV satisfy prospective axioms A.1-7 but do not satisfy the Wightman axioms [31]. With the modification to VEV, the scalar product is not of the simplified form (13) of the Wightman axioms. While $\underline{\mathcal{S}}$ is $*$-involutive, the variation of VEV with k precludes Hermiticity of the field operators (7), section 3.1.4. The development based on support limited functions $\underline{\mathcal{P}}$ is connected with the development based on the tempered functions $\underline{\mathcal{S}}$ by an equivalence

$$\frac{\omega_j \pm p_{j0}}{2\omega_j} = \theta(\pm p_{j0}) \tag{20}$$

as multipliers [21] of generalized functions supported solely on mass shells. The alternative RQP constructions present the same puzzlement as the Reeh-Schlieder theorem in RQFT [45], section 3.4.2: although functions with local support label states, the spatial support of quantum fields is global in relativistic quantum physics.

Not necessarily Hermitian quantum field operators (4) and technically relaxed quantum-classical correspondences are studied in this note. Concerns include:

1. Hermitian elevations may be inconsistent with relativity. Relativistic invariance of likelihoods implies that state describing functions with point support over space are not eigenfunctions of a Hermitian operator in RQP: due to relativity, there is no exact correspondence of location as an observable classical dynamical variable with location as an argument of state describing functions or fields [43, 66]

2. an exact, "collapse to an eigenfunction" that provides a classical description is stronger than required by our observations, and inconsistent with separable Hilbert spaces. Conditional and approximate correspondences of classical and quantum state descriptions suffice. It is neither possible to prepare all of a dense set of natural states nor to verify that, for example, there are states with locations that correspond precisely to one real number

3. quantum fields that satisfy the axioms of section 4.1 do not have any eigenfunctions in the Hilbert space of states, section 3.1.3

4. a physically equivalent development for the conventional free quantum field lacks Hermitian field operators [31]. This alternative construction demonstrates that Hermitian fields are not necessary to realize relativistic quantum physics

5. there are realizations of RQP consistent with the principles of quantum mechanics and relativity that lack Hermitian fields, section 3 and [31, 33, 35, 37]. Constructed scattering likelihoods approximate Feynman series scattering likelihoods, section 4.5 and [31, 35]. For example constructions with a single finite mass elementary particle, a short range, Yukawa-like equivalent potential suggestive of nuclear forces is associated with scattering amplitudes in first Born approximation [31, 33]; and long range $-g/r$ pair potentials suggestive of Newtonian gravity or electrostatics are associated with the evolution of significantly separated, classical particle-like concentrations in the support of states in non-relativistic approximations, section 5.

The fields $\Phi(x)_\kappa$ considered here appear in the VEV (7) and are analogous to the elevations X_ν of location x_ν. Operators with perceptible values that correspond approximately with a classical field would be Hermitian, analogously to the $\hat{X}_\nu$. The approximate and conditional correspondence of classical and quantum state descriptions substitutes for the elevations conjectured in canonical quantization. Elevations establish correspondences of classical dynamic variables with the eigenfunctions of hypothesized densely defined Hermitian operators; this exacting correspondence is contradicted by relativistic location. The revision replaces this curious extrapolation with a more justifiable conditional and approximate correspondence of classical and quantum state descriptions.

Realizations of relativistic quantum physics include: the VEV (7) that determine the scalar product (8); the basis space of test function sequences $\underline{f} \in \underline{\mathcal{P}}$ (9); and a physical understanding of the state descriptions $|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$ (4). The prospective axioms for relativistic quantum physics,

A.1-7 in section 4.1, include physical conditions from the Wightman axioms [10, 11, 56, 62] plus cluster decomposition from RQFT [61] and regularity conditions. Significantly, the assumption within the Wightman axioms that implies densely defined Hermitian fields, formal Hermiticity, is discarded. The VEV in section 3.3 satisfy axioms that revise initial constructions [31, 33, 37]. A stronger cluster decomposition property than the uniqueness of the vacuum condition used in [31, 33, 37] provides that truncated functions [10] are connected functions and this stronger condition is included as an axiom. The stronger cluster decomposition condition: provides a replacement for formal Hermiticity in demonstrations that the quantum fields (5) are Hilbert space operators; implies that states with sufficiently isolated and space-like separated dominant supports are described by free particles; provides the essential independence of the local observables of non-entangled, spatially distant bodies; and implies a unique vacuum.

3 Constructions of relativistic quantum mechanics

> In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization. – David Hilbert, in *Bulletin of the American Mathematical Society* 8 (1902), 437-479.

VEV (7) that realize relativistic quantum physics for basis function spaces $\underline{\mathcal{P}}$ are constructed in this section. These VEV generalize the single neutral scalar field example (17) introduced in section 2. The function sequences $\underline{f}$ that describe nature are elements of rigged Hilbert spaces $\mathbf{H}_{\mathcal{P}}$ that follow from basis function spaces $\underline{\mathcal{P}}$. These Hilbert space realizations of relativistic quantum physics are discussed in section 3.1. Prospective axioms A.1-7 are presented in section 4.1. The well-established VEV of relativistic free fields are discussed within the current context in section 3.2. Physically nontrivial VEV are constructed in section 3.3 as cluster expansions of split signed symmetric, connected functions. The symmetry implements local commutativity, and connectivity implements cluster decomposition. To implement relativity, the split signed symmetric, connected VEV functions are Poincaré covariant. Connected functions include free field VEV that characterize the elementary particles and interaction follows from $n \geq 4$, n-argument connected functions. The support of the VEV is limited to mass shells and Poincaré covariance provides that energy-momentum is conserved. Satisfaction of the axioms is demonstrated in sections 4.2.1-4.2.5. The basis spaces $\underline{\mathcal{P}}$ that ensure nonnegative energies are discussed in section 3.4. Also in section 3.4, the equivalence with constructions based on the unconstrained $\underline{\mathcal{S}}$ with positive energy support constrained VEV is demonstrated. The approach based upon support constrained functions $\underline{\mathcal{P}}$ is emphasized. The Hamiltonian operator

is evaluated in section 4.4. Scattering likelihoods are illustrated in section 4.5. Additional properties of the constructions are developed in sections 4.4, 5, and 6, and appendices.

3.1 Quantum mechanical description of state

Key considerations for realization of relativistic quantum physics include:

1. functions are the description of nature. State describing functions over spacetime and quantum numbers label elements of rigged Hilbert spaces, and the Fourier transforms of these functions describe momenta

2. the Hamiltonian generates time translation. To comply with relativity, the Hamiltonian must be a generator of a Hilbert space realization of the Poincaré group.

In the classical concept, observables are classical dynamical variables. The classical concept includes an observer who can describe the evolution of each body, and each body is identifiable. The description of state is considered independently of observation, or perhaps more precisely, the state is considered determined and observation need not disturb that state. This geometric description with an omniscient observer is what is meant by *classical physics* in these notes. Quantum mechanics supersedes, not "quantizes," classical descriptions. The quantum description of nature is fundamentally incompatible with classical description despite the often excellent approximations to quantum dynamics provided by classical dynamics. A description of nature as elements of rigged Hilbert spaces implements the discrete line spectra of atomic emissions, the quantized energy of photons, the entanglement of states necessary to consistent description of quantized conserved quantities, the indistinguishability of similarly described bodies that provides an extensive entropy, and manifests the observed wave-particle duality. When the dominant supports of state describing functions are well-represented by a single location and momentum, and isolated, the description provided by the state describing function is classical particle-like. For these particular state descriptions, there is a close correspondence of quantum and classical descriptions, section 5. More generally, description is inherently quantum mechanical. For widely supported states, or for significantly overlapping descriptions, or when entanglement applies, or when non-commuting observables are considered, or on small spatial scales, or for relativistic collisions, classical description is contradicted. For finite mass particles, the scale for "small" is generally set by the Compton wavelength (11). Trajectories for distinguishable bodies do not apply in quantum mechanics except approximately when a correspondence with classical bodies applies. The Heisenberg uncertainty principle is characteristic of the quantum description of nature. The likelihoods of perceptions of the location of natural objects are described by functions in a Hilbert space, and the likelihoods of perceptions of momentum are described by Fourier transforms of the state describing functions. The quantum description of state precludes classical description; states of nature are never classically described exactly. There are no functions of point support with

Fourier transforms of point support. The intrinsic understanding of quantum mechanics is the relative state (Everett-Wheeler-Graham) interpretation, [12] and appendix 8.2.8. The resulting understanding of observation explains how common perceptions of nature differ from their description, and resolves the Einstein-Podolosky-Rosen (EPR) [17], Schrödinger's cat [51] and Wigner's friend [65] paradoxes associated with earlier understandings of quantum mechanics. Quantum mechanics contradicts classical description and in this sense the concepts are disconcerting. But, quantum mechanics is necessitated to describe nature. The classical description is a perception facilitated by often excellent approximation. The correspondence of classical and quantum descriptions of nature is discussed further in section 5 and appendices.

3.1.1 States

Sequences of state describing functions (3) are elements of a rigged Hilbert space. These functions can depict the vacuum, a single particle, multiple particles, composite bodies, wave-like states, and combinations. Perception of these states by observers and the temporal evolution of states are the concerns of mechanics. This Hilbert space description fulfills essential properties of nature [14, 60] and is discussed in appendix 8.2.

The elements $|\underline{f}\rangle$ of the *rigged Hilbert spaces* of interest $\mathbf{H}_{\mathcal{P}}$ are within the completion of elements described by terminating sequences of functions from basis spaces $\underline{\mathcal{P}}$ [10, 11, 13, 56].

$$|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$$

is described by any one of an equivalence class of function sequences

$$\underline{f} = (f_0, f_1(x_1)_1, f_1(x_1)_2, \ldots f_1(x_1)_{N_c}, \ldots f_n((x)_n)_{(\kappa)_n}, \ldots) \tag{21}$$

with

1. $f_0 \in \mathbb{C}$ the component of the state in the vacuum state characterized by the sequence

$$\Omega = (1, 0, 0 \ldots)$$

2. N_c functions $f_1(x)_\kappa \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$ that describe single particle states with masses m_κ

3. $(N_c)^n$ functions $f_n((x)_n)_{(\kappa)_n} \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$,

$$f_n((x)_n)_{1\ldots11}, f_n((x)_n)_{1\ldots12}, \ldots f_n((x)_n)_{N_cN_c\ldots N_c}$$

for $n \geq 2$. $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$ is the n-argument subspace of $\mathbf{H}_{\mathcal{P}}$. Generally, the $f_n((x)_n)_{(\kappa)_n}$ are not interpretable as describing a determined set of particles. The functions $f_n((x)_n)_{(\kappa)_n}$ describe n particles of determined species in the absence of significant interaction, for example, if the dominant support of each argument is distantly space-like isolated from the dominant support of any other state describing function argument,

Each $\kappa_j \in \{1, 2, \ldots N_c\}$ labels one of N_c component fields. The basis function spaces $\underline{\mathcal{P}}$ are subspaces (9) of the space of terminating sequences of Schwartz tempered functions $\underline{\mathcal{S}}$, section 3.4.1. $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ provides that $\underline{\mathcal{S}}' \subseteq \underline{\mathcal{P}}'$ for the duals [21] but for the constructions, the constructed VEV are generalized functions in the dual $\underline{\mathcal{S}}'$ to $\underline{\mathcal{S}}$. Interpretation of the function sequences that describe the states $|\underline{f}\rangle$ is further illustrated in appendix 8.2.7.

The labels on arguments generally do not identify particular bodies. Gibb's paradox is resolved by the indistinguishability of similarly described bodies. For arguments x_k, κ_k of state describing functions $f_n((x)_n)_{(\kappa)_n}$ labeled by the same mass, charge and polarization, if the spatial supports of these similarly labeled arguments overlap, then the results of observation are not reliably associated with the support of a single argument. Isolation of support enables association of a volume of space with the state description provided by a particular argument: the volume of space reliably associates with the isolated support of an argument and this support associates with perceptions interpreted as properties of a classically described particle. From causality, the evolution of a perceived object is described as a trajectory and this description persists until significant support overlap occurs. The association of classical particles with state describing functions is developed further in section 5. Indistinguishability is implemented as split signed symmetry of the squared magnitude of VEV with interchange of argument labels, section 3.3.2.

To describe momentum, functions in the basis function spaces $\underline{\mathcal{P}}$ are required to have Fourier transforms. $\tilde{f}_n((p)_n)_{(\kappa)_n}$ denotes the Fourier transform of $f_n((x)_n)_{(\kappa)_n}$. The Fourier transform adopted here applies in four-dimensional spacetime and is the evident multiple argument extension of

$$\tilde{\psi}(p) = \int \frac{dx}{(2\pi)^2}\, e^{-ipx}\psi(x) \tag{22}$$

using wave number p, energy-momentum $P = \hbar p$, $E = cP_0$ and the Lorentz invariants $px = p_0 ct - \mathbf{p}\cdot\mathbf{x}$ and four dimensional spacetime volume element $dx = dx_0 dx_1 dx_2 dx_3$. $\hbar$ is Planck's constant h divided by 2π. To describe relativity, spacetime coordinates in four dimensions are designated $x = (ct, \mathbf{x})$ with $\mathbf{x} = (x, y, z)$ and energy-momentum vectors are $p = (E/(\hbar c), \mathbf{p})$ and $\mathbf{p} = (p_x, p_y, p_z)$. c is the speed of light and px is without units. $Px = \hbar px = Et - \mathbf{P}\cdot\mathbf{x}$. $x, p \in \mathbb{R}^4$ are Lorentz 4-vectors and $\mathbf{x}, \mathbf{p} \in \mathbb{R}^3$ are three-dimensional Euclidean vectors. $x^2 = (ct)^2 - \mathbf{x}^2$, $p^2 = (E/c)^2 - \mathbf{p}^2$ use the Minkowski signature. $\mathbf{p}\cdot\mathbf{x}$ is the dot product and $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x}$ is the squared Euclidean length. The units of spacetime coordinates are length, and wave numbers p have the units of inverse length. Mass m is in natural units and a relevant length associated with a mass m is the reduced Compton wavelength λ_c from (11). The Fourier transforms of generalized functions are defined [20] to satisfy Parseval's equality

$$\tilde{T}(\tilde{\psi}) = T(\psi). \tag{23}$$

As a consequence and when applicable, the Fourier transforms of generalized functions are

$$\tilde{T}(p) = \int \frac{dx}{(2\pi)^2}\, e^{ipx}T(x)$$

with the sign reversal of the exponent in the exponential function relative to (22). The Fourier transform is invertible [48].

$$\psi(x) = \int \frac{dp}{(2\pi)^2}\, e^{ipx}\tilde{\psi}(p).$$

Among the properties of Fourier transforms are the Fourier transform pairs

$$\begin{aligned} \psi(\Lambda(x-a)) &\leftrightarrow e^{-ipa}\widetilde{\psi}(\Lambda p) \\ \frac{d\psi(x)}{dx} &\leftrightarrow igp\widetilde{\psi}(p) \\ e^{-\alpha\|x\|^2} &\leftrightarrow \frac{1}{4\alpha^2}e^{-\|p\|^2/(4\alpha)} \end{aligned} \tag{24}$$

with $\|x\|^2$ the Euclidean (sum of squares) length squared of the Lorentz 4-vector x, Λ a Lorentz transformation, a a spacetime translation, and α a complex parameter with $\Re e(\alpha) > 0$. g is the Minkowski signature matrix,

$$gp = (p_0, -\mathbf{p}). \tag{25}$$

The third property in (24) suggests that functions $\psi(x)$ with broad spacetime support have Fourier transforms with concentrated energy-momentum support, and vice versa. Indeed, in general, the standard errors of $|\psi(x)|^2$ in spacetime and of $|\tilde{\psi}(p)|^2$ in energy-momenta considered as probability distributions satisfy $\sigma_x\sigma_p \geq \frac{1}{2}$ in each dimension. $\sigma_x\sigma_p \geq \frac{1}{2}$ illustrates the Heisenberg uncertainty principle discussed in appendix 8.7. More precise knowledge of the location of a body implies degraded knowledge of the time rate of change of the location, and this effect is more pronounced for low mass bodies than for heavy bodies due to $\hbar\mathbf{p} = \mathbf{P} \approx m\mathbf{v}$ at non-relativistic velocities $\mathbf{v}$. Then

$$\sigma_x\sigma_v \geq \frac{\hbar}{2m}$$

in each of the three dimensions.

3.1.2 The scalar product

Hilbert spaces have a scalar product that is complex in the cases of interest,

$$\langle \underline{f}|\underline{g}\rangle \in \mathbb{C},$$

for every pair of elements $|\underline{f}\rangle, |\underline{g}\rangle \in \mathbf{H}_{\mathcal{P}}$ described by function sequences $\underline{f}, \underline{g}$. The scalar product provides the norm,

$$\|\underline{f}\| = \sqrt{\langle \underline{f}|\underline{f}\rangle}. \tag{26}$$

To achieve Poincaré invariance of likelihoods and limit the support of states to nonnegative energies, a degenerate scalar product of function sequences $\underline{f}, \underline{g}$ is implemented with generalized functions

$$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \in \mathcal{S}'(\mathbb{R}^{4n})$$

[10, 21, 56]. The degenerate scalar product is

$$\begin{aligned}\mathcal{W}(\underline{f}^*, \underline{g}) &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \mathcal{W}_{n,m}(f^*_{n,(\kappa)_n}\, g_{m,(\kappa)_{n+1,n+m}})_{(\kappa)_{n+m}} \\ &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\ (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}\, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned} \tag{27}$$

with formal summation notation for generalized functions [20] in the last line. Each spacetime Lorentz vector x_k is summed over $\mathbb{R}^4$ and each $\kappa_j \in \mathbb{N}$ is summed from 1 to N_c. The indices n and m are summed over the nonnegative integers. The VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ generalize the Wightman functions $W_n((x)_n)_{(\kappa)_n}$ [10, 11, 56, 62]. Multiple arguments are denoted

$$(x)_{j,k} = x_j, x_{j+1}, \ldots x_k \in \mathbb{R}^{4(k-j+1)}$$

in the ascending case, $(x)_{j,k} = x_j, x_{j-1}, \ldots x_k$ otherwise and $(x)_n = (x)_{1,n}$. The $*$-dual $\underline{f}^*$ of a function sequence $\underline{f}$ uses complex conjugation, argument transpositions, and the nonsingular $N_c \times N_c$ linear transformation D from (14). The $*$-dual functions are

$$f^*_n((x)_n)_{(\kappa)_n} = (D^T\cdot)_n \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}$$

using the matrix notation (15) and then the Fourier transform[1] of $f^*_n((x)_n)_{(\kappa)_n}$ is (14). D, designated here as Dirac conjugation, is determined by representation of the Lorentz group and satisfies

$$\overline{D} D = \mathbb{I}_{N_c} \tag{28}$$

with $\mathbb{I}_{N_c}$ the $N_c \times N_c$ identity and as a consequence, the $*$-dual satisfies

$$\begin{aligned}\underline{f}^{**} &= \underline{f} \\ (\underline{f} + \underline{g})^* &= \underline{f}^* + \underline{g}^* \\ (\underline{g} \times \underline{f})^* &= \underline{f}^* \times \underline{g}^*.\end{aligned}$$

The linear space of function sequences $\underline{\mathcal{P}}$ becomes an algebra with the $\times$-product (6). The $*$-dual would be an involution of the algebra of function sequences if it were an automorphism.

[1] $\int dx\ e^{-ipx}\overline{\psi(x)} = \overline{\int dx\ e^{ipx}\psi(x)} = \overline{\tilde{\psi}(-p)}$.

The $*$-dual maps $\underline{\mathcal{S}} \mapsto \underline{\mathcal{S}}$ but, the $*$-dual is not an automorphism for the $\underline{\mathcal{P}}$ selected in the constructions: $\underline{\mathcal{P}}^* \neq \underline{\mathcal{P}}$ due to the nonnegative energy constraint (9) on the energy support of the elements of $\mathcal{P}(\mathbb{R}^{4n})$. From the representation of the $*$-dual (14), $\underline{\mathcal{P}} \cup \underline{\mathcal{P}}^* \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{c\Omega\}$ with $c \in \mathbb{R}$ and Ω the vacuum. Discussed in sections 2 and 3.4.1, the elements of $\mathcal{P}(\mathbb{R}^{4n})$ have zeros on the negative energy mass shells and the $*$-dual maps these zeros to the positive energy mass shells.

The sequence of generalized functions is denoted

$$\underline{\mathcal{W}} = (1, \mathcal{W}_{1,0}, \mathcal{W}_{0,1}, \ldots, \mathcal{W}_{n,0}, \mathcal{W}_{n-1,1} \ldots, \mathcal{W}_{0,n}, \ldots). \tag{29}$$

The arguments of $\mathcal{W}_{k,n-k}$ are $(x)_n, (\kappa)_n$ and similarly to (21), there are $(N_c)^n$ functions in the subsequence $\mathcal{W}_{k,n-k}$ distinguished by $(\kappa)_n$. The n arguments of $\mathcal{W}_{k,n-k}$ are denoted as k $*$-dual arguments and $n-k$ function arguments.

The scalar product of $\mathbf{H}_{\mathcal{P}}$ results from the isometry

$$\langle \underline{f} | \underline{g} \rangle = \mathcal{W}(\underline{f}^*, \underline{g})$$

that associates elements of the Hilbert space with equivalence classes of function sequences [13]. Then the norm (26) is

$$\|\underline{f}\| = (\mathcal{W}(\underline{f}^*, \underline{f}))^{1/2}.$$

An abbreviated notation $\langle f_n | g_m \rangle$ is used for a scalar product when the constituent j-argument functions $f_j = 0$ for $j \neq n$ and k-argument functions $g_k = 0$ when $k \neq m$ in function sequences $\underline{f}$ and $\underline{g}$.

$$\begin{aligned} \langle f_n | g_m \rangle \quad = \quad & \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ & \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1} \, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned} \tag{30}$$

The alternative developments for state describing functions $\underline{f}$ as either elements of $\underline{\mathcal{P}}$ or $\underline{\mathcal{S}}$, sections 2 and 3.4, illustrate an ambiguity with canonical quantization of relativistic physics. With the scalar products necessary for relativistic physics (27), an assignment of multiplier functions to VEV functions $\underline{\mathcal{W}}$ and state describing functions $\underline{f}$ is ambiguous. Eigenvectors $\underline{e}_q$ are determined given the canonically quantized operator Q for a classical dynamical variable q. In this note, approximations $\underline{\hat{e}}_q$ to the eigenvectors are within $\underline{\mathcal{P}}$ or $\underline{\mathcal{S}}$. Then, scalar products $\langle \underline{f} | \underline{\hat{e}}_g \rangle$ and state transition likelihoods vary with the assignment of multiplier functions to $\underline{\mathcal{W}}$. In nonrelativistic physics, the scalar product is determined as $\mathcal{L}^2$ for state describing functions over space.

3.1.3 The quantum field and VEV

The quantum field derives from the expression (8) for the scalar product. This definition relates the VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ in the scalar product (27) with the VEV of fields (8).

The quantum field is multiplication (6) in the algebra of function sequences. Borchers' [11] definition is

$$\Phi(\underline{f})\,\underline{g} = \underline{f} \times \underline{g} \tag{31}$$

with $\underline{f}, \underline{g} \in \mathbf{H}_{\mathcal{P}}$ but in this note, the discussion of fields $\Phi(\underline{f})$ is limited to sequences $\underline{f}$ with only single argument functions $f_1(x)_\kappa$ in the sequence $\underline{f}$ nonzero. The Schwartz kernel theorem [22] then provides the more general definition. With this convention, the quantum field consists of N_c component fields $\Phi(x)_\kappa$

$$\begin{aligned}\Phi(\underline{f}) &= \sum_{\kappa=1}^{N_c} \Phi(f_\kappa)_\kappa \\ &= \sum_{\kappa=1}^{N_c} \int dx\ \Phi(x)_\kappa f_1(x)_\kappa .\end{aligned}$$

with component field $\Phi(x)_\kappa$ associated with mass m_κ and formal summation notation [20] for the operator-valued distributions [10]. Comparison of the scalar products (8) and (27) identifies

$$(D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} = \langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1}\Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n}\Omega \rangle \tag{32}$$

in the matrix notation (15).

If the multiplication (6) preserves Hilbert space norm-equivalence classes of function sequences, then the definition (31) of field as multiplication in the algebra of function sequences elevates to Hilbert space operators. In Wightman's development [11, 56, 62], the basis function spaces are the $*$-algebra of tempered functions $\underline{\mathcal{S}}$. Then, $\underline{g}^* \times \underline{h} \in \underline{\mathcal{S}}$ for $\underline{g}, \underline{h} \in \underline{\mathcal{S}}$ and if the scalar product satisfies formal Hermiticity (16) ($\mathcal{W}(\underline{f}^*, \underline{g}) = \underline{W}(\underline{f}^* \times \underline{g})$), then the Cauchy-Schwarz-Bunyakovsky inequality demonstrates that the field (31) preserves equivalence classes. But, neither a $*$-involutive algebra of function sequences nor formally Hermitian VEV are necessary to realize relativistic quantum physics. The $*$-dual (14) is not an involution of the algebra $\underline{\mathcal{P}}$ used in this development, section 3.4.1 and [33, 37]. Nevertheless, the constructed quantum fields are unbounded Hilbert space operators, section 4.4. The constructed quantum fields are not Hermitian if physically nontrivial. In physically trivial free field and related instances, the basis space extends to $\underline{\mathcal{S}}$ and Hermitian field operators are realized.

If $\underline{e}$ were an eigenfunction sequence of the field, then for any sequence of single argument functions $\underline{f} \in \mathbf{H}_{\mathcal{P}}$

$$|\Phi(\underline{f})\underline{e}\rangle = A(\underline{f})|\underline{e}\rangle$$

with $A(\underline{f}) = \sum_\kappa \int dx \;\; A(x)_\kappa f(x)_\kappa$. The $A(x)_\kappa$ are solutions to the canonically quantized classical field equations. However, there are no eigenfunctions of the quantum field (31). An eigenfunction would be labeled by a sequence (21) of functions $\underline{e}$ such that

$$\Phi(\underline{f})\,\underline{e} = A(\underline{f})\,\underline{e}.$$

With no vacuum polarization,

$$\begin{aligned}\Phi(\underline{f})\,\underline{e} &= \underline{f} \times \underline{e} \\ &= (0, \underline{f}_1, 0, \ldots) \times (\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3 \ldots) \\ &= (0, \underline{f} \times \underline{e}_0, \underline{f} \times \underline{e}_1, \underline{f} \times \underline{e}_2 \ldots) \\ &= (A(\underline{f})\underline{e}_0, A(\underline{f})\underline{e}_1, A(\underline{f})\underline{e}_2, A(\underline{f})\underline{e}_3 \ldots)\end{aligned}$$

from the $\times$-product (6) and the definition of field (31) with $\underline{e}_n$ the sequence consisting of only n-argument functions. If $\underline{e}$ describes an eigenfunction of $\Phi(\underline{f})$ with a finite eigenvalue $A(\underline{f})$, then $A(\underline{f})\underline{e} - \underline{f} \times \underline{e}$ is a sequence in the null space of $\mathbf{H}_\mathcal{P}$. Selection of the sequence of zeroes to represent $A(\underline{f})\underline{e} - \underline{f} \times \underline{e}$ results in the recursive $A(\underline{f})\underline{e}_{n+1} = \underline{f}_1 \times \underline{e}_n$. Then $A(\underline{f})\underline{e}_0 = 0$ with $A(\underline{f}) \neq 0$ provides that $\underline{e} = 0$.

3.1.4 The adjoint of the field and Hermiticity

To canonically quantize a real classical field, field operators would be Hermitian. A definition of Hermiticity suitable for complex and multiple component fields is the field $\Phi(x)_\kappa$ is densely defined Hermitian if the composite fields (31)

$$\Phi(\underline{f}) = \sum_{\kappa=1}^{N_c} \Phi(f_\kappa)_\kappa$$

satisfy

$$\langle \underline{h} | \Phi(\underline{f})^* \underline{g} \rangle = \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle$$

for real function sequences $\underline{f} = \underline{f}^*$ and with

$$\langle \Phi(\underline{f})^* \underline{h} | \underline{g} \rangle = \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle$$

on a dense domain of function sequences $\underline{h}, \underline{g} \in \mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^*} \subset \mathbf{H}_\mathcal{P}$.

Real function sequences are dense in $*$-involutive spaces of function sequences such as the completion $\mathbf{H}_\mathcal{S}$ of $\underline{\mathcal{S}}$. The $*$-dual (14) function sequences $\underline{f}^*$ include an $N_c \times N_c$ linear transformation D from a representation of the Lorentz group, section 3.2.2. With complex coefficients, real functions are dense in $\underline{\mathcal{S}}$: every $\underline{h} \in \underline{\mathcal{S}}$ decomposes as

$$\underline{h} = \underline{h}_1 + i\underline{h}_2$$

for real $\underline{h}_1, \underline{h}_2 \in \underline{\mathcal{S}}$, $2\underline{h}_1 = \underline{h} + \underline{h}^*$ and $2\underline{h}_2 = -i(\underline{h} - \underline{h}^*)$. For VEV and function sequences that satisfy Wightman's axioms, the field is densely defined and Hermitian. These VEV include the physically trivial free fields. However, for the support limited (9) non-$*$-involutive basis space $\underline{\mathcal{P}}$, there are no nontrivial real function sequences. The adjoint functions $\underline{\mathcal{P}}^*$ are generally disjoint from the basis space of function sequences $\underline{\mathcal{P}}$. Every element $\underline{g} \in \underline{\mathcal{P}}$ has an adjoint $\underline{g}^* \in \underline{\mathcal{P}}^*$ defined by (14), but as a result of the support constraint (9), $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{(c, 0, 0 \ldots)\}$ with $c \in \mathbb{R}$.

Unless the basis space of function sequences is $*$-involutive and the VEV (7) satisfy formal Hermiticity (13), the quantum fields (31) are not Hermitian. For a densely defined Hermitian field,

$$\begin{aligned} \langle \Phi(\underline{f})\underline{h}|\underline{g}\rangle &= \langle \underline{h}|\Phi(\underline{f})^*\underline{g}\rangle \\ &= \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle \end{aligned}$$

with $\underline{f}, \underline{g}$ from a common, dense domain $\mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^*}$. In the generalized function notation (27) for fields (31),

$$\begin{aligned} \mathcal{W}((\underline{f} \times \underline{h})^*, \underline{g}\rangle &= \mathcal{W}(\underline{h}^*, \underline{f} \times \underline{g}) \\ &= \mathcal{W}(\underline{h}^* \times \underline{f}^*, \underline{g}\rangle \end{aligned}$$

from the properties of the $*$-dual (14). For the evident selection of functions $\underline{f}, \underline{g}$ within subspaces with selected numbers of arguments,

$$\begin{aligned} \mathcal{W}(\underline{h}^*, \underline{f} \times \underline{g}) - \mathcal{W}(\underline{h}^* \times \underline{f}, \underline{g}\rangle = & \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \\ & \times (D\cdot)_n \left(\mathcal{W}_{n-1,m+1}((x)_{n+m})_{(\kappa)_{n+m}} f_1(x_n)_{\kappa_n} - \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} (D\overline{f_1}(x_n))_{\kappa_n} \right) \\ & \times \overline{h_{n-1}}(x_{n-1}, \ldots x_1)_{\kappa_{n-1} \ldots \kappa_1} \, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}} \\ = 0. & \end{aligned}$$

This applies for all selections of $f_1 \in \mathbf{H}_\mathcal{P}$ and all h_{n-1}, g_m in the common, dense domain $\mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^*}$. As a consequence,

$$\mathcal{W}_{n-1,m+1}((x)_{n+m})_{(\kappa)_{n+m}} \, f_1(x_n)_{\kappa_n} = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \, (D\overline{f_1}(x_n))_{\kappa_n} \tag{33}$$

is effectively an equality within the scalar product (27) of the constructed Hilbert space.

If there are real functions $\underline{f} = \underline{f}^*$ in the Hilbert space, then the selection

$$f_1(x_n)_{\kappa_n} = (D\overline{f_1}(x_n))_{\kappa_n}$$

is valid and (33) is satisfied by VEV that are formally Hermitian VEV (16). For formally Hermitian VEV and if the basis space of function sequences is $*$-involutive, then Hermitian

fields follow. More generally, there are no nontrivial solutions to (33), and consequently no Hermitian field operators. This is the case if the basis spaces of function sequences lacks real sequences or if the VEV do not satisfy formal Hermiticity. Every linear Hilbert space operator has an adjoint, appendix 8.2.5, but in the case of fields constructed on the support limited (9) space of function sequences $\underline{\mathcal{P}}$, the field is generally not equal to this adjoint.

For VEV that satisfy formal Hermiticity (13), the adjoint of the field is

$$\Phi(\underline{f})^* = \Phi(\underline{f}^*)$$

with $\underline{f}^*$ from (14). Indeed, in Wightman's development [10, 56] or if a construction in this note satisfies formal Hermiticity (13), the definition of an adjoint results in the identification

$$\begin{aligned} \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle &= \underline{W}(\underline{h}^* \times \underline{f} \times \underline{g}\rangle \\ &= \underline{W}((\underline{f}^* \times \underline{h})^* \times \underline{g}\rangle \\ &= \langle \Phi(\underline{f}^*)\underline{h}|\underline{g}\rangle \\ &= \langle \Phi(\underline{f})^*\underline{h}|\underline{g}\rangle. \end{aligned} \tag{34}$$

This follows for the scalar product (13), the quantum field (31), the product of function sequences (6), and the properties of the $*$-dual of sequences (14). However, for the physically nontrivial constructions with VEV that satisfy formal Hermiticity, spectral support A.4 is generally not satisfied without a support limitation (9) on the basis function space $\underline{\mathcal{P}}$. And then, for $\underline{f}, \underline{g} \in \underline{\mathcal{P}}$ and VEV that satisfy formal Hermiticity, the adjoint of the field does not apply to function sequences.

$$\Phi(\underline{f}^*)\underline{g} = \underline{f}^* \times \underline{g} \notin \underline{\mathcal{P}}$$

unless $\underline{f}^* \times \underline{g} = 0$. $\underline{f}^* \times \underline{g}$ does not satisfy the support condition (9) unless $\underline{f}^* \times \underline{g}$ is in the equivalence class of zero. One-argument functions with Fourier transforms that vanish on both mass shells to satisfy the support conditions to be elements of $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^*$ are of the form $\underline{\tilde{f}} = (p^2 - m^2)\underline{\tilde{h}}$ with $\underline{h} \in \mathcal{S}(\mathbb{R}^4)$ and are in the equivalence class of zero due to the factors of $\delta(p^2 - m^2)$ in the constructed VEV.

3.1.5 Likelihoods: Born's rule

Physical understanding follows from Born's rule that determines the likelihoods of perceptions of the state describing functions. A separable Hilbert space has a denumerable basis of orthonormal elements designated $|\underline{e}_n\rangle$ here [3, 40]. These elements provide a resolution of the identity operator in $\mathbf{H}_{\mathcal{P}}$,

$$\mathbb{I} = \sum_n Q_n$$

with projection operators

$$Q_n = |\underline{e}_n\rangle\langle \underline{e}_n|$$

in bra-ket notation. Then the expansion of any state in this selected basis is

$$|\underline{f}\rangle = \sum_n \langle \underline{e}_n|\underline{f}\rangle \; |\underline{e}_n\rangle.$$

The projection operators Q_n are the elementary propositions [7, 60] of observation and for normalized states, $\|\underline{f}\| = 1$, the squared magnitudes $|\langle \underline{e}_n|\underline{f}\rangle|^2$ of the coefficients are the likelihoods that proposition n is answered affirmatively. The elementary propositions are the queries "will the state described by $\underline{f}$ be perceived as the state described by $\underline{e}_n$." This identification of likelihoods is *Born's rule*.

Born's rule provides that the likelihood of observing the state described by $|g\rangle$ for a state initially described by $|\psi\rangle$ is the squared magnitude of the scalar product,

$$\text{likelihood} = |\langle g|\psi\rangle|^2 \leq 1$$

for normalized state descriptions, $\|g\| = \|\psi\| = 1$. Born's rule requires no additional assumptions for the forms or properties of operators to evaluate likelihoods: a scalar product and the description (21) of states $|\underline{f}\rangle$ are inherent to the separable Hilbert space.

3.2 Relativistic free fields

3.2.1 Free field VEV

The well-known free field VEV satisfy both the original Wightman axioms [10, 11, 56, 62] and the prospective axioms A.1-7. Free field VEV functions describe elementary particles and are included in the constructions. Also, the $n \geq 4$, n-argument connected VEV functions that characterize interactions are based on structures from the free field VEV functions. The physically trivial free field VEV satisfy formal Hermiticity W.a and involutivity W.b.

Satisfaction of the Wightman axioms suffices to determine that the neutral scalar field two-point function $W_2(x_1, x_2)$ is a Källén-Lehmann form, a summation over a mass spectrum of the Pauli-Jordan function [54]. The two-point functions used in the constructions are extensions of the Pauli-Jordan function to include nonzero spins. The two-point functions used in the constructions have Fourier transforms

$$\begin{aligned} \tilde{W}_2(p_1, p_2)_{\kappa_1\kappa_2} &= \delta(p_1 + p_2)\; \delta_2^+ \; M(p_2)_{\kappa_1\kappa_2} \\ &= \delta(\mathbf{p}_1 + \mathbf{p}_2)\; \delta_1^- \delta_2^+ \; 2\sqrt{\omega_1\omega_2}\; M(p_2)_{\kappa_1\kappa_2} \end{aligned} \tag{35}$$

with

$$\delta_k^{\pm} = \theta(\pm E_k)\delta(p_k^2 - \lambda_{ck}^{-2})$$

and λ_{ck} is the reduced Compton wavelength (11) for mass m_{κ_k}. These wavelengths λ_{ck} are functions of κ_k and the $\delta_k^{\pm}$ are functions of both p_k and κ_k. The supports of $\delta_k^{\pm}$ are on the positive $(+)$ or negative $(-)$ energy mass shells. A mass m_κ is associated with each component

field $\Phi(x)_\kappa$. $M(p_2)_{\kappa_1\kappa_2}$ is an $N_c \times N_c$ array of multinomials in the components of the energy-momentum p_2.

The properties of the free field VEV suffice to imply that the quantum field (31) is a densely defined Hermitian Hilbert space operator and indeed, the VEV generate algebraically from the Hermitian field operator [24, 25, 52, 61]. The field operator

$$\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa + \Phi_o^-(f)_\kappa$$

has creation and annihilation components with commutation relations

$$[\Phi_o^-(f_1)_{\kappa_1}, \Phi_o^+(f_2)_{\kappa_2}]_\pm = W_2(f_1\, f_2)_{\kappa_1\kappa_2}$$

and that otherwise commute. The notation is that Φ_o^+ is the creation and Φ_o^- is the annihilation component. Whether the commutator or anti-commutator is used depends on the values of κ_1, κ_2 that determine the statistics type of the component field, boson or fermion, as specified below in (39). There is a cyclic vacuum state $|\Omega_o\rangle$ with

$$|\dots \Phi_o^-(f)_\kappa \Omega_o\rangle = 0.$$

In this algebraic evaluation of VEV, each $f_j \in \mathcal{S}(\mathbb{R}^4)$ [31]. From (32), the VEV functions and VEV of the field operators are related

$$(D\cdot)_k \mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n} = \langle \Phi_o(x_k)_{\kappa_k} \dots \Phi_o(x_1)_{\kappa_1} \Omega | \Phi_o(x_{k+1})_{\kappa_{k+1}} \dots \Phi_o(x_n)_{\kappa_n} \Omega \rangle.$$

Contrast of the two representations for the scalar product (8) and (27) with the definition of an adjoint operator, and the invertibility (28) of D provide that

$$\Phi_o(x)^*_{\kappa_1} = (D\Phi_o(x))_{\kappa_1}$$

and

$$\mathcal{F}_{k,2n-k}((x)_n)_{(\kappa)_n} = \sum_{pairs} \sigma(S,(\kappa)_n) \prod_{j=1}^{n} W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}} \tag{36}$$

with D from (14), and with the matrix notation (15). $\mathcal{F}_{k,2n-k}((x)_n)_{(\kappa)_n}$ is the result of the algebraic evaluations for the VEV of the free field and the summation is over all $(2n)!/(2^n n!)$ ways of pairing the indices $i_j, \ell_j \in \{1, 2n\}$ without regard to order and the indices are ordered $i_j < \ell_j$ within $W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}$. The sign of each term $\sigma(S,(\kappa)_n) = \pm 1$ is determined by particle statistics from the types of the indices $(\kappa)_n$. The $\sigma(S,(\kappa)_n)$ are positive if all indices $(\kappa)_n$ are boson indices; fermions introduce sign changes from the anti-commutation of $\Phi_o(x_j)^\pm_{\kappa_j}$ that achieve the pairing of indices $S = \{(i_1, \ell_1), \dots (i_n, \ell_n)\}$. The $\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ with an odd number of arguments n are zero. $\mathcal{F}_{0,0} = 1$.

The array $M(p)_{\kappa_1\kappa_2}$ in the two-point function (35) is used to construct the n-point connected functions. The elements of the $N_c \times N_c$ array $M(p)_{\kappa_1\kappa_2}$ are multinomials in the energy-momentum components [10] and in matrix notation, the array satisfies

$$DM(p) = C^*(p)C(p) \tag{37}$$

with D the Dirac conjugation matrix from (14) and $C^*(p)$ is the complex conjugate transpose of $C(p)$. The matrix nonnegativity [27] of $DM(p)$ provides the degenerate scalar product (27) [37]. Since $DM(p)$ is a nonnegative matrix, it is Hermitian and it follows from the condition (28) for D that

$$M(p)^* = DM(p)D^T. \tag{38}$$

Local commutativity A.5 is satisfied if the $M(p)$ are a direct sum decomposition into two components,

$$M(p) = \begin{pmatrix} B(p) & 0 \\ 0 & F(p) \end{pmatrix}, \tag{39}$$

that satisfy locality conditions

$$B(p)^T = B(-p), \quad \text{and} \quad F(p)^T = -F(-p). \tag{40}$$

The + sign applies for the boson constituent $B(p)$ and the − sign applies for the fermion constituent $F(p)$. (39) assigns a type, boson or fermion, to the value of each index $\kappa_j \in \{1, N_c\}$ for each j. From (39), the array $M(p)$ and consequently W_2 vanishes unless both indices κ_1, κ_2 are the same type, boson or fermion. Poincaré invariance of the scalar product is implied by two additional conditions on $M(p)$ and D.

$$\begin{aligned} S(A)M(p)S(A)^T &= M(\Lambda^{-1}p) \\ \overline{S(A)}D &= DS(A) \end{aligned} \tag{41}$$

with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz group (77). $A \in \mathrm{SL}(2,\mathbb{C})$, the group of determinant one, complex matrices. Poincaré invariance is demonstrated in section 4.2.3. Example $M(p)$, D and $S(A)$ that satisfy (28), (37), (39), (40) and (41) are illustrated in section 3.2.2.

$\underline{\mathcal{F}}$ exhibits local commutativity. Conditions (39) and (40) imply commutation or anti-commutation of free component fields. The two-point function (35), the Fourier transform convention (22), and translation invariance of the VEV express the local commutativity condi-

tion for free fields as

$$\begin{aligned}\langle\Omega|\left(\Phi_o(x)_{\kappa_1}\Phi_o(0)_{\kappa_2} \mp \Phi_o(0)_{\kappa_2}\Phi_o(x)_{\kappa_1}\right)\Omega\rangle &= \int dp\ \delta^+(p)\left(e^{-ipx}M(p)_{\kappa_1\kappa_2} \mp e^{ipx}M(p)_{\kappa_2\kappa_1}\right)\\ &= \int dp\ e^{-ipx}\left(\delta^+(p)M(p)_{\kappa_1\kappa_2} \mp \delta^-(p)M(-p)_{\kappa_2\kappa_1}\right)\\ &= \int dp\ e^{-ipx}M(p)_{\kappa_1\kappa_2}\left(\delta^+(p) - \delta^-(p)\right)\\ &= M(ig\tfrac{d}{dx})_{\kappa_1\kappa_2}\int dp\ e^{-ipx}\left(\delta^+(p) - \delta^-(p)\right)\end{aligned}$$

with $x = x_2 - x_1$ and $p = p_2$, after reflection of the summation variable $p \mapsto -p$ in the second term, substitution of (40), and the properties (24) of the Fourier transform applied to the elements of the array $M(p)$ that are multinomials in the components of p. In $M(p)$, the components of $p = p_0, p_1, p_2, p_3 \in \mathbb{R}^4$ are replaced by

$$p_\nu \mapsto ig_{\nu\nu}\frac{\partial}{\partial x_\nu}$$

with g the Minkowski signature matrix (25). $\delta^\pm(p_k) = \delta_k^\pm$ from (35). If $x^2 < 0$, then there is a Lorentz transformation Λ with $\Lambda x = -x$ and as a consequence, the Pauli-Jordan commutator function vanishes for space-like x. It then follows that the commutator/anti-commutator of free component fields generally vanishes for space-like x. This local commutativity is satisfied for functions from $\underline{\mathcal{S}}$. $\underline{\mathcal{S}}$ includes functions with bounded support. $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}}$ includes none of the functions with bounded support.

If $f(x) \in \mathbf{H}_\mathcal{P}$, then $f(x)$ is supported only on positive energies for energy-momenta on the mass shells, and $\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa$ and $\Phi_o(f^*)_\kappa = \Phi_o^-(f^*)_\kappa$. As a consequence, if $g_k, f_{n-k} \in \underline{\mathcal{P}}$, the only contributions $\mathcal{F}_{k,n-k}(g_k^* f_{n-k})$ to scalar products are from

$$(D\cdot)_k\mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_S \sigma(S,(\kappa)_{2k})\ \prod_{j=1}^k W_2(x_j, x_{i_j})_{\kappa_j\kappa_{i_j}}. \tag{42}$$

The summation $\sum_S$ includes the $k!$ distinct pairings j, i_j with $j \in \{1,k\}$ and $i_j \in \{k+1, 2k\}$. The scalar product (8) and definition (36) provide that $\sigma(S_o,(\kappa)_{2k}) = 1$ for the pairing

$$S_o = \{k, k-1, \ldots 1, k+1, k+2, \ldots 2k\}.$$

For $\underline{f} \in \underline{\mathcal{P}} \subset \underline{\mathcal{S}}$, only $k!$ of the $(2k)!/(2^k k!)$ terms for $n = 2k$ in (36) contribute to the free field VEV: terms in evaluations of the scalar product (27) that include two-point function argument pairings with a second argument from $\{1,k\}$ or a first argument from $\{k+1, 2k\}$ do not contribute. For the two-point function (35) and scalar product (27), and because functions in $\mathbf{H}_\mathcal{P}$ lack support on the negative energy mass shells,

$$W_2(f^*\, g^*)_{\kappa_1\kappa_2} = W_2(f\, g)_{\kappa_1\kappa_2} = W_2(f\, g^*)_{\kappa_1\kappa_2} = 0$$

and only $W_2(f^*\, g)_{\kappa_1\kappa_2}$ contributes if $f, g \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$. These properties of the free field VEV for function sequences $\underline{\mathcal{P}}$ provide that $\underline{\mathcal{F}}$ exhibits an unconditional split signed symmetry, (48) in section 3.3.2, that satisfies local commutativity A.5.

It is well-established [10, 56] that the sesquilinear form (27) defines a degenerate scalar product for function sequences from $\underline{\mathcal{S}}$ with free field VEV (36). $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$. The methods of this construction provide another demonstration of the nonnegativity of (27) for function sequences from $\underline{\mathcal{P}}$ with free field VEV (36). The cluster expansion (68) introduced below in section 3.3.4 establishes that

$$\begin{aligned} \underline{\mathcal{F}} &= \exp\circ({}^C\underline{\mathcal{F}}) \\ &= \mathbb{I} + {}^C\underline{\mathcal{F}} + \ldots + \tfrac{1}{n!}\, {}^C\underline{\mathcal{F}} \circ {}^C\underline{\mathcal{F}} \circ \ldots \circ {}^C\underline{\mathcal{F}} + \ldots \end{aligned}$$

with the sequence

$${}^C\underline{\mathcal{F}} = (0, 0, W_2, 0, \ldots).$$

Then, identifying that positively weighted linear combinations of VEV function sequences are nonnegative if the terms are individually nonnegative, and that $\circ$-products preserve the nonnegativity of individually nonnegative, split signed symmetric VEV sequences, section 4.2.1, demonstrates the nonnegativity of $\underline{\mathcal{F}}$ if $\sum_{\kappa_1,\kappa_2} W_2(f^*_{\kappa_1}\, f_{\kappa_2}) \geq 0$. This is manifestly nonnegative,

$$\begin{aligned} \sum_{\kappa_1,\kappa_2} W_2(f^*_{\kappa_1}\, f_{\kappa_2}) &= \sum_{\ell} \int dp\; \delta^+(p) \left| \sum_{\kappa} C_{\ell\kappa} \tilde{f}(p)_\kappa \right|^2 \\ &\geq 0 \end{aligned}$$

from the Parseval equality definition of the Fourier transform of generalized functions, the two-point function (35), the $*$-dual sequences (14), the nonnegativity (37) of the matrix DM and simplification.

The free field VEV (42) result in scattering likelihoods with particle number conserved, and incoming and outgoing momenta equal in pairs, that is, no exhibition of interaction. Constructed from the two-point function (35) and the basis function space $\underline{\mathcal{S}}$, the free fields $\Phi_o(\underline{f})$ (31) are Hermitian Hilbert space operators that realize the Wightman and Gårding-Wightman axioms [10, 11, 56, 62]. Constructed from (35) and the basis function spaces $\underline{\mathcal{P}}$, the free fields $\Phi_o(\underline{f})$ are not Hermitian operators but an available extension of the basis function spaces from $\underline{\mathcal{P}}$ to $\underline{\mathcal{S}}$ achieves Hermitian field operators for the free field [31]. Hermitian Hilbert space field operators appear to be peculiar to physically trivial realization. Physically trivial relativistic realizations include generalized free and Wick polynomial (monomial) fields [10, 56].

3.2.2 Example $M(p)$, D and $S(A)$

Realizations of free fields are available in [8, 10, 23, 24, 25, 52, 61]. These realizations provide Dirac conjugation D and two-point function arrays $M(p)$ with a realization of the Lorentz group

$S(A)$, $A \in \mathrm{SL}(2,\mathbb{C})$. These $N_c \times N_c$ matrices $D, M(p), S(A)$ satisfy (28), (37), (39), (40), (41) and (79):

$$\begin{aligned} \overline{D}D &= \mathbb{I}_{N_c} \\ DM(p) &= C^*(p)C(p) \\ M(\Lambda^{-1}p) &= S(A)M(p)S(A)^T \\ DS(A) &= \overline{S(A)}D \\ M(p) &= B(p) \oplus F(p) \\ B(p)^T &= B(-p) \\ F(p)^T &= -F(-p). \end{aligned} \tag{43}$$

with an $N_c \times N_c$ array $C(p)$ and $S(A)$ a representation of the Lorentz group (79). $N_c = N_B + N_F$ with $B(p)$ an $N_B \times N_B$ array of functions that describe bosons and $F(p)$ an $N_F \times N_F$ array of functions that describe fermions.

The conditions (43) are linear in $M(p)$ and therefore any nonnegative real multiple $|c|^2 M(p)$ also satisfies (43).

For a neutral scalar field, $N_c = 1$, the scalar representation of the Lorentz group applies and

$$M(p) = S(A) = D = 1.$$

A complex-valued (charged) scalar field has two component fields, $N_c = 2$.

$$\begin{aligned} M(p) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M(-p)^T \\ D &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S(A) &= \mathbb{I}_2 \end{aligned}$$

the 2×2 identity.

The charged scalar field has a symmetry associated with a charge, $S_\phi M(p) S_\phi^T = M(p)$ with

$$S_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$. The transformations S_ϕ commute with the Lorentz transformations $S(A)$ and the generators of translations,

$$[S_\phi, S(A)] = [S_\phi, \omega] = [S_\phi, \mathbf{p}] = 0$$

with ω from (10). S_ϕ introduces a symmetry in addition to Poincaré symmetry. The transformations S_ϕ leave two subspaces of one-argument state describing functions invariant,

$$S_\phi \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix} = e^{i\phi} \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix}, \text{ and } \quad S_\phi \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix}.$$

Invariance up to a phase under S_ϕ is physical equivalence and these functions are eigenfunctions of the Hermitian generator of S_ϕ. The commutation of S_ϕ with the generators of Poincaré transformations implies that labels assigned to state describing functions $f(x_1)_1, 0$ and $0, f(x_2)_2$ apply for all observers. The label is determined by classical correspondence and not axioms A.1-7. Labeling the two subspaces of states with the eigenvalues of the generator for S_ϕ, as charge 1 or -1, a charge operator

$$Q = -i \, \frac{d}{d\phi} S_\phi \Big|_{\phi=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{44}$$

results in

$$Q \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix} = \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix}, \text{ and } \quad Q \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix} = - \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix}.$$

If the spatial supports of each argument of the state describing functions are widely space-like separated and sufficiently localized at one time, a classical particle-like interpretation applies and the extension of the charge operator to the n-argument subspaces

$$Q f((x)_n)_{(\kappa)_n} = \sum_{j=1}^{n} (\delta_{1,\kappa_j} - \delta_{2,\kappa_j}) f((x)_n)_{(\kappa)_n}$$

results in a summation of the charges in the state described by $f((x)_n)_{(\kappa)_n}$. $\delta_{j,\ell}$ is the Kronecker delta. The extended Q commutes with the Hamiltonian $\omega_1 + \ldots \omega_n$ (103) in the n-particle subspaces and as a consequence, charge is conserved over time.

A massive, neutral vector boson field is realized with four component fields, $N_c = 4$. This spin-1 boson field uses a real 4×4 representation of the Lorentz group.

$$\begin{aligned} M(p)_{jk} &= \lambda_c^2 \, p_j p_k - g_{jk} \\ &= C^*(p) C(p) \\ D &= \mathbb{I}_4 \\ S(A) &= \Lambda \end{aligned}$$

with the components of p denoted p_0, p_1, p_2, p_3 and $p^2 = \lambda_c^{-2}$. g is the 4×4 Minkowski signature matrix and Λ is a Lorentz transformation determined by $A \in \mathrm{SL}(2, \mathbb{C})$ (78). $M(p) = M(-p)^T$.

In four dimensions with the components of p denoted p_0, p_x, p_y, p_z,

$$C^*(p) = \begin{pmatrix} \lambda_c\sqrt{p_0^2 - \lambda_c^{-2}} & 0 & 0 & 0 \\ \lambda_c p_0 \frac{p_x}{\sqrt{p_0^2-\lambda_c^{-2}}} & \sqrt{\frac{p_y^2+p_z^2}{p_0^2-\lambda_c^{-2}}} & 0 & 0 \\ \lambda_c p_0 \frac{p_y}{\sqrt{p_0^2-\lambda_c^{-2}}} & \frac{-p_x p_y}{\sqrt{(p_y^2+p_z^2)(p_0^2-\lambda_c^{-2})}} & \frac{p_z}{\sqrt{p_y^2+p_z^2}} & 0 \\ \lambda_c p_0 \frac{p_z}{\sqrt{p_0^2-\lambda_c^{-2}}} & \frac{-p_x p_z}{\sqrt{(p_y^2+p_z^2)(p_0^2-\lambda_c^{-2})}} & \frac{-p_y}{\sqrt{p_y^2+p_z^2}} & 0 \end{pmatrix}$$

with λ_c^{-1} the inverse of the Compton wavelength $\lambda_c^{-1} = mc/\hbar$. For both the neutral and charged scalar fields, (79) is trivially satisfied and for the neutral vector boson, (79) is $\Lambda = \Lambda_2\Lambda_1$, the composition of Lorentz transformations.

A massive, charged fermion field is also realized with four component fields, $N_c = 4$. This spin-1/2 fermion field uses a 4×4 complex spinor representation of the Lorentz group.

$$\begin{aligned} M(p) &= \begin{pmatrix} 0 & \beta(p) \\ \beta(p)^T & 0 \end{pmatrix} \\ \beta(p) &= \begin{pmatrix} p_0 + p_z & p_x + ip_y \\ p_x - ip_y & p_0 - p_z \end{pmatrix} \\ D &= \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \\ S(A) &= \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \end{aligned}$$

with 2×2 $\beta(p) = \beta(p)^*$, $\mathbb{I}_2$ and $A \in \mathrm{SL}(2,\mathbb{C})$. $M(-p) = -M(p)^T$.

$$DM(p) = C^*(p)C(p) = \begin{pmatrix} c^T(p) & 0 \\ 0 & c^*(p) \end{pmatrix} \begin{pmatrix} \overline{c}(p) & 0 \\ 0 & c(p) \end{pmatrix}$$

is a nonnegative matrix with p, $p^2 = \lambda_c^{-2}$, denoted p_0, p_x, p_y, p_z within the forward cone and

$$c(p) = \begin{pmatrix} \sqrt{p_0 + p_z} & \frac{p_x+ip_y}{\sqrt{p_0+p_z}} \\ 0 & \frac{\lambda_c^{-1}}{\sqrt{p_0+p_z}} \end{pmatrix}.$$

For massive, charged fermion field, the group property and composition of Poincaré transformations (79) follows from $A = A_2A_1 \in SL(2,\mathbb{C})$ if $A_1, A_2 \in SL(2,\mathbb{C})$ as a consequence of the multiplication of determinants [27].

A massive, charged fermion field also has a symmetry $S_\phi M(p) S_\phi^T = M(p)$ associated with a charge,

$$S_\phi = \begin{pmatrix} e^{i\phi}\mathbb{I}_2 & 0 \\ 0 & e^{-i\phi}\mathbb{I}_2 \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$. Two invariant up to a phase subspaces are spanned by

$$S_\phi \begin{pmatrix} f(x)_1 \\ f(x)_2 \\ 0 \\ 0 \end{pmatrix} = e^{i\phi} \begin{pmatrix} f(x)_1 \\ f(x)_2 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \quad S_\phi \begin{pmatrix} 0 \\ 0 \\ f(x)_3 \\ f(x)_4 \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 0 \\ 0 \\ f(x)_3 \\ f(x)_4 \end{pmatrix}.$$

Similarly to the charged scalar field, there is a charge operator in the one-argument subspace of $\mathbf{H}_\mathcal{P}$,

$$Q = -i \left. \frac{d}{d\phi} S_\phi \right|_{\phi=0} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}.$$

Real ($\overline{V} = V$) and imaginary ($\overline{V} = -V$) orthogonal ($V^T = V^{-1}$) similarity transforms $VM(p)V^T, VS(A)V^T, VDV^T$ preserve satisfaction of (43) if there is no $B(p)$ or no $F(p)$ constituent in $M(p)$, appendix 8.12. The direct sum decomposition (39) and (40) are preserved if the similarity transform preserves superselection sectors,

$$V = V_B \oplus V_F$$

with V_B of the same dimension as B in the direct sum decomposition (39).

The physical characteristics of a representation $M(p), D, S(A)$ may be changed by similarity transformation V. For the example of the massive, charged spin-1/2 fermion field above, the 4×4 real, orthogonal transformation

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}$$

transforms the charged field to a neutral field. For a real and orthogonal V,

$$V^{-1} = V^T = V^*.$$

The transformed representation is

$$\begin{aligned} M(p) &= \frac{1}{2} \begin{pmatrix} -(\beta(p) + \beta(p)^T) & \beta(p) - \beta(p)^T \\ -(\beta(p) - \beta(p)^T) & \beta(p) + \beta(p)^T \end{pmatrix} \\ D &= \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \\ S(A) &= \frac{1}{2} \begin{pmatrix} A + \overline{A} & A - \overline{A} \\ A - \overline{A} & A + \overline{A} \end{pmatrix} \end{aligned}$$

with the same 2×2 $\beta(p)$ and $A \in \mathrm{SL}(2, \mathbb{C})$ as the massive, charged spin-1/2 fermion field. $\beta(p) + \beta(p)^T$ is real and $\beta(p) - \beta(p)^T$ is imaginary. The unitarily implemented symmetry S_ϕ associated with a conserved charge for the charged fermion field is transformed to

$$S_\phi = \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix}.$$

This S_ϕ indicates a symmetry that implies identity of the neutral spin-1/2 particle with its antiparticle. The neutral spin-1/2 field is associated with Majorana fermions [61].

Direct sum and Kronecker product compositions, appendix 8.12, of these example and additional representations of

$$M(p), D, S(A)$$

and subrepresentations provide a rich class of realizations of relativistic quantum physics.

3.3 Physically nontrivial relativistic fields

VEV that exhibit nontrivial relativistic physics are displayed in this section. VEV follow from generalized VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ in (32).

$$\langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle = (D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$$

with Dirac conjugation D introduced in the $*$-dual of function sequences (14). The matrix notation (15) applies. Construction generalizes the single scalar field example (17) to multiple component fields. The constructed VEV are cluster expansions of split signed symmetric, connected VEV functions ${}^C\underline{\mathcal{W}}$ and free field VEV functions $\underline{\mathcal{F}}$ composed from two-point functions (35), section 3.2. The following sections describe connectivity, split signed symmetry, the composition of ${}^C\underline{\mathcal{W}}$ from functions and measures, and cluster expansion. The VEV inherit their Poincaré transformation properties from a free field realization. Earlier studies [2, 39] and Feynman series, [31, 35] and appendix 8.9, suggest appropriate forms for the constructions. However, without Hermitian field operators, the analytic extensions of VEV functions to imaginary time are not necessarily symmetric with transpositions of their arguments. The constructions generalize random process methods [2, 39].

3.3.1 Connected functions

Connected VEV functions include free field two-point functions (35) and $n \geq 4$, n-argument connected VEV functions illustrated in (17).

A *connected function* ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ attenuates with great space-like separation of its arguments. A connected function does not significantly contribute to scalar products if the

support of an $f_n((x)_n) \in \mathcal{S}(\mathbb{R}^{4n})$ consists of two greatly space-like separated volumes. That is, ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ is connected if

$$\lim_{\rho\to\infty} {}^C\mathcal{W}_{k,n-k}(f_n(\rho))_{(\kappa)_n} = 0 \tag{45}$$

with

$$f_n(\rho) = f_n(x_1, \ldots x_j, x_{j+1} - \rho a, \ldots, x_n - \rho a).$$

The Lorentz vector a is space-like ($a^2 < 0$), $\rho \in \mathbb{R}$, and $1 \leq j < n$. (45) applies for all $(N_c)^n$ components labeled by the κ_j. In a scalar product (27),

$$f_n((x)_n) = h_k^*(((x)_k)\, g_{n-k}((x)_{k+1,n}) \in \mathcal{S}(\mathbb{R}^{4n})$$

with $h_k \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4k})$ and $g_{n-k} \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4(n-k)})$, and the $*$-dual from (14).

In the constructions, connected VEV functions are designated

$$\begin{aligned} \mathcal{W}_{1,1}(x_1, x_2)_{\kappa_1\kappa_2} &= \mathcal{F}_{1,1}(x_1, x_2)_{\kappa_1\kappa_2} \\ &= W_2(x_1, x_2)_{\kappa_1\kappa_2} \\ {}^C\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} &= c_n\delta(p_1 + \ldots p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_{cj}^{-2})\, Q_{k,n-k}((p)_n)_{(\kappa)_n} \end{aligned} \tag{46}$$

with $2 \leq k \leq n-2$ and $n = 2\ell \geq 4$ for $\ell \in \mathbb{N}$. W_2 is a free field two-point function (35) and $\widetilde{\mathcal{F}}_{k,n-k}((x)_n)_{(\kappa)_n}$ are the designations for the VEV functions of a free field VEV, section 3.2. The λ_{cj} are Compton wavelengths (11) for the finite elementary particle masses m_{κ_j} associated with the component fields $\Phi(x_j)_{\kappa_j}$. The array of functions $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ characterizes the interaction for multiple component fields. $Q_{k,2n-k}$ is constructed to be physically nontrivial, section 4.5 and appendix 8.8, satisfy Lorentz covariance, section 4.2.3, and exhibit normal statistics (48). From section 3.3.3, to satisfy Lorentz covariance A.3 for nonzero spins, odd orders of the VEV vanish.

$${}^C\mathcal{W}_{k,2\ell-1-k}((x)_{2\ell-1})_{(\kappa)_{2\ell-1}} = 0$$

for all k. Nonzero spin elementary particles are created or annihilated in pairs although odd order VEV may contribute for scalar representations of the Lorentz group [31, 33]. To satisfy elemental stability A.7,

$${}^C\mathcal{W}_{0,k} = {}^C\mathcal{W}_{k,0} = {}^C\mathcal{W}_{1,k} = {}^C\mathcal{W}_{k,1} = 0$$

for $k \geq 3$ and

$$\mathcal{W}_{0,2} = \mathcal{W}_{2,0} = 0$$

evaluated for functions from $\underline{\mathcal{P}}$ and for all κ_j. For the initial construction of VEV functions, component field means (vacuum polarizations) are set to zero,

$$\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0,$$

for all component fields κ without loss of generality since nonzero means are independently assigned, section 3.3.5. The remaining connected VEV is

$$\mathcal{W}_{0,0} = 1,$$

the normalization of the vacuum.

Elements of the array $Q_{k,2n-k}$ are constructed in section 3.3.3 to be polynomial bounded growth, locally absolutely Lebesgue summable functions of the momenta $\mathbf{p}_j$ within the domains included in the summations that evaluate the scalar products (27). After evaluation of the energy-momentum conservation and mass shell delta functions, the remaining Lebesgue summations that evaluate (45) include a factor $\exp(i\rho\,\mathbf{a}\cdot(\mathbf{p}_1+\dots\mathbf{p}_k))$ with $1 \le k < n-1$. Then, the Riemann-Lebesgue lemma and Poincaré invariance of the scalar products demonstrate connectedness (45) for the connected VEV (46).

3.3.2 Split signed symmetric functions

The constructed VEV are split signed symmetric. The VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are signed symmetric with transpositions among either the k $*$-dual function arguments, or transpositions among the $n-k$ function arguments. If fields are not Hermitian operators, then no symmetry of the VEV with transposition of $*$-dual and function arguments is necessary. Local commutativity of the field $\Phi(x)_\kappa$ follows from split signed symmetry of the VEV functions. Argument type, $*$-dual or function, follows from definition of the degenerate scalar product (27).

A convenient shorthand notation for VEV is to designate

$$\mathcal{W}_{k,n-k}(A) = \mathcal{W}_{k,n-k}((x)_A)_{(\kappa)_A} \tag{47}$$

for a set of n arguments with indices designated $A = i_1, \dots i_n$. That is, $(x)_A = x_{i_1}, \dots x_{i_n}$ and similarly $(\kappa)_A = \kappa_{i_1}, \dots \kappa_{i_n}$.

In the notation (47), a sequence (29) of generalized functions $\underline{\mathcal{V}}$ is designated *split signed symmetric* if the elements satisfy

$$\mathcal{V}_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) = \sigma(\pi_1,(\kappa)_n)\sigma(\pi_2,(\kappa)_n)\mathcal{V}_{k,n-k}(\{1,n\}) \tag{48}$$

for every k, n. $\pi_1(\{1,n\})$ represents one of the $k!$ permutations of the sequence $\{1,k\}$ of $*$-dual function arguments. $\pi_2(\{1,n\})$ represents one of the $(n-k)!$ permutations of the sequence $\{k+1,n\}$ of function arguments. The sequences $\{1,k\}$ and $\{k+1,n\}$ are within the sequence of n elements $\{1,n\}$. In a notation with permutations denoted

$$\pi_\nu(A) = \pi_\nu(i_1), \pi_\nu(i_2), \dots \pi_\nu(i_n),$$

and $\nu = 1, 2$,

$$\begin{aligned}\pi_1(\{1,n\}) &= \pi_1(1), \pi_1(2) \dots \pi_1(k), k+1, k+2, \dots n\\ \pi_2(\{1,n\}) &= 1, 2, \dots k, \pi_2(k+1), \pi_2(k+2) \dots \pi_2(n).\end{aligned}$$

The signs $\sigma(\pi_\nu,(\kappa)_n) = \pm 1$ are assigned to each permutation to satisfy local commutativity A.5 with normal commutation relations [10].

$$\begin{array}{lll} \sigma(\pi_1,(\kappa)_n) = 1 & \text{if} & \pi_1(\{1,n\}) = \{1,n\} \\ \sigma(\pi_2,(\kappa)_n) = 1 & \text{if} & \pi_2(\{1,n\}) = \{1,n\}. \end{array}$$

and

$$\sigma(\pi'_\nu,(\kappa)_n) = -\sigma(\pi_\nu,(\kappa)_n)$$

if the permutations π'_ν and π_ν differ by one transposition of two adjacent fermion indices, and

$$\sigma(\pi'_\nu,(\kappa)_n) = \sigma(\pi_\nu,(\kappa)_n),$$

if the one transposition is either two adjacent boson indices or transposition of a boson with an adjacent fermion index. $\nu = 1,2$. In (39) of section 3.2, arguments x_j, κ_j are assigned as boson or fermion by the value of $\kappa_j \in \{1, N_c\}$. The sign associated with an ordering of indices $A = i_1, i_2, \ldots i_n = \pi_2(\pi_1(A_o))$ is with respect to a reference order $A_o = j_1, j_2, \ldots j_n$. The sign for an A with respect to A_o is accumulated over the sequence of adjacent index transpositions that result in the index order A starting from A_o. The sign is determined by the product of signs from the sequence of transpositions, and the sign is independent of the choice for a sequence of transpositions that result in A from A_o [10].

The *split signed symmetrization* $\underline{\mathcal{V}}$ of a sequence (29) of generalized functions $\underline{v}$ is

$$\mathcal{V}_{k,n-k}(A_o) = \sum_{\pi_1} \sigma(\pi_1,(\kappa)_{A_o}) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_{A_o})\, v_{k,n-k}(\pi_2(\pi_1(A_o))) \right). \tag{49}$$

The summations $\sum_{\pi_1}$ and $\sum_{\pi_2}$ include all $k!$ permutations of the indices labeled $\{1,k\}$ and all $(n-k)!$ permutations of the indices labeled $\{k+1,n\}$, respectively. The $\underline{\mathcal{V}}$ constructed from a generalized function $\underline{v}$ in (49) is split signed symmetric.

The free field VEV are split signed symmetric for function sequences from $\underline{\mathcal{P}}$. The expression (42) for $\underline{\mathcal{F}}$ is equivalent to

$$\mathcal{F}_{k,k}(\{1,2k\}) = \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_n)\, \frac{1}{k!} \prod_{j=1}^{k} W_2(x_j, x_{2k+1-j})_{\kappa_j \kappa_{k+j}} \right).$$

From the vanishing of two-point functions with indices of distinct type (39), the pair of indices of each contributing two-point function are necessarily the same type, and since $n = 2k$, transpositions among the last k indices are redundant with transpositions among the first k indices. The factor $k!$ compensates for the redundant transpositions.

If $\underline{w}$ provides a degenerate scalar product, then its split signed symmetrization $\underline{\mathcal{W}}$ also provides a degenerate scalar product since

$$\mathcal{W}(\underline{f}^*, \underline{f}) = w(\underline{g}^*, \underline{g}) \geq 0 \tag{50}$$

with

$$\underline{g} = (f_0, f_1, \ldots \sum_{\pi_1} \sigma(\pi_1, (\kappa)_n) f_k(\pi_1(\{1,k\})), \ldots)$$

from (49) for the scalar product (27).

If $\underline{\mathcal{V}}$ provides a degenerate scalar product, then a *scaled sequence* of functions $\underline{\mathcal{V}_a}$ generated from $\underline{\mathcal{V}}$ as

$$\underline{\mathcal{V}_a} = (|a_0|^2, \overline{a}_1 a_0 \mathcal{V}_{1,0}, \ldots \overline{a}_k a_{n-k} \mathcal{V}_{k,n-k}, \ldots), \tag{51}$$

also provides a degenerate scalar product. Each $a_k \in \mathbb{C}$ and $|a_k| > 0$. That $\mathcal{V}_a(\underline{f}^*, \underline{f}) \geq 0$ follows from $\mathcal{V}(\underline{g}^*, \underline{g}) \geq 0$ by setting the sequence $\underline{f} = (g_0/a_0, g_1/a_1, \ldots) \in \underline{\mathcal{P}}$ in the scalar product (27).

3.3.3 Connected VEV functions that exhibit interaction

Connected $n \geq 4$, n-argument VEV functions introduce interaction to the RQP constructions. The $n \geq 4$, n-argument connected functions are designated ${}^C\underline{\mathcal{W}}$ in (46). The VEV functions ${}^C\mathcal{W}_{k,n-k}$ are zero for $n < 4$. The constructed ${}^C\mathcal{W}_{k,n-k}$ are split signed symmetric (48), physically nontrivial, section 4.5 and appendix 8.8, Lorentz covariant, section 4.2.3 and nonnegative,

$${}^C\mathcal{W}(\underline{f}^*, \underline{f}) \geq 0$$

in the notation of the scalar product (27).

From (46), the Fourier transforms of the connected functions are

$${}^C\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} = c_n \delta(p_1 + \ldots p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_{cj}^{-2})\, Q_{k,n-k}((p)_n)_{(\kappa)_n}.$$

The support of these connected VEV functions over energy-momenta is the submanifold of $(p)_n \in \mathbb{R}^{4n}$ with energy-momentum conservation and each p_j on a mass shell, $p_j^2 = \lambda_{cj}^{-2}$. Energy-momentum conservation provides translation invariance of the scalar product (27). The construction of ${}^C\underline{\mathcal{W}}$ follows from the triple of $N_c \times N_c$ matrices $D, M(p), S(A)$ from the free field two-point function $W_2(x_1, x_2)_{\kappa_1 \kappa_2}$ in the description (46) of the connected VEV functions.

With substitutions,

$$\delta(p_1 + \ldots p_n) = \int \frac{du}{(2\pi)^4} \prod_{j=1}^{n} e^{-ip_j u}$$

and

$$c_n = \int d\sigma(\lambda)\, \lambda^n, \tag{52}$$

the Fourier transforms of the connected functions ${}^C\mathcal{W}_{k,2n-k}$ are expressed

$${}^C\widetilde{\mathcal{W}}_{k,2n-k}(\{1,2n\}) = \iint \frac{d\sigma(\lambda)\, du}{(2\pi)^4}\, \lambda^{2n} \prod_{j=1}^{2n} e^{-ip_j u}\, \delta(p_j^2 - \lambda_{cj}^{-2})\, Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}} \tag{53}$$

in the shorthand notation (47). The summation du is over $\mathbb{R}^4$ and $d\sigma(\lambda)$ denotes a summation with nonnegative weight over $\mathbb{R}$. Then $c_{2n} \geq 0$. For example, $c_n = \rho^n$ for $d\sigma(\lambda) = \delta(\lambda - \rho)d\lambda$, or $c_n = n!/\rho^{n+1}$ for $d\sigma(\lambda) = \theta(\lambda)\exp(-\rho\lambda)d\lambda$ with $0 < \rho \in \mathbb{R}$.

The $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ are split signed symmetrizations (49) of function arrays $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$.

$$\begin{aligned} Q_{k,n-k}((p)_n)_{(\kappa)_n} &= \sum_{\pi_2} \sigma(\pi_2, (\kappa)_n) \left(\sum_{\pi_1} \sigma(\pi_1, (\kappa)_n)\ \mathfrak{q}_{k,n-k}(\pi_2(\pi_1(\{1,n\})))\right) \\ \mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n} &= \prod_{j=1}^{n} \left(\frac{d}{d\rho_j}\right) \exp\Big(\sum_{a,b\in\mathbb{J}_{k,n}} \rho_a\rho_b\, \mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}\Big) \end{aligned} \tag{54}$$

for $2 \leq k \leq n-2$ and with the $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$ evaluated at $(\rho)_n = 0$ after the differentiation. The $(\rho)_n \in \mathbb{R}^n$ are independent parameters. To eliminate the divergent term from extrapolation of (54) to $n = 2$,

$$Q_{k,2n-k}(\{1,2n\}) = 0 \qquad \text{if} \quad k = 0, 1, 2n-1, 2n.$$

The $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$ are functions of products of pairs $\rho_a\rho_b$ with $a \neq b$ and as a consequence of evaluation at $(\rho)_n = 0$,

$$Q_{k,2n+1-k}(\{1,2n+1\}) = 0$$

for $0 \leq k \leq 2n-1$. The summations over argument transpositions π_1 and π_2 in the split signed symmetrization (49) are described in section 3.3.2. Split signed symmetry (48) of ${}^C\mathcal{W}_{k,n-k}(\{1,2n\})$ follows from the split signed symmetry of $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ together with invariance of the preceding factors in (53) to argument transpositions. The summation over $a, b \in \mathbb{J}_{k,n}$ with $a, b \in \{1,n\} \subset \mathbb{N}$ is defined below in (57). There is significant freedom in selection of this summation over $a, b \in \mathbb{J}_{k,n}$. The $N_c \times N_c$ array of functions $\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}$ over energy-momenta p_j is described by two distinct $N_c \times N_c$ matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$ varying with the values of a, b.

$$\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b} = \begin{cases} \mathfrak{B}(\alpha_o p_a + p_b)_{\kappa_a\kappa_b} & \text{if } a,b \in \{k+1,n\} \\ \mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a\kappa_b} & \text{if } a,b \in \{1,k\} \\ \Upsilon(-p_a + p_b)_{\kappa_a\kappa_b} & \text{if } a \in \{1,k\}, b \in \{k+1,n\} \\ 0 & \text{if } a \in \{k+1,n\}, b \in \{1,k\}. \end{cases} \tag{55}$$

Considered as matrices, $\mathfrak{h} = \mathfrak{B}$ if arguments (p_a,κ_a) and (p_b,κ_b) are both $*$-dual function arguments or if both are function arguments, and $\mathfrak{h} = \Upsilon$ or zero if the arguments are split, one from a $*$-dual function and one from a function. Indices $a, b \in \{1,k\}$ are $*$-dual function arguments, and indices $a, b \in \{k+1,n\}$ are function arguments in a scalar product (54). $\alpha_o \geq 0$ is a nonnegative real parameter that determines dynamics. Addressed in section 4.2.5,

this form for $\mathfrak{h}(p_1,p_2)$ satisfies regularity A.1 for nonconstant $\mathfrak{h}(p_1,p_2)_{\kappa_1\kappa_2}$. Due to Lorentz covariance, nonconstant $\mathfrak{h}(p_1,p_2)_{\kappa_1\kappa_2}$ are necessarily singular for $\mathfrak{B}(0)$ or $\Upsilon(0)$, and (55) is selected to exclude these singularities from the submanifold of joint support of the VEV functions ${}^C\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}$ and state describing functions from $\underline{\mathcal{P}}$. The summations in the scalar products (27) are over submanifolds of $(p)_n \in \mathbb{R}^{4n}$ described in (46). The matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$ that define $\mathfrak{h}(p_1,p_2)_{\kappa_1\kappa_2}$ in (55) are Fourier or Laplace transforms of the array $M(p)$ from the free field two-point connected function (35) in the specification of connected functions (46).

$$\begin{aligned} \mathfrak{B}(p) &= \int d\mu_{\mathfrak{B}}(s)\, e^{isp} M(s) \\ \Upsilon(p) &= \int d\mu_{\Upsilon}(s)\, e^{-sp} M(s) \end{aligned} \tag{56}$$

with $s \in \mathbb{R}^4$, and s_0 is limited to the Poincaré invariant $s_0 > 0$ for Υ. $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$ are real-valued Lorentz invariant measures, $d\mu_{\Upsilon}(s)$ is a nonnegative measure, and $sp = s_0 p_0 - \mathbf{s}\cdot\mathbf{p}$ is a Lorentz invariant. The measures [54] are limited to achieve polynomial bounded growth, locally absolutely Lebesgue summable functions $\mathfrak{B}(p)$ and $\Upsilon(p)$ over the momenta $\mathbf{p}_j$ when the $p_{j0} = \omega_j$ within the domains included in the summations that evaluate the scalar products (27). The $N_c \times N_c$ matrix triple $M(p), D, S(A)$ satisfy (43). The evaluation of $Q_{2,2}((p)_4)_{(\kappa)_4}$ in terms of constituents $\mathfrak{B}(p)$ and $\Upsilon(p)$ is illustrated in appendix 8.8. $Q_{2,2}((p)_4)_{(\kappa)_4}$ provides boson scattering, fermion scattering, fermion pair production from bosons, fermion pair annihilation to bosons, and fermion-boson scattering amplitudes. Like $M(p)$, the $\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}$ determined in (55) and (56) are zero if κ_a and κ_b are distinct index types, a fermion with a boson index.

The scalar product follows from a factorization of the constructed $Q_{k,n-k}((p)_n)_{(\kappa)_n}$. Three factors follow from the discrimination of terms in (55) considering the number of factors of D contributed from $*$-dual functions (14). From (54),

$$\begin{aligned} (D\cdot)_k \exp\Big(\sum_{a,b\in\mathbb{J}_{k,n}} \rho_a\rho_b \mathfrak{h}_{\kappa_a\kappa_b}\Big) &= \exp\Big(\sum_{b_1>a_1=1}^{k} \rho_{a_1}\rho_{b_1} (D\mathfrak{B}(p_{a_1}+\alpha_o p_{b_1})D^T)_{\kappa_{a_1}\kappa_{b_1}}\Big) \\ &\times \exp\Big(\sum_{b_2>a_2=k+1}^{n} \rho_{a_2}\rho_{b_2} \mathfrak{B}(\alpha_o p_{a_1}+p_{b_1})_{\kappa_{a_2}\kappa_{b_2}}\Big) \\ &\times \exp\Big(\sum_{\ell=1}^{\mathfrak{I}_{k,n}} \rho_{k+1-\ell}\rho_{k+\ell} (D\Upsilon(-p_{k+1-\ell}+p_{k+\ell})_{\kappa_{k+1-\ell}\kappa_{k+\ell}}\Big). \end{aligned} \tag{57}$$

$\sum_{b>a=1}^{k}$ designates $\sum_{a=1}^{k-1}\sum_{b=a+1}^{k}$. The limit on the third summation is

$$\mathfrak{I}_{k,n} = \min(k, n+1-k).$$

(57) determines the summation of $a, b \in \mathbb{J}_{k,n}$ in (54). Additional forms for $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ also provide realizations. The description of interaction includes the summation over $\mathbb{J}_{k,n}$, the c_n, α_o and the symmetrization (49), appendix 8.8.

The upper limit in the summation to $\beth_{k,n}$ in (57) can be extended arbitrarily. The additional terms are set to zero after the differentiations with respect to $(\rho)_n$ indicated in (54). Only terms generated by differentiations with $\rho_j \in \{\rho_1, \rho_2, \ldots \rho_n\}$ contribute. The additional terms from setting $\beth_{k,n} \to \infty$ and including ρ_j for any integer j in (57) do not contribute to the connected VEV functions. This notation simplifies demonstration of the nonnegativity of scalar products in section 4.2.2.

3.3.4 Cluster expansion of VEV

Cluster expansions compose the connected functions (46) into VEV (32). The VEV consist of the connected functions from sections 3.2 and 3.3.3. Connectedness and the cluster expansion satisfy imply the cluster decomposition property of VEV, A.6 in section 4.1.

Cluster expansion is illustrated by the two- and four-point arrays of VEV functions $\mathcal{W}_{1,1}$ and $\mathcal{W}_{2,2}$. With designations B for the boson and F for the fermion blocks of the block diagonal $\mathcal{F}_{1,1}$ from (39), in the abbreviated notation (47), and expressing B and F as $N_c \times N_c$ arrays,

$$\begin{aligned} B(12) &= \mathcal{F}_{1,1}(12) \quad \text{if } 1 \le \kappa_1, \kappa_2 \le N_b \\ &= 0 \quad \text{otherwise} \\ F(12) &= \mathcal{F}_{1,1}(12) \quad \text{if } N_b + 1 \le \kappa_1, \kappa_2 \le N_c \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and $\mathcal{F}_{1,1}(12) = 0$ if the index types of κ_1 and κ_2 differ, then the cluster expansion is

$$\begin{aligned} \mathcal{W}_{1,1}(12) &= \mathcal{F}_{1,1}(12) \\ &= B(12) + F(12) \\ \mathcal{W}_{2,2}(1234) &= B(13)B(24) + B(14)B(23) \\ &\quad + B(14)F(23) + B(13)F(24) + F(13)B(24) + F(14)B(23) \\ &\quad - F(13)F(24) + F(14)F(23) + {}^{C}\mathcal{W}_{2,2}(1234). \end{aligned} \tag{58}$$

Particular VEV functions and a sign follow from the selection of indices κ_j. In (58), vacuum polarization is zero. N_b is the number of boson component fields and $N_b \le N_c$, the number of component fields. The index type of each index κ_j, boson or fermion, is determined in (39) and the free field contribution to (58) is determined in section 3.2 by the normal commutation/anti-commutation relations of free field operators. The cluster expansion (58) satisfies both local commutativity A.5 with normal commutation relations [10], and cluster decomposition A.6. ${}^{C}\mathcal{W}_{2,2}$ is split signed symmetric (48). Satisfaction of local commutativity A.5 and normal statistics (48) follows from the split signed symmetry of the connected functions for function sequences from $\underline{\mathcal{P}}$.

$$\begin{aligned} \mathcal{W}_{2,2}(1234) &= \pm\mathcal{W}_{2,2}(2134) \\ &= \pm\mathcal{W}_{2,2}(1243) \end{aligned}$$

where $\pm$ is the sign from transposition of the arguments x_1, κ_1 with x_2, κ_2, or from the transposition of the arguments x_3, κ_3 with x_4, κ_4, respectively. The strong cluster decomposition condition A.6 is satisfied due to the connectivity (45) of the ${}^{C}\mathcal{W}_{k,n-k}$. For arguments 1 and 3 space-like distant from arguments 2 and 4 , the support of the functions $g_1^* f_1^* f_1 g_1$ has the support of the f's arbitrarily space-like distant from the support of the g's),

$$\mathcal{W}_{2,2}(1234) \to \mathcal{F}_{1,1}(14)\mathcal{F}_{1,1}(23).$$

The remaining cases in axiom A.6 result in the four-point functions $\mathcal{W}_{2,2}$ asymptotically approaching zero.

For the construction, it is convenient to segregate the free field VEV $\underline{\mathcal{F}}$ from the cluster expansion of connected functions ${}^{C}\underline{\mathcal{W}}$. The cluster expansion of ${}^{C}\underline{\mathcal{W}}$ lacks a contribution of a connected two-point function. The cluster expansion of ${}^{C}\underline{\mathcal{W}}$ is designated $\underline{\mathcal{U}}$.

$$\underline{\mathcal{U}} = \text{cluster expansion of } {}^{C}\underline{\mathcal{W}}. \tag{59}$$

The VEV functions are the component functions of the sequence of functions $\underline{\mathcal{W}}$ resulting from a $\circ$-product of free field VEV functions $\underline{\mathcal{F}}$ with the $n \geq 4$, n-argument VEV functions $\underline{\mathcal{U}}$.

$$\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}. \tag{60}$$

$\underline{\mathcal{F}}$ from (42) results from cluster expansion of the two-point connected function (35) and $\underline{\mathcal{U}}$ results from cluster expansion of the sequence of connected functions ${}^{C}\underline{\mathcal{W}}$. To satisfy Poincaré invariance A.3, the sequences $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$ composed in the construction (60) transform with the same representation of the Lorentz group.

The $\circ$-product $\underline{\mathcal{W}}$ of two VEV function sequences (29) $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ is

$$\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}, \tag{61}$$

the result of the split signed symmetrization (49) of a function sequence $\underline{w}$ constructed from the component functions of $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$.

$$\begin{aligned} \mathcal{W}_{k,n-k}(\{1,n\}) &= \sum_{\pi_1} \sigma(\pi_1, (\kappa)_n) \left(\sum_{\pi_2} \sigma(\pi_2, (\kappa)_n)\, w_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) \right) \\ w_{k,n-k}(\{1,n\}) &= \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!} \end{aligned} \tag{62}$$

with a reference argument argument order A_o, A'_o

$$\begin{aligned} A_o &= \{1, \ell\} \cup \{k+1, k+j\} \\ A'_o &= \{1,n\} \setminus A_o = \{\ell+1, k\} \cup \{k+j+1, n\}. \end{aligned} \tag{63}$$

The terms in the summation (62) with the arguments in the reference order have positive sign. In section 4.2.1 it is demonstrated that this choice of sign results in a nonnegative degenerate scalar product (27). Signs are determined in section 3.3.2 to exhibit normal statistics. $\ell = 0$ provides that there are no $*$-dual arguments in $\mathcal{T}_{0,j}$ and $\ell = j$ provides that there are no function arguments in $\mathcal{T}_{j,j}$. Similarly for $\mathcal{V}_{k-\ell,n-k-j}$ if $\ell = k$ or $\ell = 2k+j-n$. The terms in the expansion $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ are products of elements from $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$, factors include no arguments in common, and all arguments are present in each term of the expansion (62). As a consequence, the sequence $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ consists of generalized functions if $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are sequences of generalized functions.

A convenient alternative expression for the elements of $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ follows if the component functions in the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are split signed symmetric (48). Use of (48) to transpose the arguments into natural order results in

$$\mathcal{W}_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \left(\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2} \sigma_1 \sigma_2 \, \mathcal{T}_{\ell,j}(A) \, \mathcal{V}_{k-\ell,n-k-j}(A') \right) \tag{64}$$

with the abbreviated notation $\sigma_\nu = \sigma(\pi_\nu, (\kappa)_n)$ for $\nu = 1, 2$, and

$$A = \pi_1(\pi_2(A_o)) \quad \text{and} \quad A' = \pi_1(\pi_2(A'_o))$$

with the reference order A_o and A'_o from (63). The A' are the set complements with respect to $\{1, n\}$ of A. The summations $\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2}$ include only the subset of the permutations π_1, π_2,

$$\{\mathfrak{s}_\nu\} \subset \{\pi_\nu\}$$

that result in naturally ordered A and A'. With

$$A = \pi_1(\pi_2(A_o)) = i_1, i_2, \ldots i_{\ell+j},$$

the permutation is included in the summation $\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2}$ only if $i_1 < i_2 < \ldots i_{\ell+j}$ and the $n-\ell-j$ argument indices in A' are similarly ordered. Equality of (64) and (62) follows from that after the indicated argument orderings, there are $\ell! j! (k-\ell)! (n-k-j)!$ of each distinct term. The normalization in (62) results in a magnitude unity coefficient for each term $\mathcal{T}_{\ell,j}(A) \, \mathcal{V}_{k-\ell,n-k-j}(A')$ with distinct, naturally ordered sets of arguments.

Demonstrated in section 4.2, axioms A.1-7 are satisfied if the factor sequences in the construction (60) are split signed symmetric (48) and each factor sequence satisfies A.1-7. The free field contributions $\underline{\mathcal{F}}$ associate the constructions with particles, section 4.3, and the higher-order connected functions $\underline{\mathcal{U}}$ exhibit interaction, sections 4.5 and 5. In the construction (60) of VEV, $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$, only the $\mathcal{F}_{\ell,\ell}$ contribute for function sequences from $\underline{\mathcal{P}}$. Consequently, only terms with $j = \ell$ contribute to the summation in (62) and then

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \frac{\mathcal{F}_{\ell,\ell}(A_o)}{(\ell!)^2} \, \frac{\mathcal{U}_{k-\ell,n-k-\ell}(A'_o)}{(k-\ell)!(n-k-\ell)!}. \tag{65}$$

The $\circ$-product (62) is commutative, associative and distributive with addition of VEV function sequences. The commutativity of, $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is demonstrated by relabeling the summations in the $\circ$-product (62) $\ell' = k - \ell$ and $j' = n - k - j$. With the relabeling,

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \, \frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!}$$

becomes

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell'=0}^{k} \sum_{j'=0}^{n-k} \frac{\mathcal{V}_{\ell',j'}(A'_o)}{\ell'! j'!} \, \frac{\mathcal{T}_{k-\ell',n-k-j'}(A_o)}{(k-\ell')!(n-k-j')!}.$$

This is (62) for $\underline{\mathcal{V}} \circ \underline{\mathcal{T}}$ with the same terms and signs as $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$. That (62) is a summation of arithmetic multiplications provides that the $\circ$-product is associative and distributive with addition of sequences.

The exponential of a sequence $\underline{\mathcal{T}}$ utilizing the $\circ$-product (61) provides its *cluster expansion*. The function sequence (29) resulting from the series

$$\exp\circ(\underline{\mathcal{T}}) = \Omega + \sum_{j=1}^{\infty} \frac{1}{j!} \, \underline{\mathcal{T}} \circ \underline{\mathcal{T}} \circ \ldots \underline{\mathcal{T}} \tag{66}$$

is the cluster expansion of $\underline{\mathcal{T}}$. The series has j factors of $\underline{\mathcal{T}}$ in the jth term.

The cluster expansion (66) of the connected functions ${}^{C}\underline{\mathcal{W}}$ from (53) was designated $\underline{\mathcal{U}}$,

$$\underline{\mathcal{U}} = \exp\circ({}^{C}\underline{\mathcal{W}}) \tag{67}$$

from (59). The VEV functions $\underline{\mathcal{W}}$ result from (60) with this $\underline{\mathcal{U}}$.

The free field functions $\underline{\mathcal{F}}$ are the cluster expansion of the connected two-point function.

$$\underline{\mathcal{F}} = \exp\circ({}^{C}\underline{\mathcal{F}}) \tag{68}$$

with

$${}^{C}\mathcal{F}_{1,1} = W_2$$

and all other ${}^{C}\mathcal{F}_{k,n-k} = 0$. W_2 is the two-point function (46) from (35). The nonzero elements of $\underline{\mathcal{F}}$ are

$$\mathcal{F}_{k,k} = \frac{1}{k!}({}^{C}\underline{\mathcal{F}}\circ)^k.$$

Equality of this cluster expansion (68) and the sequence $\underline{\mathcal{F}}$ constructed algebraically from the commutation relations of the creation and annihilation components of the free field in (42)

is demonstrated by induction. Inspection of the algebraic construction (42) and the cluster expansion (68) verifies agreement for $k = 1$. Then, assertion that

$$\mathcal{F}_{k,k}(\{1,2k\}) = \sum_S \sigma(S,(\kappa)_{2k}) \prod_{j=1}^{k} W_2(j,i_j)$$

and application of the cluster expansion to evaluate the next element provides

$$\begin{aligned}\mathcal{F}_{k+1,k+1}(\{1,2k+2\}) &= \frac{1}{k+1}\, \mathcal{F}_{k,k}(\{1,k\}\cup\{k+2,2k+1\}) \circ W_2(k+1,2k+2) \\ &= \frac{1}{k+1} \sum_S \sigma_S \prod_{j=1}^{k} W_2(j,i_j) \circ W_2(k+1,2k+2)\end{aligned}$$

with the notation from (42), (47) and (66), and the function argument labels $\{i_j\}$ adjusted to $i_j \in \{k+2, 2k+1\}$ for $j \in \{1,k\}$.

$$\sigma_S = \sigma(S,\{1,k\}\cup\{k+2,2k+1\})$$

from the algebraic construction (42). Substitution of the $\circ$-product (64) provides

$$\mathcal{F}_{k+1,k+1}(\{1,2k+2\}) = \sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}\frac{\sigma_1\sigma_2}{k+1}\left(\sum_S \sigma_S \prod_{j=1}^{k} W_2(\mathfrak{s}_2(\mathfrak{s}_1(j)),\mathfrak{s}_2(\mathfrak{s}_1(i_j)))W_2(\mathfrak{s}_2(\mathfrak{s}_1(k+1)),\mathfrak{s}_2(\mathfrak{s}_1(2k+2)))\right)$$

with the abbreviated notation $\sigma_\nu = \sigma(\pi_\nu,(\kappa)_{2k+2})$ for $\nu = 1,2$. The sums over selected permutations $\mathfrak{s}_1, \mathfrak{s}_2$ are from (64). From the free field sequence $\underline{\mathcal{F}}$ in the algebraic construction (42), the summation $\sum_S$ includes the $k!$ distinct pairings j, i_j with $j \in \{1,k\}$ and $i_j \in \{k+2, 2k+1\}$. $S = \{1,\ldots k, i_1, \ldots i_k\}$. From section 3.3.2, signs accumulate as the product of signs from adjacent argument transpositions that accumulate to achieve a final argument order. Then,

$$\sigma(S',(\kappa)_{2k+2}) = \sigma(\pi_1,(\kappa)_{2k+2})\sigma(\pi_2,(\kappa)_{2k+2})\sigma(S,(\kappa)_{2k})$$

with $S' = \{1,\ldots k+1, i_1, \ldots i_k, 2k+2\}$ with reference argument order $\sigma(S'_o) = 1$ for $S'_o = \{k+1,\ldots 1, k+2, \ldots 2k+2\}$. The terms are each a product of identical two-point functions with distinct pairings of arguments, and the indicated permutations $\mathfrak{s}_1, \mathfrak{s}_2$ result in an accumulation of $k+1$ of each distinct term. Then

$$\mathcal{F}_{k+1,k+1}(\{1,2k+2\}) = \sum_{S'} \sigma_{S'} \prod_{j=1}^{k+1} W_2(j,i_j)$$

with summation over distinct pairings of arguments. Finally, induction verifies that the two expressions for the free field VEV, (42) and (68), are equivalent.

From (66), the commutativity, associativity and distributivity with addition of sequences of the $\circ$-product (61), and the binomial expansion provide

$$\begin{aligned}
\exp\circ(\underline{\mathcal{T}}+\underline{\mathcal{V}}) &= \sum_{N=0}^{\infty}\frac{1}{N!}\left((\underline{\mathcal{T}}+\underline{\mathcal{V}})\circ\right)^N \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\nu=0}^{N}\binom{N}{\nu}(\underline{\mathcal{T}}\circ)^{\nu}\circ(\underline{\mathcal{V}}\circ)^{N-\nu} \\
&= \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\frac{1}{N_1!N_2!}(\underline{\mathcal{T}}\circ)^{N_1}\circ(\underline{\mathcal{V}}\circ)^{N_2} \\
&= (\exp\circ(\underline{\mathcal{T}}))\circ(\exp\circ(\underline{\mathcal{V}}))
\end{aligned} \tag{69}$$

from relabeling terms in summations. The expansions for $\underline{\mathcal{U}}$ and $\underline{\mathcal{F}}$, (67) and (68) respectively, substituted into the construction (60) provides that

$$\begin{aligned}
\underline{\mathcal{W}} &= \exp\circ({}^{C}\underline{\mathcal{F}})\circ\exp\circ({}^{C}\underline{\mathcal{U}}) \\
&= \exp\circ({}^{C}\underline{\mathcal{F}}+{}^{C}\underline{\mathcal{U}})
\end{aligned} \tag{70}$$

from the identity (69). The sequences of connected functions add in the construction (60).

Inversion of (66) defines truncated functions ${}^{T}\underline{\mathcal{W}}$ given a sequence $\underline{\mathcal{W}}$.

$$\underline{\mathcal{W}} = \exp\circ({}^{T}\underline{\mathcal{W}}). \tag{71}$$

Similarly to RQFT [10], truncated functions are connected in the constructions if vacuum polarization is zero. If both ${}^{1}\underline{\mathcal{W}}$ and ${}^{2}\underline{\mathcal{W}}$ are split signed symmetric and satisfy A.1-7, then ${}^{1}\underline{\mathcal{W}}\circ{}^{2}\underline{\mathcal{W}}$ realizes relativistic quantum physics and this leads to consideration of prime and divisible realizations [26]. From (69), truncated functions add.

This completes construction of example, physically nontrivial VEV. Together with specification of the basis function space $\underline{\mathcal{P}}$, section 3.4, a construction follows from specification of the connected functions (46). The high order connected functions ${}^{C}\underline{\mathcal{W}}$ follow from (53) with (54), (55), (56) and (57) in section 4.2. The cluster expansions (67) and (68) with specification (35) of the two-point function in section 3.2 complete a construction. Satisfaction of the properties of RQP, prospective axioms A.1-7 in section 4.1, is verified in section 4.2.

3.3.5 Vacuum polarizations

Mean values of the VEV of the field $\langle\Omega|\Phi(x)_{\kappa}\Omega\rangle$ are set to zero in the development without loss of generality. Nonzero means are freely specified for the boson field constituents without impact

to satisfaction of the prospective axioms A.1-6 in section 4.1. The statement of A.7 presented in section 4.1 assumes no vacuum polarization. The addition of a constant to the quantum field defined by (31) implements fields $\Phi(x)_\kappa$ with finite means except, if κ is a fermion index, then addition of a finite constant to the field violates anti-commutation.

The VEV

$$(D\cdot)_k \, \mathcal{V}_{k,n-k}(\{1,n\}) = \prod_{\ell=1}^{k} \overline{a}_{\kappa_\ell} \prod_{j=k+1}^{n} a_{\kappa_j}$$

satisfy A.1-6 and an appropriately restated A.7. The constants

$$a_\kappa = \langle \Omega | \Phi(x)_\kappa \Omega \rangle$$

and a_κ is finite only for boson indices κ. Applications of Dirac conjugations D use the matrix notation (15) and the abbreviated notation (47) is used for the VEV functions.

$\underline{\mathcal{V}}$ is trivially split signed symmetric (48) and as a consequence of the properties of the $\circ$-product (61), and from the definition of truncated functions (71) and the composition (69) of cluster expansions,

$$\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}} \circ \underline{\mathcal{V}} = \exp \circ ({}^{C}\underline{\mathcal{F}} + {}^{C}\underline{\mathcal{U}} + {}^{T}\underline{\mathcal{V}})$$

satisfies axioms A.1-6 and an appropriately restated A.7.

3.4 The basis function spaces

3.4.1 The basis function spaces $\underline{\mathcal{P}}$

A choice of support constrained (9) basis function spaces $\underline{\mathcal{P}}$ admits nontrivial VEV realizations that are unavailable in Wightman's original development [56, 62]. Wightman selected the Schwartz tempered test functions $\underline{\mathcal{S}}$. In this development, apparently unrealizable constraints on VEV are relaxed by placing constraints on the basis function space.

$\underline{\mathcal{P}}$ consists of those tempered functions with Fourier transforms that vanish on the appropriate negative energy mass shells. The appropriate mass shell is determined for each component field labeled κ_j by the mass m_{κ_j}.

$$\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}(\mathbb{R}^{4n})$$

if

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = 0 \qquad \text{when any energy} \qquad p_0 = -\omega_j$$

for $j = 1, \ldots n$ with

$$\hbar c \omega_j = \hbar c \omega(\mathbf{p}_j) = \sqrt{(m_{\kappa_j} c^2)^2 + (\hbar \mathbf{p}_j)^2 c^2}$$

from (10). The zeros are implemented using multiplier functions for tempered functions [21]. The functions $\varphi_n \in \mathcal{P}(\mathbb{R}^{4n})$ are constructed in (9) as

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n}(p_{j0} + \omega_j)\, \tilde{h}_n((p)_n)_{(\kappa)_n}$$

with $h_n \in \mathcal{S}(\mathbb{R}^{4n})$. $\varphi_0 = h_0 \in \mathbb{C}$. The infinitely differentiable and polynomially bounded in magnitude factors $(p_{k0} + \omega_k)$ are multipliers of tempered test functions [21]. The remaining issue is whether A.3 is satisfied: whether $\underline{\mathcal{P}}$ is stable under Poincaré transformations (77). The Lorentz invariance of p^2 and that $(a, \Lambda)\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ for every $\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ provides that $\underline{\mathcal{P}}$ is stable under Lorentz transformations. Indeed, the Lorentz invariance of p^2 provides that a zero of the appropriate form (9) is stable with proper, orthochronous Lorentz transformation,

$$p'_{0,j} + \omega'_j = (0, \Lambda)(p_{j0} + \omega_j) = (p_{j0} + \omega_j)\frac{(p_{j0} - \omega_j)}{(0, \Lambda)(p_{j0} - \omega_j)}$$

for the Poincaré transformation (a, Λ). The final factor is regular in a neighborhood of the negative energy mass shell and $p'_{0,j} + \omega'_j = 0$ if $p_{j0} + \omega_j = 0$. Then $(0, \Lambda)\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}(\mathbb{R}^{4n})$ if $\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}(\mathbb{R}^{4n})$.

$\mathbf{H}_{\mathcal{P}}$ includes Cauchy sequences of functions in $\underline{\mathcal{P}}$ convergent in the Hilbert space norm (26). For the constructions, these include functions

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n}(p_{j0} + \omega_j)\, \tilde{g}_n((\mathbf{p})_n)_{(\kappa)_n})$$

with $g_n \in \mathcal{S}(\mathbb{R}^{3n})$ and, as a consequence, $\mathbf{H}_{\mathcal{P}}$ includes states characterized by state describing functions (262) with point support over time [33]. The functions used by Lehmann, Symanzik and Zimmermann to isolate the creation component of a field operator in their developments of scattering for RQFT [10] are included in $\mathbf{H}_{\mathcal{P}}$. As a consequence, LSZ expressions for scattering amplitudes readily adapt to the constructions. Explicit scattering likelihoods are presented in section 4.5 and [31, 33, 35, 37].

The Hilbert space completion of $\underline{\mathcal{P}}$ includes no strictly localized states [33] but there are essentially localized states, appendix 8.17. Functions of the form (9) are anti-local [45, 53] and not of bounded spatial support. Anti-local functions do not equal zero within any finite volume. Tails of essentially localized state describing functions are physically negligible since minor support corresponds to unlikely and effectively unrepeatable observations.

Selection of the basis function spaces $\underline{\mathcal{P}}$ overcomes the challenge of discovering VEV within $\underline{\mathcal{S}}'$ that satisfy the physical conditions of RQP for any function from $\underline{\mathcal{S}}$. This long-standing problem remains without a resolution [2, 5, 10, 30, 39, 41, 56, 59]. The realizations of relativistic quantum physics discussed here are admitted by elimination of canonical quantization's

assertion that quantum fields (31) are elevations of corresponding classical fields and, as a consequence, must be Hermitian. The lack of real functions within the completion of $\underline{\mathcal{P}}$ precludes Hermitian field operators but does not preclude an appropriate correspondence of classical and quantum dynamics.

3.4.2 An alternative formulation based on $\underline{\mathcal{S}}$

An equivalent formulation modifies the constructed VEV from section 3.3 and selects the Schwartz tempered functions $\underline{\mathcal{S}}$ as the basis function space. In this equivalent formulation, the VEV are necessarily not formally Hermitian if interaction is exhibited. The equivalent formulation includes the physically trivial, conventional free field VEV.

The equivalence derives from that multiplier functions

$$\frac{\omega \pm p_0}{2\omega} = \theta(\pm p_0)$$

on mass shells $\hbar^2 p^2 = m^2 c^2$ with $\theta(x)$ the Heaviside step function. The zeros in (9) that distinguish the subspace $\underline{\mathcal{P}}$ from the tempered functions $\underline{\mathcal{S}}$ are provided by these multiplier functions that may alternatively be applied to the VEV. In this equivalent, alternative formulation, involutivity W.b is satisfied but formal Hermiticity W.a is abandoned to achieve interaction. With the multipliers applied to the VEV rather than to the test functions, the generalized functions become

$$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \mapsto \prod_{j=1}^{k} \prod_{\ell=k+1}^{n} \left(\frac{-p_{j0} + \omega_j}{2\omega_j}\right) \left(\frac{p_{\ell 0} + \omega_\ell}{2\omega_\ell}\right) \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}. \tag{72}$$

From section 3.3, the VEV have point support on energies on mass shells, $\delta(p_j^2 - m_{\kappa_j}^2 c^2/\hbar^2)$ for each $j \in \{1, n\}$. The resulting equivalence of formulations is:

$$\begin{array}{cc} \underline{\text{VEV}} & \underline{\text{Basis space}} \\ \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{P}} \\ \prod_{j=1}^{k} \theta(-p_{j0}) \prod_{\ell=k+1}^{n} \theta(p_{\ell 0})\, \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{S}}. \end{array} \tag{73}$$

The equivalent formulations are interesting for an analogy with a result from Reeh and Schlieder [45, 66]. A result of Reeh and Schlieder is that states described by (4) with Hermitian field operators, states described by operating on the vacuum with powers of field operators and functions $\varphi_n((x)_n)_{(\kappa)_n}$ supported solely within an open and bounded region $\mathcal{O}$ are dense in the entire Hilbert space and not just dense for states supported within the same bounded region $\mathcal{O}$. Although the theorem of Reeh and Schlieder is demonstrated for RQFT and not axioms

A.1-7 from section 4.1, the equivalent formulations in (73) display the same puzzlement. The formulation with states labeled by the anti-local functions in $\underline{\mathcal{P}}$ is equivalent to a formulation that includes states labeled by functions of bounded support in $\underline{\mathcal{S}}$. Anti-local functions have global support: anti-local functions do not vanish over any finite volume. The equality (73) demonstrates that the more evident anti-locality of the formulation based on $\underline{\mathcal{P}}$ is also exhibited in the formulations based on $\underline{\mathcal{S}}$: this suggests that anti-locality and the direction to time that establishes positive energies are inherent to relativistic quantum physics.

4 Characteristics of the relativistic fields

From Born's rule, scalar products determine state transition likelihoods and these likelihoods satisfy our observations of nature. The scalar product (27) follows from the VEV (32) and state describing function sequences (21). The constructed state descriptions satisfy established properties of quantum mechanics and relativity [10, 14, 56, 60, 61]. These properties are prospective axioms A.1-7 in section 4.1. Realization of the axioms by the constructions is demonstrated in section 4.2. Additional properties including correspondences with classical particles, the Hamiltonian and scattering amplitudes are demonstrated in sections 4.3-4.5.

4.1 Axioms for the VEV of relativistic fields

Early study of mathematical structures that realize quantum physics includes studies by John von Neumann, Paul Dirac, Huzihiro Araki, Nikolay Bogolubov, Hans-Jürgen Borchers, Rudolf Haag, Raphael Høegh-Krohn, Léon van Hove, Res Jost, Arthur Wightman and Eugene Wigner among many others. Early studies did not identify structures that demonstrably realize non-trivial relativistic quantum physics [8, 9, 10]. Methods include the proposal of axioms to characterize general properties of relativistic quantum physics. Axioms are used to define concepts, derive general results, and assess the consistency of the axioms with additional assumptions. Significantly, the only realizations discovered for the established prospective axioms for relativistic quantum physics are physically trivial. Established axioms include the Wightman functional analytic axioms [10, 11, 56, 62], the Gårding-Wightman axioms for field operators [10, 56, 63], and the Haag-Kastler (Araki-Haag-Kastler) algebraic axioms for bounded, local Hermitian operators [10, 66, 67]. These axioms all consider Hilbert space realizations in addition to Fock space but the realizations that have been constructed do not exhibit interaction. The realizations of relativistic quantum physics in section 3.3 and [31, 33, 37] satisfy a revision to the Wightman axioms that preserves physical characteristics but relaxes technical properties.

Seven prospective axioms adapted from the Wightman functional analytic axioms [10, 11, 56, 62] and RQFT [61] characterize the constructed realizations of relativistic quantum physics. The axioms are stated for the emphasized constructions adopting energy support constrained (9) functions $\underline{\mathcal{P}}$.

A.1) *Regularity:* the vacuum expectation values (VEV) of quantum fields are generalized functions dual to the Schwartz tempered functions $\underline{\mathcal{S}}$.

A.2) *Nonnegativity:* the state describing functions are elements of a Hilbert space. VEV define (27) a degenerate scalar product $\mathcal{W}(\underline{g}^*, \underline{f})$ for sequences of state describing functions $\underline{g}, \underline{f}$ with $\underline{g}^*$ from (14).

A.3) *Relativistic invariance:* inertial observers agree on transition likelihoods. The degenerate scalar product (27) is invariant to proper orthochronous Poincaré transformations of the state describing function sequences.

A.4) *Spectral support:* energy-momenta lie within the closed forward (nonnegative energy) cone. The VEV satisfy axiom A.4 in a subspace $\underline{\mathcal{P}} \subseteq \underline{\mathcal{S}}$.

A.5) *Local commutativity:* field operators commute if not causally related and similarly described particles are indistinguishable. The magnitudes of VEV are invariant with interchange of argument values if the argument values are space-like separated.

A.6) *Cluster decomposition:* transition likelihoods for non-entangled, distantly space-like separated state descriptions are independent. The degenerate scalar product for functions supported on distantly space-like separated volumes factors,

$$\mathcal{W}(\underline{g_1}^* \times \underline{g_2}^*, \underline{f_1} \times \underline{f_2}) \longrightarrow \mathcal{W}(\underline{g_1}^*, \underline{f_1})\mathcal{W}(\underline{g_2}^*, \underline{f_2}), \tag{74}$$

as the supports of $\underline{g_1}, \underline{f_1}$ become arbitrarily distantly space-like separated from the supports of $\underline{g_2}, \underline{f_2}$.

A.7) *Elemental stability:* Neglecting vacuum polarization, the vacuum and one particle states are orthogonal to multiple argument states. If $g_0 = g_1(x)_\kappa = 0$, $\kappa \in \{1, N_c\}$ and $f_n((x)_n)_{(\kappa)_n} = 0$ unless $n = 1$, then $\langle \underline{g}|\Omega\rangle = 0$ and $\langle \underline{g}|\Phi(\underline{f})\Omega\rangle = 0$.

The notation is developed in section 3.1 and [11]. Axioms A.2-6 are designated here as the *physical conditions* [31]. Regularity and nonnegativity imply that states are realized as elements of a rigged Hilbert space and satisfaction of the physical conditions applies in the Hilbert space. In the emphasized development, the Hilbert space, based on support constrained (9) function sequences $\underline{\mathcal{P}}$, section 3.4.1, is designated $\mathbf{H}_{\mathcal{P}}$. In the alternative development, section 3.4.2 , the basis space of function sequences is $\underline{\mathcal{P}} = \underline{\mathcal{S}}$.

Given generalized functions $\mathcal{U}_{n,m}$ that satisfy A.1-6, generalized functions $\mathcal{W}_{n,m}$ that also satisfy A.7 follow from

$$\mathcal{W}(\underline{f}^*, \underline{f}) = |f_0|^2 + \mathcal{F}_{1,1}(f_1^*, f_1) + \mathcal{U}((\underline{f} - {}^A\underline{f})^*, \underline{f} - {}^A\underline{f}) \tag{75}$$

with the abbreviated sequence ${}^A\underline{f}$ derived from $\underline{f}$,

$${}^A\underline{f} = (f_0, f_1(x)_1, \ldots f_1(x)_{N_c}, 0, 0, \ldots).$$

$\mathcal{F}_{1,1}$ is a free field two-point function, (35) in section 3.2. In (75), vacuum polarizations

$$\langle\Omega|\Phi(x)_\kappa\Omega\rangle$$

are asserted to vanish without loss of generality: the addition of finite vacuum polarizations in section 3.3.5 preserves satisfaction of the axioms. Satisfaction of A.1-6 is preserved in the transformation (75). The decoupling of the contributions from the $n \geq 4$, n-argument VEV in $\underline{\mathcal{U}}$ from the two-point function illustrates that with the revised axioms, a two-point function equal to the free field two-point function no longer implies that the quantum field is a free field. Without densely defined Hermitian field operators, that is, without VEV that satisfy the additional constraints of formal Hermiticity W.a and involutivity W.b introduced below, the RQFT lemma [10] that provides that the Pauli-Jordan function can be a two-point VEV only for a free field does not apply to the constructions.

Using Born's rule, all inertial observers perceive the same likelihood of events if the scalar products are invariant to Poincaré transformations.

$$\langle(a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle\underline{g}|\underline{f}\rangle, \tag{76}$$

for Poincaré transformations

$$\begin{aligned}(a,\Lambda)f_n((x)_n)_{(\kappa)_n} &= (S(A)^T\cdot)_n f_n((\Lambda^{-1}(x-a))_n)_{(\kappa)_n}\\ (a,\Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} &= \prod_{k=1}^{n} e^{-ip_k a}\,(S(A)^T\cdot)_n \tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n}\end{aligned} \tag{77}$$

from the Fourier transform (22) with properties (24), with the matrix notation (15) and with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz group. Λ is the proper orthochronous Lorentz transformation determined by $A \in \mathrm{SL}(2,\mathbb{C})$, and $a \in \mathbb{R}^4$ is a translation in spacetime. The 4×4 Lorentz transformation Λ determined from A has components

$$\Lambda_{jk} = \frac{1}{2}\mathrm{Trace}(\sigma_j A \sigma_k A^*) \tag{78}$$

with σ_j designating one of the four Pauli spin matrices [10]. $\Lambda^T g \Lambda = g$ with g the Minkowski signature matrix (25).

From the reconstruction theorem [10, 56], the invariance (76) provides that the spacetime transformation (77) is unitarily implemented: the Hilbert space operator adjoint of (a,Λ) is the inverse of (a,Λ),

$$(a,\Lambda)^* = (a,\Lambda)^{-1} = (-\Lambda a, \Lambda^{-1}).$$

The transform (77) identifies that

$$\begin{aligned}(a_2,\Lambda_2)(a_1,\Lambda_1)\underline{f} &= S(A_2)^T S(A_1)^T \underline{f}(\Lambda_2^{-1}\Lambda_1^{-1}(x-a_1) - \Lambda_2^{-1}a_2)\\ &= (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)\underline{f}\\ &= S(A)^T \underline{f}(\Lambda_2^{-1}\Lambda_1^{-1}(x - a_1 - \Lambda_1 a_2))\end{aligned}$$

from the group property and composition of Poincaré transformations (77). Then

$$S(A) = S(A_1)S(A_2) \tag{79}$$

results in the appropriate composition of $S(A)$ for a representation of the Poincaré group.

In a non-relativistic development, time is a universal parameter and many Hamiltonians are compatible with the non-relativistic scalar product [52]. In relativistic physics, time translations of states, and hence, the Hamiltonian, must comply with Poincaré invariance (76) of likelihoods. With the Fourier transform (22) and Poincaré transformations (77), time translations by λ are

$$\tilde{f}_n((p)_n)_{(\kappa)_n} \to \prod_{j=1}^{n} e^{-ip_{j0}\lambda}\, \tilde{f}_n((p)_n)_{(\kappa)_n}$$

in the n-argument subspace of the momentum domain. A state describing function $f(x_0 - \lambda, \mathbf{x})$ evaluates the field at an advanced time.

$$\int dx\ \Phi(x) f(x_0 - \lambda, \mathbf{x}) = \int dx\ \Phi(x_0 + \lambda, \mathbf{x}) f(x).$$

A subgroup of the Poincaré group, temporal translation is unitarily implemented,

$$U(\lambda) f((x)_n)_{(\kappa)_n} = f((x_0 - \lambda, \mathbf{x})_n)_{(\kappa)_n}. \tag{80}$$

The Fourier transform (22) has properties (24), and as a consequence,

$$e^{-ipy} \tilde{f}(p - q) \tag{81}$$

and

$$e^{iq(x-y)} f(x - y) \tag{82}$$

are a Fourier transform pair. The support for the function in (81) is translated from $f(x)$ in spacetime by $y \in \mathbb{R}^4$ and in energy-momentum by $q \in \mathbb{R}^4$. A change in sign of spatial and momentum translations follows from the Minkowski signature in Lorentz vector products $px = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}$.

A.4 follows from the observation that energy-momenta Lorentz vectors lie within the closed forward cone

$$\overline{V}^+ = \{p :\ p^2 \geq 0 \text{ and } p_0 \geq 0\}.$$

If $|p\rangle$ is a generalized eigenfunction of momentum with eigenvalue p, then for any state describing function g_k with k arguments,

$$\langle g_k | p \rangle = 0$$

if $p \notin \overline{V}^+$. As a consequence and illustrated below, Fourier transforms of the generalized functions

$$\tilde{T}_n((p)_n)_{(\kappa)_n} = \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \widetilde{g_k^*}((p)_k)_{(\kappa)_k} \tilde{f}_{n-k}((p)_{k+1,n})_{(\kappa)_{k+1,n}} \tag{83}$$

are limited to $\mathcal{E}_n^+$ [10, 11, 56, 62]. $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are the VEV functions (32), $g_k \in \mathcal{P}(\mathbb{R}^{4k})$, $f_{n-k} \in \mathcal{P}(\mathbb{R}^{4n-4k})$ and

$$\mathcal{E}_n^+ = \{(p)_n : \ p_n \in \overline{V}^+, \ p_{n-1} + p_n \in \overline{V}^+, \ldots, \ p_2 + \ldots p_n \in \overline{V}^+, \ p_1 + p_2 + \ldots p_n = 0\}. \quad (84)$$

The support limitation follows from spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [22], lemma 5.6.7 [40], chapters 7-10 [25]). The unitary spacetime translation operator in $\mathbf{H}_\mathcal{P}$ is

$$U(a) = \int dE(p) \ e^{-ipa}$$

with

$$dE(p) \sim dp \ |p\rangle\langle p|$$

a resolution of the identity in $\mathbf{H}_\mathcal{P}$. Every $p \in \overline{V}^+$. Unitarity of $U(a)$ follows from Poincaré invariance A.3 of the scalar product and Hermiticity of the densely defined generators of translations, the energy-momentum, appendix 8.3 and Stone's theorem. Translations of the fields are

$$U(a)\Phi(x_j)_{\kappa_j}U(a)^{-1} = \Phi(x_j + a)_{\kappa_j},$$

and the vacuum is translation invariant.

$$U(a)\Omega = \Omega.$$

Then, translation of arguments ℓ though n is expressed

$$\begin{aligned}\langle\Phi_k \ldots \Phi_1\Omega|\Phi_{k+1} \ldots \Phi_{\ell-1}U(a)\Phi_\ell U(a)^{-1} \ldots U(a)\Phi_n U(a)^{-1}\Omega\rangle \\ = \ \langle\Phi_k \ldots \Phi_1\Omega|\Phi_{k+1} \ldots \Phi_{\ell-1}U(a)\Phi_\ell \ldots \Phi_n\Omega\rangle\end{aligned}$$

with an abbreviated notation for fields $\Phi_j = \Phi(x_j)_{\kappa_j}$. The scalar product (8) with functions g_{n-k} of translated support for arguments ℓ through n provides

$$\begin{aligned}F(q) \ &= \int da \ e^{iqa} \int d(x)_n \ \langle\Phi_k \ldots \Phi_1\Omega|\Phi_{k+1} \ldots \Phi_{\ell-1}U(a)\Phi_\ell \ldots \Phi_n\Omega\rangle \ \overline{f_k}g_{n-k} \\ &= \int da \int d(x)_n \int e^{iqa-ipa}\langle\Phi_k \ldots \Phi_1\Omega|\Phi_{k+1} \ldots \Phi_{\ell-1}dE(p)\Phi_\ell \ldots \Phi_n\Omega\rangle \ \overline{f_k}f_{n-k} \\ &= (2\pi)^4 \int d(x)_n \int \delta(q-p) \ \langle\Phi_k \ldots \Phi_1\Omega|\Phi_{k+1} \ldots \Phi_{\ell-1}dE(p)\Phi_\ell \ldots \Phi_n\Omega\rangle \ \overline{f_k}g_{n-k}.\end{aligned}$$

$F(q) = 0$ if $q \notin \overline{V}^+$ for each choice of n, k, ℓ and $(\kappa)_n$ from A.4 and the properties of the translation operator $U(a)$. From the equivalent expression (27) for scalar product, Parseval's equality (23) and the properties of Fourier transforms (24),

$$\begin{aligned}F(q) \ &= \int da \ e^{iqa} \int d(x)_n \ \mathcal{W}_{k,n-k}((x)_{\ell-1}, (x+a)_{\ell,n})_{(\kappa)_n} \ f_k^* g_{n-k} \\ &= \int da \int d(p)_n \ e^{iqa - i\sum_{j=\ell}^n p_j a} \ \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \ \widetilde{f_k^*}\tilde{g}_{n-k} \\ &= (2\pi)^4 \int d(p)_n \ \delta(q - \textstyle\sum_{j=\ell}^n p_j) \ \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \ \widetilde{f_k^*}\tilde{g}_{n-k}.\end{aligned}$$

Then, $F(q) = 0$ unless

$$\sum_{j=\ell}^{n} p_j \in \overline{V}^+$$

implies that the support of (83) includes only p_j from $\mathcal{E}_n^+$ (84). Development follows similarly for $\ell \leq k$ from the sesquilinearity of the scalar product (8). As a consequence, to satisfy the observation that all energy-momentum lie in the closed forward cone, the support of (83) includes only p_j from $\mathcal{E}_n^+$.

In Wightman's original axioms, the support of the state describing functions $\tilde{f}_k, \tilde{g}_{n-k}$ are not constrained and as a consequence, to satisfy the support limitation (83), the support of the VEV functions $\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}$ is limited to $\mathcal{E}_n^+$. In the constructions, the supports of the constructed VEV are limited to mass shells and the supports (18) of elements of $\mathcal{P}(\mathbb{R}^{4n})$ limit the energy-momentum support to positive energies. Together, the joint support of VEV functions and functions in $\underline{\mathcal{P}}$ satisfies A.4 [33]. Support constrained functions enable joint satisfaction of local commutativity, positive energy support and Poincaré covariance, an unmet and apparently unattainable task within RQFT developments when interaction is manifest [5, 10, 31, 41].

Local commutativity is that linear combinations of VEV with transpositions of arguments conditionally vanish.

$$\langle \underline{f} | \Phi_1 \dots (\Phi_k \Phi_{k+1} \pm \Phi_{k+1} \Phi_k) \dots \Phi_n \Omega \rangle = 0$$

if the points x_k and x_{k+1} are space-like, $(x_k - x_{k+1})^2 < 0$, $1 \leq k < n$, and the notation is abbreviated, $\Phi_k = \Phi(x_k)_{\kappa_k}$. Space-like separations specify that the points in the supports of the quantum fields are not causally related. The sign is determined by particle statistics to satisfy normal commutation relations [10].[m] Verification of local commutativity uses functions of bounded support with space-like separated support, for example, tempered test functions $\underline{\mathcal{S}}$. Local commutativity applies for constructions based on the unconstrained basis function space $\underline{\mathcal{S}}$, and in constructions based on $\underline{\mathcal{P}}$, local commutativity reduces to commutativity or anti-commutativity of the component field operators. The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ lack functions of bounded support and the constructed VEV exhibit unconditional split signed symmetry if based upon $\underline{\mathcal{P}}$, section 3.3 and [31, 33]. For the realizations emphasized in these notes, A.5 can be tightened to unconditional split signed symmetry of VEV but to include free field developments with the basis function spaces extended to $\underline{\mathcal{S}}$, only conditional local commutativity is required in A.5 [31]. Inclusion of parastatistics [10] would extend this development.

Condition A.6 is another manifestation of causality: great spatial separation implies independence of nonentangled local observables. A.6 is stronger than the condition used in [31, 33, 37] that was also designated as cluster decomposition. Earlier constructions in [31, 33, 37] satisfy formal Hermiticity described below. However, in section 3.3 it is illustrated that formal Hermiticity conflicts with cluster decomposition A.6 in constructions with nonzero spin and in some scalar field examples. Both conditions, A.6 or the cluster decomposition condition used

[m] For brevity, the possibility of *parastatistics* [10] is not included here.

in [31, 33, 37], imply that the vacuum is the sole translation-invariant state. In common with Wightman's development, formal Hermiticity was maintained in [37] and resulted in truncated functions that are not connected. In the earlier development that satisfied formal Hermiticity, the strong cluster decomposition condition A.6 is not satisfied. The constructions in section 3.3 abandon formal Hermiticity to establish the causal cluster decomposition property A.6. Cluster decomposition is a nonlinear condition and relates the description of interaction across orders of the VEV.

Condition A.6 implies that: the vacuum $|\Omega\rangle$ is the only translationally invariant state [33, 56]; that states with sufficiently isolated and space-like separated support are described by free particles, section 4.3; that the quantum field (31) elevates to Hilbert space operators, section 4.4; and the essential independence of the local observables of non-entangled, spatially distant bodies, appendices 8.2.8 and 8.2.9.

A.7 provides satisfaction of nonnegativity A.2 together with regularity A.1 for the constructions. However, satisfaction of A.7 is not necessary for regularity. Regularity and nonnegativity generally require that $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ for $n \geq 2$ in the constructions, section 4.2.5. Satisfaction of formal Hermiticity precludes setting $\mathcal{W}_{0,n} = \mathcal{W}_{1,n} = 0$ for $n \geq 2$ but *kinematic constraints* can result in $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ if $f_1 \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$ and $f_n \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$ for nonzero $\mathcal{W}_{0,n}$ and $\mathcal{W}_{1,n}$. For realizations with only a single mass, $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ if $n \geq 2$ due to conservation of energy-momentum: neither the vacuum nor a single particle can create a cascade of particles with each particle of the same finite mass as the decaying particle.

A.7 applies for free field theory constructed with the basis function space $\underline{\mathcal{P}}$ [31] or with the modified free field VEV (73) and the basis function space $\underline{\mathcal{S}}$. A.7 does not hold for free field theory constructed with the basis function space $\underline{\mathcal{S}}$.

Confined particles, those that do not appear with their support isolated from the support of other arguments, have $\mathcal{F}_{1,1} = 0$ for the κ_j included in their description. A $|f_0|^2$ term ($\mathcal{W}_{0,0} = 1$) is required in any construction that includes a vacuum state. Elemental stability A.7 expresses that if a state is prepared with only one elementary particle, or together with A.6 that if one elementary particle is greatly isolated from the support of every other body, then the elementary particle is stable until it encounters another body. Whether or which elementary particles are confined is not determined by the axiom and a discussion of confinement awaits better understanding of bound states within the constructions. Confined species extends this development.

Wightman's original axioms [10, 11, 56, 62] are distinguished here by two additional assumptions.[n] Wightman's axioms support the canonical formalism's conjectured correspondence that quantizes classical fields to Hermitian quantum field operators, Wightman's prospective axioms include assumptions that imply the fields in the VEV (7) are Hermitian Hilbert space field

[n] The conditions are not called out as separate axioms but formal Hermiticity and involutivity are assumed within a statement of the axioms [10, 11, 56, 62].

operators:

W.a) *Formal Hermiticity:* the generalized functions that define the degenerate scalar product (27) satisfy

$$\mathcal{W}_{k,n-k}((x)_n))_{(\kappa)_n} = W_n((x)_n))_{(\kappa)_n}$$

independently of k. In this case, the degenerate scalar product is

$$\mathcal{W}(\underline{g}^*, \underline{f}) = \underline{W}(\underline{g}^* \times \underline{f})$$

for a Wightman functional $\underline{W}$ and the product (6).

W.b) *Involutivity:* $\underline{\mathcal{P}} = \underline{\mathcal{S}}$. Satisfaction of the axioms applies for all sequences of Schwartz tempered functions.

Together with A.1-2, satisfaction of W.a and W.b implies that the field is realized as densely defined Hermitian Hilbert space operators. Condition W.a results in the simplified form (13) for the scalar product. Sequences of tempered functions $\underline{\mathcal{S}}$ are a $*$-involutive algebra for the product (6) and involution (14). However, satisfaction of both conditions W.a-b preclude the constructed nontrivial realizations of relativistic quantum physics. Physically trivial examples, free fields, Wick polynomials and generalized free fields, satisfy both W.a and W.b. Either, but not both W.a and W.b can be satisfied by the physically nontrivial constructions in section 3.3. In section 3.4.2, it is demonstrated that involutivity W.b can be satisfied if formal Hermiticity W.a is abandoned: with a non-Hermitian embellishment to the VEV presented in section 3.3, the constructions can be based on $\underline{\mathcal{S}}$. In [31], it is demonstrated that W.a can be satisfied for a physically nontrivial neutral scalar field if involutivity W.b is abandoned.

The revised axioms A.1-7 for RQP result in substantial departures from established RQFT results. Without densely defined Hermitian field operators, axioms A.1-7 are demonstrably realizable, several RQFT "no go" results [10, 47] do not apply, and the RQFT demonstrations of PCT and spin-statistics theorems [10, 56] require review to include the physically nontrivial RQP constructions.

4.2 Satisfaction of the axioms

4.2.1 Nonnegative $\circ$-products

The $\circ$-product preserves the nonnegativity of sequences. The $\circ$-product of two split signed symmetric VEV function sequences (29) that each provide a degenerate scalar product (27) provides a degenerate scalar product. The demonstration follows in this section.

From the definition of $\circ$-product (62), $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is the split signed symmetrization (49) of a sequence $\underline{w}$. From (50), if $\underline{w}$ provides a degenerate scalar product, then $\underline{\mathcal{W}}$ provides a degenerate scalar product. With $\underline{w}$ from (62), it is demonstrated that $w(\underline{f}^*, \underline{g})$ provides a degenerate scalar product for elements in the tensor product of two linear vector spaces.

These linear vector spaces each have degenerate scalar products derived from the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$, respectively. The preservation of nonnegativity for scaled sequences (51) provides that $\underline{\mathcal{T}_a}(\underline{f}^*, \underline{g}) \geq 0$ and $\underline{\mathcal{V}_a}(\underline{f}^*, \underline{g}) \geq 0$ with $\underline{\mathcal{T}}_a$ and $\underline{\mathcal{V}}_a$ scaled versions of the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$. Linear vector spaces with degenerate scalar products are also designated *pre-Hilbert spaces*.

If both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ provide degenerate scalar products, then the tensor product of the pre-Hilbert spaces that result from scaled $\underline{\mathcal{V}}$ and $\underline{\mathcal{T}}$ has elements labeled $\underline{f} \otimes \underline{g}$ and a degenerate scalar product

$$w((\underline{f}_1 \otimes \underline{f}_2)^*, \underline{g}_1 \otimes \underline{g}_2) = \mathcal{T}_a(\underline{f}_1^*, \underline{g}_1)\, \mathcal{V}_a(\underline{f}_2^*, \underline{g}_2). \tag{85}$$

The scales (51) on the degenerate scalar products are $a_\ell = 1/\ell!$ in both instances. In the pre-Hilbert spaces of function sequences based on $\underline{\mathcal{P}}$ for generalized functions $\underline{\mathcal{T}_a}$ and $\underline{\mathcal{V}_a}$, fields $\Phi_\mathcal{T}$ and $\Phi_\mathcal{V}$ are defined (31). These operations extend to the tensor product space. The elevations $\Phi_\mathcal{T} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_\mathcal{V}$ to the tensor product space are

$$(\Phi_\mathcal{T} \otimes \mathbb{I})\underline{f} \otimes \underline{g} = (\Phi_\mathcal{T} \underline{f}) \otimes \underline{g}$$

and

$$(\mathbb{I} \otimes \Phi_\mathcal{V})\underline{f} \otimes \underline{g} = \underline{f} \otimes (\Phi_\mathcal{V} \underline{g}).$$

The composite vacuum is $\Omega = \Omega_\mathcal{T} \otimes \Omega_\mathcal{V}$. To condense notation, the arguments of the fields are understood. The construction is a variation of a construction due to Borchers and Uhlmann [26, 47]. $|O(f_n)\Omega\rangle$ with

$$O(f_n) = \sum_{(\kappa)_n} \int d(x)_n \left(\sum_{\ell=0}^{n} \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!} \right) f_n((x)_n)_{(\kappa)_n}$$

are elements of the tensor product space with

$$(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell} = \left(\prod_{j=1}^{\ell} (\Phi_\mathcal{T}(x_j)_{\kappa_j} \otimes \mathbb{I}) \right) \left(\prod_{j=\ell+1}^{n} (\mathbb{I} \otimes \Phi_\mathcal{V}(x_j)_{\kappa_j}) \right).$$

In the tensor product space, $\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle$ is a degenerate scalar product for function sequences from $\underline{\mathcal{P}}$.

$$\begin{aligned} \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \\ \times \langle \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!}\, \Omega | \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^j (\mathbb{I} \otimes \Phi_\mathcal{V})^{m-j}}{j!(m-j)!}\, \Omega \rangle \\ \times \overline{f_n((x)_n)}_{(\kappa)_n}\, g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned}$$

The definition of scalar product, the operators $\Phi_{\mathcal{T}}(x_j)_{\kappa_j} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_{\mathcal{V}}(x_j)_{\kappa_j}$, the $*$-dual (14) and the relation between VEV and VEV functions (32) result in

$$\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \frac{\mathcal{V}_{n-\ell,m-j}(A'_o)}{(n-\ell)!(m-j)!} \times f_n((x)_n)^*_{(\kappa)_n} g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

for A_o, A'_o from (63). Then substitution of the expression (62) for functions $w_{k,n-k}$ identifies that

$$w(\underline{f}^*, \underline{g}) = \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle. \tag{86}$$

As a consequence. $w(\underline{f}^*, \underline{f}) \geq 0$ since it is the squared norm of a vector in the tensor product space.

Satisfaction of nonnegativity A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ follows from (50), (51) and (62) if both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are split signed symmetric and satisfy A.1-2 for function sequences $\underline{\mathcal{P}}$. For $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$,

$$\mathcal{W}(\underline{f}^*, \underline{f}) \geq 0$$

if $\underline{\mathcal{U}}$ satisfies nonnegativity since it is established that $\underline{\mathcal{F}}$ is positive [10, 56]. $\underline{\mathcal{U}}$ is constructed as split signed symmetric and from (42), $\underline{\mathcal{F}}$ is split signed symmetric. Satisfaction of nonnegativity A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ reduces to demonstration that the $\underline{\mathcal{U}}$ in the cluster expansion (67) of the connected functions (53) satisfies nonnegativity.

4.2.2 Nonnegativity of $\underline{\mathcal{U}}$

In this section it is demonstrated that the constructed split signed symmetric sequences (29) of generalized functions $\underline{\mathcal{U}}$ from the cluster expansion (67) of the connected functions (53) provide degenerate scalar products (27) for state describing function sequences from $\underline{\mathcal{P}}$. With the result of section 4.2.1 that the $\circ$-products of split signed symmetric sequences that each provide a degenerate scalar product (27) provides a degenerate scalar product, the nonnegativity of $\underline{\mathcal{U}}$ provides that $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ satisfies nonnegativity A.2.

The construction (67) of $\underline{\mathcal{U}}$ as $\exp \circ ({}^{C}\underline{\mathcal{W}})$ and that a positively weighted summation of generalized functions that provide degenerate scalar products provides a degenerate scalar product,

$$(\alpha\mathcal{T} + \beta\,\mathcal{V})(\underline{f}^*, \underline{f}) = \alpha\mathcal{T}(\underline{f}^*, \underline{f}) + \beta\,\mathcal{V}(\underline{f}^*, \underline{f}) \geq 0$$

if $\alpha, \beta > 0$ provides that the degenerate scalar product based on $\underline{\mathcal{U}}$ satisfies nonnegativity if the degenerate scalar product based on ${}^{C}\underline{\mathcal{W}}$ is nonnegative. If ${}^{C}\underline{\mathcal{W}}$ provides a degenerate scalar product, then $\circ$-products of ${}^{C}\underline{\mathcal{W}}$ provide degenerate scalar products from the result of section 4.2.1. The $\circ$-product is split signed symmetric. A demonstration of the nonnegativity

of ${}^{C}\underline{\mathcal{W}}(\underline{f}^*, \underline{f})$ suffices to demonstrate the nonnegativity of $\underline{\mathcal{U}}(\underline{f}^*, \underline{f})$. From section 3.3.4, the sequences $\underline{f} \in \underline{\mathcal{P}}$ are terminating and the number of contributing terms in the expansion (66) of $\mathcal{U}_{k,n-k}$ is finite.

From substitution of the description (54) of $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ into the construction (53) of the connected VEV functions, the connected VEV functions ${}^{C}\mathcal{U}_{n,m}(\{1, n+m\})$ are the split signed symmetrization (49) of $\tilde{\mathfrak{u}}_{n,m}(\{1, n+m\})$ with

$$\begin{aligned}\tilde{\mathfrak{u}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} &= \iint d\sigma(\lambda) \frac{du}{(2\pi)^4}\, \lambda^{n+m} \prod_{j=1}^{n+m} \left(e^{-ip_j u}\, \delta(p_j^2 - \lambda_{cj}^{-2})\, \frac{d}{d\rho_j} \right) \\ &\qquad \times \exp\Big(\sum_{a,b \in \mathbb{J}_{n,n+m}} \rho_a \rho_b\, \mathfrak{h}_{\kappa_a \kappa_b}(p_a, p_b) \Big).\end{aligned} \tag{87}$$

Then from that split signed symmetrization preserves positivity (50), ${}^{C}\mathcal{U}(\underline{f}^*, \underline{f}) \geq 0$ if $\mathfrak{u}(\underline{f}^*, \underline{f}) \geq 0$ with

$$\begin{aligned}\mathfrak{u}(\underline{f}^*, \underline{f}) &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(p)_{n+m}\, (D\cdot)_n \tilde{\mathfrak{u}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times \overline{\tilde{f}_n}(-p_n, \ldots - p_1)_{\kappa_n \ldots \kappa_1}\, \tilde{f}_m(p_{n+1}, \ldots p_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned} \tag{88}$$

from the scalar product (27). This scalar product is expressed using Fourier transforms (22) from Parseval's equality (23), and the definition (14) for $*$-dual functions. The Fourier transform (22) of a $*$-dual function is

$$\int \frac{d(x)_n}{(2\pi)^{2n}}\, e^{-ip_1 x_1} \ldots e^{-ip_n x_n} f^*((x)_n)_{(\kappa)_n} = (D^T\cdot)_n \overline{\tilde{f}_n}((-p)_{n,1})_{(\kappa)_{n,1}}.$$

The factorization (57) of $\tilde{\mathfrak{u}}_{m,n+m}((p)_{n+m})_{(\kappa)_{n+m}}$ in section 3.3.3,

$$\begin{aligned}(D\cdot)_n \exp\Big(\sum_{a,b \in \mathbb{J}_{n,n+m}} \rho_a \rho_b \mathfrak{h}_{\kappa_k + \ell \kappa_b} \Big) &= \exp\Big(\sum_{b_1 > a_1 = 1}^{n} \rho_{a_1} \rho_{b_1} (D\mathfrak{B}(p_{a_1} + \alpha_o p_{b_1}) D^T)_{\kappa_{a_1} \kappa_{b_1}} \Big) \\ \times \exp\Big(\sum_{b_2 > a_2 = n+1}^{n+m} \rho_{a_2} \rho_{b_2} \mathfrak{B}(\alpha_o p_{a_2} + p_{b_2})_{\kappa_{a_2} \kappa_{b_2}} \Big)\, \exp\Big(\sum_{\ell=1}^{\infty} \rho_{n+1-\ell} \rho_{n+\ell} (D\Upsilon)_{\kappa_{n+1-\ell} \kappa_{n+\ell}} \Big),\end{aligned} \tag{89}$$

results in display of $\mathfrak{u}(\underline{f}^*, \underline{f}) \geq 0$ [37]. Substitution of the identity (38) for transformations of $M)p)$ into the definition (56) of $\mathfrak{B}(p)$ provides that

$$D\mathfrak{B}(p) D^T = \mathfrak{B}(-p)^* \tag{90}$$

with

$$\mathfrak{B}^*(-p) = \overline{\mathfrak{B}(-p)}^T.$$

Then, substitution of (90) results in

$$\exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}(D\mathfrak{B}D^T)_{\kappa_{a_1}\kappa_{b_1}})=\exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}\overline{\mathfrak{B}(-p_{a_1}-\alpha_o p_{b_1})}_{\kappa_{b_1}\kappa_{a_1}}).$$

If the arguments, $a_1, b_1 \in \{1, n\}$ are reordered from the order of a $*$-dual function (14), $p_n, \kappa_n, p_{n-1}, \kappa_{n-1}, \ldots p_1, \kappa_1$, to

$$p_1, \kappa_1, p_2, \kappa_2, \ldots p_n, \kappa_n,$$

then the first factor in the factorization (89) becomes

$$\exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}(D\mathfrak{B}(p_{a_1}+\alpha_o p_{b_1}))D^T)_{\kappa_{a_1}\kappa_{b_1}})=\exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}\overline{\mathfrak{B}(-\alpha_o p_{a_1}-p_{b_1}))}_{\kappa_{a_1}\kappa_{b_1}}.$$

This is the complex conjugate in the form of the second term in (89). Except for the summation from 1 to n rather than summation from $n+1$ to $n+m$ and reflection of the energy-momenta $p_j \mapsto -p_j$ appropriate for the $*$-dual function, the first and second factors of (89) are complex conjugates.

The nonnegative matrix $DM(s) = C^*(s)C(s)$ from (37) and then substitution into (56) provides that

$$D\Upsilon(p) = \int d\mu_\Upsilon(s)\; e^{-sp}C^*(s)C(s). \tag{91}$$

The factor

$$\exp(\sum_{\ell=1}^{\infty}\rho_{n+1-\ell}\rho_{n+\ell}(D\Upsilon)_{\kappa_{n+1-\ell}\kappa_{n+\ell}})$$
$$=\sum_{N=0}^{\infty}\frac{1}{N!}\left(\sum_{\ell=1}^{\infty}\rho_{n+1-\ell}\rho_{n+\ell}(D\Upsilon(-p_{n+1-\ell}+p_{n+\ell}))_{\kappa_{n+1-\ell}\kappa_{n+\ell}}\right)^N.$$

Then, the identity for summation,

$$\sum_a\sum_b x_a y_b = \left(\sum_a x_a\right)\left(\sum_b y_b\right), \tag{92}$$

and using linearity to reorder finite summations with the integration in (91) results in

$$\sum_{\ell=1}^{\infty}\rho_{n+1-\ell}\rho_{n+\ell}(D\Upsilon(-p_{n+1-\ell}+p_{n+\ell})))_{\kappa_{n+1-\ell}\kappa_{n+\ell}}$$
$$=\sum_{\ell=1}^{\infty}\sum_{\ell=1}^{N_c}\int d\mu_\Upsilon(s)\left(\rho_{n+1-\ell}e^{sp_{n+1-\ell}}\overline{C(s)}_{\ell\kappa_{n+1-\ell}}\right)\left(\rho_{n+\ell}e^{-sp_{n+1-\ell}}C(s)_{\ell\kappa_{n+\ell}}\right)$$

using the expression for Hermitian transpose $C(s)^*$. Similarly,

$$\left(\sum_{\ell=1}^{\infty}\rho_{n+1-\ell}\rho_{n+\ell}(D\Upsilon(-p_{n+1-\ell}+p_{n+\ell}))_{\kappa_{n+1-\ell}\kappa_{n+\ell}}\right)^N = \sum_{\ell_1=1}^{\infty}\sum_{\jmath_1=1}^{N_c}\int d\mu_\Upsilon(s_1)\ldots$$
$$\times\sum_{\ell_N=1}^{\infty}\sum_{\jmath_N=1}^{N_c}\int d\mu_\Upsilon(s_N)\prod_{\nu=1}^{N}\left(\rho_{n+1-\ell_\nu}e^{s_\nu p_{n+1-\ell_\nu}}\overline{C(s_\nu)}_{\jmath_\nu\kappa_{n+1-\ell_\nu}}\right)\left(\rho_{n+\ell_\nu}e^{-s_\nu p_{n+\ell_\nu}}C(s_\nu)_{\jmath_\nu\kappa_{n+\ell_\nu}}\right).$$

Substitution of the factors in (57) display $\mathfrak{u}(\underline{f}^*,\underline{f})$ as a summation with positive weights of manifestly nonnegative terms. Collection of factors from (87) and the factorization of the exponential form (57) results in a convenient definition.

$$F_N(\lambda,u,(s,\ell,\jmath)_N) = \sum_n\sum_{(\kappa)_n}\lambda^n\int d(p)_n\,\tilde{f}_n((p)_n)_{(\kappa)_{1,n}}\prod_{j=1}^{n}\left(e^{-ip_ju}\,\delta(p_j^2-\lambda_{cj}^{-2})\,\frac{d}{d\rho_j}\right)$$
$$\times\exp(\sum_{b_2>a_2=1}^{n}\rho_{a_2}\rho_{b_2}\mathfrak{B}(\alpha_o p_{a_2}+p_{b_2})_{\kappa_{a_2}\kappa_{b_2}})\prod_{\nu=1}^{N}\left(\rho_{\ell_\nu}e^{-s_\nu p_{\ell_\nu}}C(s_\nu)_{\jmath_\nu\kappa_{\ell_\nu}}\right).$$

F_N is evaluated with all $\rho_j = 0$ after the indicated differentiations. Then, reordering the either finite or uniformly convergent summations and improper integrations, and collecting summations and sets of factors in the abbreviated notation $F_N(\lambda,u,(s,\ell,\jmath)_N)$, it follows that the expression (88) for $\mathfrak{u}(\underline{f}^*,\underline{f})$ becomes

$$\mathfrak{u}(\underline{f}^*,\underline{f}) = \sum_{N=0}^{\infty}\frac{1}{N!}\iint d\sigma(\lambda)\frac{du}{(2\pi)^4}\sum_{\jmath_1=1}^{N_c}\sum_{\ell_1=1}^{\infty}\int d\mu_\Upsilon(s_1)\ldots$$
$$\times\sum_{\jmath_N=1}^{N_c}\sum_{\ell_N=1}^{\infty}\int d\mu_\Upsilon(s_N)\;|F_N(\lambda,u,(s,\ell,\jmath)_N)|^2.$$

The identity (92), and with the order of the arguments of the $*$-dual of $\tilde{f}_n$ returned to the order

$$p_1,\kappa_1,p_2,\kappa_2,\ldots p_n,\kappa_n$$

from $p_n,\kappa_n,p_{n-1},\kappa_{n-1},\ldots p_1,\kappa_1$ and with momenta reflected $p_j\mapsto -p_j$ for the $*$-dual, $j\in\{1,n\}$, by relabeling of summation variables, inspection provides the result. The second factor of F_N includes variables labeled by indices $n+1$ to $n+m$ that are relabeled as independent variables with indices 1 to m.

The factoring of the connected functions (57) expresses $\mathfrak{u}(\underline{f}^*,\underline{f})$ as a positively weighted summation of the nonnegative magnitude squared. The measures du, $d\sigma(\lambda)$ and $d\mu_\Upsilon(s)$ are nonnegative. $\mathfrak{u}(\underline{f}^*,\underline{f})$ is nonnegative and as a consequence, from (50), ${}^C\underline{\mathcal{W}}(\underline{f}^*,\underline{f})$ is nonnegative. Finally, $\mathcal{U}(\underline{f}^*,\underline{f})$ is nonnegative and split signed symmetric as a consequence of the expansion (66), and the nonnegativity and split signed symmetry preserving properties of the $\circ$-product.

4.2.3 Poincaré invariance

The Poincaré covariance of the constructed VEV is established in [37]. Invariance of the degenerate scalar product (27) follows from the conservation of energy-momentum and properties (41) of the representations of the Lorentz group. Compliant example representations of the Lorentz group are included in section 3.2.2.

Poincaré invariance of likelihoods is that

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle \underline{g}|\underline{f}\rangle,$$

for Poincaré transformations (77),

$$(a,\Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} = \prod_{k=1}^{n} e^{-ipa}\,(S(A)^T\cdot)_n\tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n}.$$

The scalar product (27), the Fourier transform (22), the Parseval's equality definition for the Fourier transform of a generalized function (23), and the dual of functions (14) result in

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\,(D\cdot)_n\widetilde{\mathcal{W}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}}\prod_{\ell=1}^{n+m}e^{-ip_\ell a}$$
$$\times\overline{(S(A)^T\cdot)_n\tilde{g}_n((-\Lambda^{-1}p)_{n,1})}_{\kappa_{n,1}}(S(A)^T\cdot)_{n+1,n+m}\tilde{f}_m((\Lambda^{-1}p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

in the matrix notation (15). Translation invariance is verified by noting that the support of $\widetilde{\mathcal{W}}_{k,n-k}$ includes only the surface with energy-momentum conserved; $p_1+p_2+\ldots p_n = 0$ from the connected VEV functions (46) and the cluster expansion (70). Energy-momentum conservation, $p_1+p_2+\ldots p_n = 0$, implements translation invariance.

Reordering summations, substitution of $\overline{S(A)}D = DS(A)$ from (41), the indicated substitutions $\Lambda^{-1}p_j \mapsto p_j$ for summation variables, and that the determinant of the Lorentz transformation Λ is unity results in

$$\begin{aligned}\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\,(D\cdot)_n(S(A)\cdot)_{n+m}\widetilde{\mathcal{W}}_{n,m}((\Lambda p)_{n+m})_{(\kappa)_{n+m}}\\ &\qquad\times\overline{\tilde{g}_n}((-p)_{n,1})_{\kappa_{n,1}}\tilde{f}_m((p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}\\ &= \langle \underline{g}|\underline{f}\rangle\end{aligned}$$

if

$$(S(A)\cdot)_n\widetilde{\mathcal{W}}_{k,n-k}((p)_n) = \widetilde{\mathcal{W}}_{k,n-k}((\Lambda^{-1}p)_n).$$

There is exactly one factor of $S(A)$ for each component field argument κ_j and consequently two factors of $S(A)$ for every factor of $M(p_\ell)_{\kappa_j\kappa_\ell}$ or $\mathfrak{h}(p_j,p_\ell)_{\kappa_j\kappa_\ell}$ in each term in $\mathcal{W}_{k,n-k}$ from the cluster expansion (70) of connected functions from (42) and (67) with (53) and (54).

By construction, the $N_c \times N_c$ matrices $M(p)$ in (35) and the $\mathfrak{h}(p_1, p_2)$ from (55) and (56) transform with the same representation of the Lorentz group.

$$((S(A)\cdot)_2 M(p)) = S(A)M(p)S(A)^T$$

from (15), and from condition (41),

$$S(A)M(p)S(A)^T = M(\Lambda^{-1}p).$$

Then if $\mathfrak{h}(p_1, p_2) = \mathfrak{B}(\alpha_o p_1 + p_2)$,

$$\begin{aligned} S(A)\mathfrak{h}(p_1, p_2)S(A)^T &= \int d\mu_{\mathfrak{B}}(s)\; e^{is(\alpha_o p_1 + p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_{\mathfrak{B}}(s)\; e^{is(\alpha_o p_1 + p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_{\mathfrak{B}}(s')\; e^{is'\Lambda^{-1}(\alpha_o p_1 + p_2)} M(s') \\ &= \mathfrak{h}(\Lambda^{-1}p_1, \Lambda^{-1}p_2) \end{aligned}$$

and similarly if $\mathfrak{h}(p_1, p_2) = \mathfrak{B}(p_1 + \alpha_o p_2)$. If $\mathfrak{h}(p_1, p_2) = \Upsilon(-p_1 + p_2)$,

$$\begin{aligned} S(A)\mathfrak{h}(p_1, p_2)S(A)^T &= \int d\mu_{\Upsilon}(s)\; e^{s(p_1 - p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_{\Upsilon}(s)\; e^{s(p_1 - p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_{\Upsilon}(s')\; e^{s'\Lambda^{-1}(p_1 - p_2)} M(s') \\ &= \mathfrak{h}(\Lambda^{-1}p_1, \Lambda^{-1}p_2). \end{aligned}$$

These results follow from Lorentz invariance of the measures $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$, the substitution $s' = \Lambda^{-1}s$ for the summation variable, the Lorentz invariance of the Minkowski signature $\Lambda^T g \Lambda = g$, and that $ps = p^T g s$ is a Lorentz scalar. The generalized functions $\delta(p_{i_1} + \ldots p_{i_n})$ and $\delta(p_k^2 - \lambda_{ck}^{-2})$ are Lorentz scalars. In all three cases, each κ_j uniquely associates with a factor of M, $\mathfrak{B}$ or Υ, apply in every term in the cluster expansion (70) with (42), (67), (53) and (54). Each factor of M or $\mathfrak{h}$ introduces paired indices κ_a, κ_b with $a \neq b$, and as a consequence, with the exception of Lorentz scalar fields, only even order VEV appear in the constructions.

The constructed VEV provide a Poincaré invariant scalar product (27) that satisfies relativistic invariance A.3.

4.2.4 Cluster decomposition

Satisfaction of the strong form of cluster decomposition A.6 follows from the connectivity of the truncated functions (71) and the cluster expansion (70). This is an established result in

RQFT [10] but the constructed $\underline{\mathcal{U}}$ do not generally satisfy the Wightman axioms. Satisfaction of cluster decomposition A.6 for the constructions is demonstrated in this section. A.6 provides that

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \longrightarrow \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f})$$

as the supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the supports of $\underline{g}, \underline{f}$. If any one of $\underline{\psi}, \underline{\varphi}, \underline{g}, \underline{f}$ have no support, $\|\underline{\psi}\| = 0$ for example, then cluster decomposition A.6 is satisfied trivially, 0=0.

Linearity of summations and relabeling summation variables in the scalar product (27) with $\mathcal{W}(\underline{\psi}^*, \underline{\varphi})$ labeled by n, m and $(x, \kappa)_{n+m}$, and $\mathcal{W}(\underline{g}^*, \underline{f})$ labeled by ℓ, k and $(y, \mu)_{\ell+k}$ results in

$$\begin{aligned}\mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f}) = \sum_{n.m} \sum_{\ell,k} \sum_{(\kappa)_{n+m}} \sum_{(\mu)_{\ell+k}} \iint d(x)_{n+m} d(y)_{\ell+k} \\ \times (D\cdot)_n\, \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\, (D\cdot)_\ell\, \mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}} \\ \times \overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}} \varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}} \overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}} f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{93}$$

The $\times$-product of function sequences (6), the $*$-dual of function sequences (14), and relabeling summation variables with the same designations for ψ, φ and g, f as (93) produces

$$\begin{aligned}\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) = \sum_{n.m} \sum_{\ell,k} \sum_{(\kappa)_{n+m}} \sum_{(\mu)_{\ell+k}} \iint d(x)_{n+m} d(y)_{\ell+k} \\ \times (D\cdot)_{n+\ell}\, \mathcal{W}_{n+\ell,m+k}((x)_n, (y)_\ell, (x)_{n+1,n+m}, (y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ \times \overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}} \varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}} \overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}} f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{94}$$

The assignment of summation variables is a partition of arguments into two subsets P with spacetime arguments $(x)_{n+m}$ and P' with spacetime arguments $(y)_{\ell+k}$ suitable to test satisfaction of cluster decomposition (74). To test cluster decomposition (74), the supports of the ψ_n, φ_m are taken to be arbitrarily space-like separated from the support of the g_ℓ, f_k: every $x_j \in P$ can be considered to be arbitrarily space-like separated from every $y_{j'} \in P'$.

From (71), the construction of $\underline{\mathcal{W}}$ has elements

$$\mathcal{W}_{n+\ell,m+k}((x)_n, (y)_\ell, (x)_{n+1,n+m}, (y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} = (\exp \circ ({}^T\mathcal{W}))_{n+\ell,m+k}$$

with the indicated association of arguments

$$(x, \kappa)_n, (y, \mu)_\ell, (x, \kappa)_{n+1,n+m}, (y, \mu)_{\ell+1,\ell+k} \leftrightarrow (x, \kappa)_{1,n+m+\ell+k}$$

on the right- and left-hand sides, respectively. From (70), the sequence of truncated functions ${}^T\underline{\mathcal{W}}$ includes the connected functions ${}^C\underline{\mathcal{W}}$ and the free field two-point function (35). The

number of the $*$-dual function arguments is designated $n+\ell$ and $m+k$ designates the number of function arguments in $(x,\kappa)_{n+m+\ell+k}$.

If

$$\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\,\mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}},$$

when the supports of every $x_j \in P$ is arbitrarily space-like separated from the support of every $y_{j'} \in P'$, then inspection of (93) and (94) provides that cluster decomposition is demonstrated.

The elements of the sequence ${}^T\underline{\mathcal{W}}$ from (46) are connected, and due to the great space-like separations of the $(x)_{n+m}$ from the $(y)_{\ell+k}$, any factor of ${}^T\mathcal{W}_{\nu_1,\nu_2}$ in any term of the cluster expansion (66) for $(\exp\circ({}^T\mathcal{W})))$ is zero unless the arguments are all from $(x,\kappa)_{n+m}$ or are all from $(y,\mu)_{\ell+k}$. Then, designate

$$\delta_P((x)_{n+m},(y)_{\ell+k}) = \begin{cases} 1 & \text{if all arguments } (x)_{n+m},(y)_{\ell+k} \text{ are } \in P \\ 0 & \text{otherwise} \end{cases} \tag{95}$$

and

$$\delta_{P'}((x)_{n+m},(y)_{\ell+k}) = \begin{cases} 1 & \text{if all arguments } (x)_{n+m},(y)_{\ell+k} \text{ are } \in P' \\ 0 & \text{otherwise.} \end{cases}$$

Arguments in P are arguments of ψ_n,φ_m and arguments in P' are arguments of g_ℓ,f_k in the evaluation (94) of $\mathcal{W}(\underline{\psi}^*\times\underline{g}^*,\underline{\varphi}\times\underline{f})$. From the identity (70), the truncated functions (71) equal the connected functions (46). The connectedness of the truncated functions provides that all the connected functions provide vanishing contributions unless all the arguments are from P or all are from P'. Then, in the scalar product (94) with the great space-like separations of arguments,

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^T\underline{\mathcal{W}}) \\ &= \exp\circ(\delta_P{}^T\underline{\mathcal{W}} + \delta_{P'}{}^T\underline{\mathcal{W}}) \\ &= (\exp\circ(\delta_P{}^T\underline{\mathcal{W}}))\circ(\exp\circ(\delta_{P'}{}^T\underline{\mathcal{W}})) \\ &= (\delta_P\underline{\mathcal{W}})\circ(\delta_{P'}\underline{\mathcal{W}}). \end{aligned} \tag{96}$$

from the identity (69). $\delta_P\underline{\mathcal{V}}$ designates the sequence (29)

$$(\mathcal{V}_{0.0},\delta_P(x_1)\mathcal{V}_{1,0}(x_1)_{\kappa_1},\ldots,\delta_P((x)_{n+m})\mathcal{V}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}},\ldots).$$

The factors of the elements of the sequence $\delta_P\underline{\mathcal{W}}$ have common spacetime arguments, and the designation of arguments as P or P' is established in the evaluation of (94). From (95), partitions that transpose P arguments with P' arguments do not contribute in the $\circ$-product (62) in the last line of (96). From $\mathcal{W}_{0,0}=1$ and the identity (64) for the $\circ$-product (62),

identification of the contributing elements of the sequence $\underline{\mathcal{W}} = (\delta_P \underline{\mathcal{W}}) \circ (\delta_{P'} \underline{\mathcal{W}})$ in (96) in the case of interest produces

$$\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\, \mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}}$$

from the great space-like separation of the arguments in P and P'. Connected functions with arguments from both P and P' do not contribute due to the property (45) of connectivity. Substitution into (94) and comparison with (93) completes the demonstration that for the construction (70) of $\underline{\mathcal{W}}$,

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \longrightarrow \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f})$$

as the dominant supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the dominant supports of $\underline{g}, \underline{f}$.

The cluster expansion (70) for $\underline{\mathcal{W}}$ and the connectedness (46) of ${}^T\underline{\mathcal{W}}$ satisfy cluster decomposition A.6.

4.2.5 Regularity

If the constructed VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are generalized functions from $\mathcal{S}'(\mathbb{R}^{4n})$ then regularity A.1 is satisfied. The construction, (60) with (65), (35), (42), (53) and (67), displays the VEV as finite sums of products of connected functions (46) with factors that have no arguments in common. Then, demonstration of regularity reduces to demonstration that the $n \geq 4$, n-argument connected functions are elements of $\mathcal{S}'(\mathbb{R}^{4n})$ since the two-point function (35) is an element of $\mathcal{S}'(\mathbb{R}^8)$. The $n \geq 4$, n-argument connected functions from (46), ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$, are products of elementary generalized functions but since products are not generally defined, these particular products require justification.

To satisfy regularity A.1 for the physically nontrivial realizations, elemental stability A.7 is adopted. Elemental stability is implemented by a scalar product of the form (75). Avoidance of the extrapolation of ${}^C\mathcal{U}_{k,2n-k}$ to the singular $2n = 2$ [33, 37] follows from (75) and the characterization of $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ from (54) in (46). For functions in $\underline{\mathcal{P}}$, constructions with a single finite mass m and without vacuum polarization, conservation of energy sets all contributions from $\mathcal{W}_{0,n}$ and $\mathcal{W}_{n,0}$ to zero for $n \geq 2$. If conservation of energy precludes decay of isolated elementary particles, then satisfaction of formal Hermiticity W.a is consistent with $\mathcal{W}_{1,n-1}(f_1^*\, f_{n-1}) = 0$ and $\mathcal{W}_{n-1,1}(f_{n-1}^*\, f_1) = 0$. More generally, $\mathcal{W}_{1,n-1}(f_1^*\, f_{n-1}) = 0$ and $\mathcal{W}_{n-1,1}(f_{n-1}^*\, f_1) = 0$ is required but is inconsistent with formal Hermiticity. In the constructions, regularity A.1 requires elemental stability A.7.

Whether the connected functions ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ satisfy regularity A.1 rests on whether the scalar products (27) are finite. The scalar products are determined by summations over

submanifolds of $(p)_n \in \mathbb{R}^{4n}$ determined by the energy-momentum conservation and mass shell delta functions contributed by the connected VEV functions (46) and the positive energy support limitations of functions from $\underline{\mathcal{P}}$. The functions $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ in the connected VEV functions are polynomially bounded growth, locally Lebesgue-summable functions within the domains included by the summations in the scalar product (27).

The singularities of the $N_c \times N_c$ array of functions $\mathfrak{h}(p_1, p_2)$ are removed from consideration. Regular $\mathfrak{h}(p_1, p_2)$ are selected. From (55), $\mathfrak{h}(p_a, p_b)_{\kappa_a\kappa_b}$ equals $\mathfrak{B}(\alpha_o p_a + p_b)_{\kappa_a\kappa_b}$, $\mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a\kappa_b}$ or $\Upsilon(-p_a + p_b)_{\kappa_a\kappa_b}$ depending on whether the arguments are both $*$-dual arguments, both function arguments, or one $*$-dual and one function argument in the evaluation of the scalar product (27). $\alpha_o \geq 0$ is a dimensionless, nonnegative real parameter. From (56),

$$\begin{aligned} \mathfrak{B}(p) &= \int d\mu_{\mathfrak{B}}(s)\, e^{isp} M(s) \\ \Upsilon(p) &= \int d\mu_{\Upsilon}(s)\, e^{-sp} M(s). \end{aligned}$$

There are no nontrivial finite Lorentz invariant measures [10] and as a consequence, $\mathfrak{B}(p)$ and $\Upsilon(p)$ are either constant or diverge at $p = 0$. The trivial Lorentz invariant measures are $d\mu_{\mathfrak{B}}(s), d\mu_{\Upsilon}(s) = c\delta(s)ds$. Non-constant $\mathfrak{B}(p)$ and $\Upsilon(p)$ are singular at $p = 0$ due to the divergence of the summations $\int d\mu_{\mathfrak{B}}(s)$ and $\int d\mu_{\Upsilon}(s)$. From the definition (56) of $\mathfrak{B}(p)$ and $\Upsilon(p)$, the Fourier transform property (24) and that the $M(p)$ are multinomials of the energy-momentum components, it follows that nonconstant $\mathfrak{B}(p)$ and $\Upsilon(p)$ are derivatives of Källén-Lehmann forms. Källén-Lehmann forms are at least as singular at $p^2 = 0$ as the Pauli-Jordan function [54]. From the definition (56), $\mathfrak{B}(p)$ and $\Upsilon(p)$ are as singular at energy-momenta $p^2 = 0$ as Källén-Lehmann forms are singular at spacetime points $x^2 = 0$. The energy-constrained support of VEV functions and functions from $\underline{\mathcal{P}}$, and the selection (55) of $\mathfrak{h}(p_1, p_2)$ suffice to exclude singularities at $p^2 = 0$ from consideration in satisfaction of regularity A.1. For energy-momentum p_j, p_k with either both $p_{j0}, p_{k0} > 0$ or both $p_{j0}, p_{k0} < 0$, $(\alpha_o p_j + p_k)^2 > \alpha_o^2 \lambda_j^{-2} + \lambda_k^{-2}$. The singularity of $\mathfrak{B}(\alpha_o p_j + p_k)$ would be encountered only if p_j is a $*$-dual argument ($p_{j0} j = -\omega_j < 0$) and p_k is a function argument ($p_{k0} = \omega_k > 0$), or p_k is a $*$-dual argument and p_j is a function argument. These singularites are excluded by the selection (55) of $\mathfrak{h}(p_1, p_2)$. Similarly for $\mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a\kappa_b}$. For $\Upsilon(-p_j + p_k)$, the singularities are encountered only if both arguments are associated with a function or both are associated with a $*$-dual function. The form (55) excludes the singularities of $\mathfrak{h}(p_1, p_2)$ from the intervals included in the summations that determine scalar products (27). The measures $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$ are selected to ensure convergence of the summations (56) for $p^2 > 0$ [37], and then the constructed $\mathfrak{h}(p_1, p_2)$ are regular .

In this section, the number of spacetime dimensions is considered and is designated d.

Each connected function (46) in the constructions is supported solely on submanifolds with energy and momentum conserved. The contributions of the free field VEV are regular and need not be considered in this section. After evaluation of the mass shell delta functions and with

consideration of the zeros in the energy support of functions in $\underline{\mathcal{P}}$ and $\underline{\mathcal{P}}^*$, the Fourier transforms of the $n \geq 4$, n-argument connected functions include only momenta on the manifold defined by

$$\prod_{j=1}^{n} \frac{1}{2\omega_j} \, \delta(\omega_1 \ldots + \omega_k - \omega_{k+1} \ldots - \omega_n) \, \delta(\mathbf{p}_1 + \mathbf{p}_2 \ldots + \mathbf{p}_n) \tag{97}$$

with

$$\omega_j = \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}$$

from (10). If a generalized function follows from (97) for all k, then expansion of a factor

$$1 = \prod_{\ell=1}^{n} (\theta(E_k) + \theta(-E_k))$$

times a function from $\mathcal{S}$ and relabeling indices provides that the generalized function (97) applies for sequences from $\underline{\mathcal{S}}$.

Factors of $1/(2\omega_j)$ are multipliers of tempered functions for finite masses and are not considered further. Within the submanifold of $\mathbb{R}^{3n}$ with momentum conserved,

$$\mathbf{p}_n = -\mathbf{p}_1 \ldots - \mathbf{p}_{n-1}. \tag{98}$$

Energy conservation is $\delta(E_k((\mathbf{p})_n))$ with

$$\begin{aligned} E_k((\mathbf{p})_n) &= \sum_{j=1}^{k} \omega_j - \sum_{j=k+1}^{n} \omega_j \\ &= \sum_{j=1}^{n} s_j \omega_j. \end{aligned} \tag{99}$$

$s_j = -1$ if $j \in \{k+1, n\}$ and equals 1 otherwise. $\delta(E_k((\mathbf{p})_n))$ defines a generalized function except possibly for points on the surface $E_k((\mathbf{p})_n) = 0$ with a vanishing gradient, $\nabla E_k((\mathbf{p})_n) = 0$ [20].

The components of the gradient $\nabla E_k((\mathbf{p})_n)$ within the submanifold with momentum conserved are

$$\begin{aligned} \frac{dE_k((\mathbf{p})_n)}{d\mathbf{p}_j} &= s_j \frac{d\omega_j}{d\mathbf{p}_j} - \frac{d\omega_n}{d\mathbf{p}_n} \frac{d\mathbf{p}_n}{d\mathbf{p}_j} \\ &= s_j \frac{\mathbf{p}_j}{\omega_j} + \frac{\mathbf{p}_n}{\omega_n} \end{aligned} \tag{100}$$

from (99) with $d-1$ dimensional momentum vectors $\mathbf{p}_j$, the constrained $\mathbf{p}_n$ from (98), and $j \in \{1, n-1\}$. Summing squares provides that when the gradient vanishes, $\mathbf{p}_j^2/\omega_j^2 = \mathbf{p}_n^2/\omega_n^2$.

Then, substitution of ω_j provides that the gradient vanishes if and only if

$$s_j \frac{\mathbf{p}_j}{m_{\kappa_j}} = -\frac{\mathbf{p}_n}{m_{\kappa_n}}$$

for each $j \in \{1, n-1\}$ and with the constrained $\mathbf{p}_n$ from (98). Then both the function and its gradient vanish,

$$E_k((\mathbf{p})_n) = \nabla E_k((\mathbf{p})_n) = 0,$$

only if

$$E_k((\mathbf{p})_n) = \left(\sqrt{\frac{c^2}{\hbar^2} + \frac{\mathbf{p}_n^2}{m_{\kappa_n}^2}}\right) \sum_{j=1}^{n} s_j m_{\kappa_j} = 0 \tag{101}$$

from the assignments

$$\frac{\mathbf{p}_j}{m_{\kappa_j}} = -s_j \frac{\mathbf{p}_n}{m_{\kappa_n}}$$

that set the gradient to zero, and the resultant

$$\omega_j = m_{\kappa_j} \sqrt{\frac{c^2}{\hbar^2} + \frac{\mathbf{p}_n^2}{m_{\kappa_n}^2}}.$$

A neighborhood of those points with both a zero energy and vanishing gradient is

$$\frac{\mathbf{p}_j}{m_{\kappa_j}} = s_j \frac{\mathbf{p}_1}{m_{\kappa_1}} + \mathbf{e}_j$$

with $\|\mathbf{e}_j\| < \epsilon \ll 1$, $0 < \|\mathbf{p}_1\|$ and $j \in \{2, n-1\}$. In this neighborhood,

$$\mathbf{p}_n = -\sum_{j=1}^{n-1} \mathbf{p}_j = -\frac{m_{\kappa_n}}{m_{\kappa_1}}\, \mathbf{p}_1 - \sum_{j=2}^{n-1} \mathbf{e}_j$$

using (101) and

$$\omega_j \approx \frac{m_{\kappa_j}}{m_{\kappa_1}} \omega_1 + s_j m_{\kappa_j} \frac{\mathbf{p}_1 \cdot \mathbf{e}_j}{\omega_1} + m_{\kappa_1} m_{\kappa_j} \frac{\mathbf{e}_j^2}{2\omega_1} - m_{\kappa_1} m_{\kappa_j} \frac{(\mathbf{p}_1 \cdot \mathbf{e}_j)^2}{2\omega_1^3}$$

to second order in small quantities and with $\mathbf{e}_n = -\sum_{\ell=2}^{n-1} \mathbf{e}_\ell$ and for $j \in \{2, n\}$.

On the submanifold with momentum and energy conserved, and within a neighborhood of the points where the gradient $\nabla E_k((\mathbf{p})_n)$ equals zero,

$$\begin{aligned} E_k((\mathbf{p})_n) &\approx \frac{m_{\kappa_1}}{2\omega_1^3} \sum_{j=2}^{n} s_j m_{\kappa_j} \left(\omega_1^2\, \mathbf{e}_j^2 - (\mathbf{p}_1 \cdot \mathbf{e}_j)^2\right) \\ &= \frac{R^2}{2\omega_1^3} \left(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2\right) \end{aligned}$$

from (101), polar coordinates for the $(n-2)$ spatial vectors $\mathbf{e}_j$, and with the definitions

$$R^2 = \sum_{j=2}^{n-1} \mathbf{e}_j^2,$$

$\mathbf{u}_1 = \mathbf{p}_1/\|\mathbf{p}_1\|$ and

$$\alpha R^2 = m_{\kappa_1} \sum_{j=2}^{n} s_j m_{\kappa_j} \left(\mathbf{e}_j^2 - (\mathbf{u}_1 \cdot \mathbf{e}_j)^2\right), \qquad \beta R^2 = m_{\kappa_1} \sum_{j=2}^{n} s_j m_{\kappa_j} \mathbf{e}_j^2.$$

The result is that energy conservation (97) defines a generalized function except possibly for the points with the gradient $\nabla E_k((\mathbf{p})_n)$ vanishing when $E_k((\mathbf{p})_n) = 0$, the points with $R = 0$. For $R \approx 0$,

$$\begin{aligned} \delta(E_k((\mathbf{p})_n)) &= \delta(\frac{R^2}{2\omega_1^3}\ (\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2)) \\ &= \frac{2\omega_1^3}{R^2}\ \delta(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2) + \frac{2\omega_1^3}{\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2}\ \delta(R^2). \end{aligned}$$

Since (97) is a generalized function except when both the energy (99) and gradient (100) vanish, that is, when $R > 0$, $\delta(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2)$ is regular.

For $n \geq 4$ and a sufficient number of dimensions d, the singularities of the energy-momentum conserving delta functions (97) are locally summable for the regular selection (55) of $\mathfrak{h}(p_1, p_2)$. The determinant of the Jacobian matrix to polar coordinates for $(\mathbf{e})_{2,n-1}$ contributes

$$R^{(d-1)(n-2)-1}$$

in d spacetime dimensions for the nth order connected functions. Then, the summations in evaluation of the degenerate scalar product (27) include

$$\frac{R^{(d-1)(n-2)-1}}{R^2}\ dR = R^{(d-1)(n-2)-3}\ dR$$

in the neighborhood of the singularities and $d \geq 3$ suffices for the summations to converge and the constructed $\underline{\mathcal{W}}$ to be continuous linear functionals dual to tempered functions. Terms

$$R^{(d-1)(n-2)-1}\delta(R^2)$$

do not contribute for $n \geq 4, d \geq 3$ [20]. $d \geq 3$ suffices for finite masses m_{κ_j} and $d \geq 4$ includes massless particles [36].

To include massless particles, $m_\kappa = 0$, the development follows the finite mass case except that new singularities are encountered. The positive and negative mass shells intersect at $\mathbf{p}^2 = 0$, and ω^{-1} and the derivatives of $\omega = \sqrt{\mathbf{p}^2}$ diverge at $\mathbf{p}^2 = 0$. Selection of basis

function spaces $\underline{\mathcal{P}}$ with Fourier transforms with infinite order zeroes at each $\mathbf{p}_j^2 = 0$ regularizes the development [36]. With massless particles, a neighborhood of the points with a vanishing gradient (100) is defined by

$$\frac{\mathbf{p}_j}{\sqrt{\mathbf{p}_j^2}} = s_j \frac{\mathbf{p}_1}{\sqrt{\mathbf{p}_1^2}} + \mathbf{e}_j$$

with the constraint that

$$\mathbf{e}_j^2 + 2s_j \frac{\mathbf{p}_1 \cdot \mathbf{e}_j}{\sqrt{\mathbf{p}_1^2}} = 0.$$

Unlike the finite mass case, the perturbations $\mathbf{e}_j$ must preserve the unit lengths of $\mathbf{p}_j/\sqrt{\mathbf{p}_j^2}$. The constraint to unit length reduces the number of degrees of freedom in the summation over $(\mathbf{e})_{2,n-1}$. In polar coordinates, the summation contributes $R^{(d-2)(n-2)-1}$ and as a consequence, $d = 4$ is required to satisfy regularity with the inclusion of massless particles.

Finite masses, elemental stability A.7, the regular selection (55) for the arrays $\mathfrak{h}(p_1, p_2)$ and three or more spacetime dimensions d suffice for the constructed $\underline{\mathcal{W}}$ to satisfy regularity A.1. Inclusion of massless elementary particles requires four or more dimensions [36].

4.2.6 Summary of constructions

The construction of VEV functions $\underline{\mathcal{W}}$ with connected functions (46), (60) with (35), (42), (53) and (67), suffices to satisfy axioms A.1-7. The constructions are explicit example nontrivial realizations of relativistic quantum physics. While sufficient to satisfy A.1-7, the selected forms are not necessary to satisfaction of A.1-7. The constructed VEV satisfy A.1-7 for states described by function sequences from $\mathbf{H}_{\mathcal{P}}$, the completion in the Hilbert space norm (26) of the basis function spaces $\underline{\mathcal{P}}$ (9). Or equivalently, the construction is for function sequences from $\underline{\mathcal{S}}$ with modified VEV, section 3.4.2. For the constructed VEV:

A.1: Demonstrations of regularity apply if the factors $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ in the connected functions (46) are polynomially bounded growth, locally Lebesgue-summable functions of the energy-momenta within the domains (84), masses are finite, and the number of spacetime dimensions equals or exceeds three (2+1). Massless particles require four (3+1) or more spacetime dimensions and additional constraints on $\underline{\mathcal{P}}$, [36] and section 4.2.5. Regularity requires that the singularities of nonconstant $\mathfrak{h}(p_1, p_2)$ in the VEV (67) with (53) and (54) are excluded from the support of the scalar product (27), and that the divergent extrapolation of the connected functions (53) to two-point function is eliminated (75). A demonstration of regularity is in section 4.2.5.

A.2: The nonnegativity of the scalar product (27) determined by the composition (60), $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$, follows from the nonnegativity of the split signed symmetric (48) constituent sequences of generalized functions $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, section 4.2.1. The nonnegativity of $\underline{\mathcal{F}}$ is well-established [10, 56]. The nonnegativity of $\underline{\mathcal{U}}$ is demonstrated in section 4.2.2 and

follows from the nonnegativity of ${}^{C}\underline{\mathcal{W}}$, the cluster expansion (67), and that the $\circ$-product preserves the nonnegativity and split signed symmetry of sequences.

A.3: Section 4.2.3 includes a demonstration of relativistic invariance (76). Relativistic invariance follows from the relativistic covariance of $\underline{\mathcal{F}}$ and the expansions (53), (54) and (67) for $\underline{\mathcal{U}}$. The $S(A)$, $M(p)$ and D from an underlying free field $\underline{\mathcal{F}}$ establishes the realization of the Lorentz group for a construction. Translation invariance follows from energy-momentum conservation.

A.4: The limitation of the support of VEV to mass shells together with the zeros (18) on negative energy mass shells in the support of the functions from the basis spaces $\underline{\mathcal{P}}$ implements the spectral support condition. In the evaluation of scalar products (27), either

$$p_j + p_{j+1} \ldots + p_n \in \overline{V}^+$$

explicitly since each $p_\ell \in \overline{V}^+$, or energy-momentum conservation provides that this sum of energy-momenta is the negative of $p_1 + p_2 \ldots + p_{j-1}$ with each of $p_1, \ldots p_{j-1}$ explicitly in the closed backward cone. The cones are closed under addition of elements. Demonstrations of spectral support are included in [33, 37].

A.5: For constructions based on function sequences $\underline{\mathcal{P}}$ from section 3.4, local commutativity follows from the split signed symmetry of the constructed VEV (60).

A.6: Cluster decomposition follows from the connectedness of the functions ${}^{C}\underline{\mathcal{W}}$ in (46) and the cluster expansion (70) for the VEV functions $\underline{\mathcal{W}}$. The demonstration of cluster decomposition is in section 4.2.4.

A.7: VEV function sequences that satisfy A.1-6 determine sequences $\underline{\mathcal{W}}$ that also satisfy elemental stability A.7, (75). Elemental stability follows for mean zero fields if $\mathcal{W}_{k,0} = \mathcal{W}_{0,k} = \mathcal{W}_{k,1} = \mathcal{W}_{1,k} = 0$ for $k \geq 2$. For mean zero fields, $\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0$. If kinematic constraints do not suffice to satisfy elemental stability, then the constructions must violate formal Hermiticity W.a. The introduction of vacuum polarizations preserves satisfaction of A.1-7 with the appropriately restated A.7, section 3.3.5.

The relaxed quantum-classical correspondence introduced in section 1 does not require Hermitian fields in the VEV (7). Free fields, Wick polynomials of free fields and generalized free fields satisfy formal Hermiticity W.a with a basis function space $\underline{\mathcal{S}}$. Satisfaction of A.1-6 with formal Hermiticity W.a and involutivity W.b appears to be peculiar to physically trivial VEV.

There is a realization of relativistic quantum physics for: every realization of free fields (selection of N_c and a triple of $N_c \times N_c$ matrices $M(p), D, S(A)$ with $S(A)$ a representation of the Lorentz group (79)); nonnegative measure $d\sigma(\lambda)$ (that determines the c_n, the relative contributions of the connected functions); Lorentz invariant measure $d\mu_{\mathfrak{B}}(p)$ (that characterize

interactions); and Lorentz invariant and nonnegative measure $d\mu_{\Upsilon}(p)$, and constant α_o (that characterize interactions). In constructions with multiple species, distinct α_o, $d\mu_{\mathfrak{B}}(p)$, $d\mu_{\Upsilon}(p)$ and coupling constants c_n can be introduced for each indecomposable triple $M(p), D, S(A)$. Decomposable triples $M(p), D, S(A)$ are discussed in appendix 8.12.

Generalizations to these constructions include odd order scalar field VEV [33], compositions that add connected functions $\underline{\mathcal{F}} \circ \underline{\mathcal{U}}_1 \circ \underline{\mathcal{U}}_2$, additional forms in the expansion (57) of connected functions in terms of the matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$, extension of $d\mu_{\mathfrak{B}}(s)$ to complex-valued measures [33, 35], and massless particles, [36] and section 4.2.5.

4.3 Relation to free fields

The cluster expansion (65), the selection of two-point VEV (46) with (35), and the connectedness of the VEV functions provide that the constructed realizations of RQP are described by free fields when the supports of each argument of the state describing functions are localized and widely space-like separated. The connectedness and that at least two $*$-dual arguments and at least two function arguments appear in each contributing factor of the $n \geq 4$, n-arguments connected functions ${}^{C}\mathcal{U}_{k,n-k}$ provides that the contributions of the constructed $\mathcal{W}_{k,n-k}$ reduce to contributions from the free field two-point VEVs in $\mathcal{F}_{k,n-k}$ with one $*$-dual and one function argument each (42). In these instances with the supports of each argument of the state describing functions $\underline{f}$ sufficiently isolated and widely space-like separated, the states are readily interpreted as consisting entirely of (nearly) free particles. The scalar products of states with the support of each argument sufficiently widely space-like separated and localized equal the scalar products for free fields. The two-point function in (46) is a free field two-point function.

$$\langle \Phi(\underline{f})\Omega | \Phi(g)\Omega \rangle = \langle \Phi_o(\underline{f})\Omega_o | \Phi_o(g)\Omega_o \rangle_o$$

with $\Phi(\underline{f})$ the constructed quantum field (31) and $\Phi_o(\underline{f})$ the free field with the same elementary masses m_κ, $\langle \underline{f} | \underline{g} \rangle$ is the scalar product (27) using the constructed VEV and $\langle \underline{f} | \underline{g} \rangle_o$ is the free field, Fock space scalar product. As a consequence, sufficiently isolated concentrations of support propagate as nearly free particles. In the Wigner classification [64], the irreducible representations of the Poincaré group describe the elementary particles. Mass and spin label the irreducible representations for the finite mass cases emphasized here. The masses of the elementary particles are the m_κ that determines the zeros of the state describing functions $\tilde{f}_n((p)_n)_{(\kappa)_n}$ associated with each component field $\Phi(x)_\kappa$ by (4) and (9). Massless elementary particles are included in the constructions with additional considerations.

4.4 Properties of the constructions

In this section, the Hamiltonians for the constructions are discussed and it is demonstrated that the fields $\Phi(\underline{f})$ constructed in section 3.3 are unbounded Hilbert space operators. Discussion includes that physically nontrivial fields $\Phi(\underline{f})$ are not Hermitian.

As a consequence of the limitation of the momentum support of the VEV to mass shells and that the intersection of the supports of the VEV with the support (18) of functions from $\underline{\mathcal{P}}$ includes only positive energies, the Hamiltonian, the generator of time translations of a state describing function, derives from the single-argument subspace operator (80).

$$e^{-ip_{j0}\lambda} = e^{-i\omega_j\lambda} \tag{102}$$

using (10) and the translation is by $\lambda = ct$. This follows from (80) with the Fourier transform (22). In an n-argument subspace of $\mathbf{H}_{\mathcal{P}}$,

$$U(\lambda) = \prod_{j=1}^{n} e^{-i\omega_j\lambda} \tag{103}$$

implements temporal translation. The $\sum_{j=1}^{n} p_j$ are densely defined Hermitian operators in the n-argument subspaces and correspond to energy and momentum. Hermiticity follows from Poincaré invariance of the scalar product (27) and Stone's theorem [25]. In the subspace of the vacuum, $n = 0$, the Poincaré transformations (77) are $(a, \Lambda) = \mathbb{I}$. In multiple argument subspaces, an association of single argument subspace operators with the classical dynamical variables of corresponding particles is generally not determined. An association of the arguments of state describing functions with the properties of classical particles necessarily applies only for appropriate state describing functions, sections 4.3 and 5.2, or for VEV that lack interaction. The exhibition of interaction is described by the VEV (7): VEV determine the Hilbert space scalar product (8) and consequently the likelihoods of observations. The n and k-argument state describing functions are not orthogonal for $n \neq k$ when scalar products exhibit interaction. The evolution of state describing functions and the exhibition of interaction are discussed further in sections 4.5 and 5.

Demonstrated below, satisfaction of the prospective axioms A.1-7 in section 4.1 suffices to define the constructed quantum fields $\Phi(\underline{f})$ as Hilbert space operators. The quantum fields (31) are the multiplication (6) of function sequences and this product preserves Hilbert space norm-equivalence classes. The expansion (5) for states as products of the field applies for function sequences from $\underline{\mathcal{P}}$: the domain of the field (31) includes this dense set of elements within $\mathbf{H}_{\mathcal{P}}$. Demonstrated below, the field is not a bounded operator and as a consequence, the field is at best only densely defined, [25], section 7 and appendix 8.2.5.

For a sufficiently great space-like translation T and an element $\underline{h}$ of the null space of $\mathbf{H}_{\mathcal{P}}$, satisfaction of cluster decomposition (74) in section 4.1 (axiom A.6) provides that

$$\begin{aligned}
\|\underline{f} \times (\underline{g} + T\underline{h})\|^2 &= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \times T\underline{h} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | T\underline{h} \rangle + \langle T\underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle T\underline{h} | T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | \underline{h} \rangle + \langle \underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle \underline{h} | \underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle \\
&= \|\underline{f} \times \underline{g}\|^2
\end{aligned}$$

from evaluation of the norm (26), linearity of the scalar product, translation invariance of the scalar product ($\|T\underline{g}\| = \|\underline{g}\|$), the Cauchy-Schwarz-Bunyakovsky inequality ($|\langle \underline{h}|\Omega\rangle| \leq \|\Omega\|\|\underline{h}\|$), and $\|\underline{h}\| = 0$. Elements of the null space are translationally invariant,

$$\|\underline{h} - T\underline{h}\| = 0,$$

a consequence of the Cauchy-Schwarz-Bunyakovsky inequality, the unitarity of translation $\|T\underline{h}\| = \|\underline{h}\|$, and $\|\underline{h}\| = 0$. As a consequence, the field operators $\Phi(\underline{f})$ in (31) preserve equivalence classes of the Hilbert space norm (26).

$$\|\Phi(\underline{f})\,(\underline{g} + \underline{h})\| = \|\Phi(\underline{f})\,\underline{g}\|$$

for any $\|\underline{h}\| = 0$. Repetition of this argument with $(\underline{f} + \underline{h}) \times \underline{g}$ demonstrates that the quantum field $\Phi(\underline{f})$ is also independent of the representative used for $\underline{f} \in \mathbf{H}_{\mathcal{P}}$. The constructed fields are Hilbert space operators.

A quantum field $\Phi(\underline{f})$ (31) would be bounded if there is a real constant C such that $\|\Phi(\underline{f})\underline{v}\| \leq C\,\|\underline{v}\|$ for all $\underline{v} \in \mathbf{H}_{\mathcal{P}}$, [46] and appendix 8.2.5. However, the fields $\Phi(\underline{f})$ (31) are unbounded and consequently are not continuous anywhere within $\mathbf{H}_{\mathcal{P}}$, [46] and section 7.

A demonstration of unboundedness for a free scalar boson field suffices for the constructions of physical interest. Any construction that includes unconfined scalar bosons includes the free boson field VEV. Then, the cluster decomposition axiom A.6 in section 4.1 provides that the field $\Phi(\underline{f})$ is unbounded if the free boson field is unbounded. For free bosons, the VEV functions result from

$$\mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_S \prod_{j=1}^{k} W_2(x_j, x_{i_j})_{\kappa_j \kappa_{i_j}}$$

and are otherwise zero from (42) in section 3.2. For a product function,

$$\underline{f^n} = (0, 0 \ldots, \prod_{\ell=1}^{n} f(x_\ell)_{\kappa_\ell}, \ldots),$$

application of (6) and (31) results in

$$\Phi(\underline{f})\underline{f^n} = (0, 0 \ldots, f(x_1)_{\kappa_1} \prod_{\ell=2}^{n+1} f(x_\ell)_{\kappa_\ell}, \ldots).$$

Then the norm (26) for the free boson field provides

$$\|\underline{f^n}\|^2 = n!\; W_2(\underline{f}^*, \underline{f})^n$$

and

$$\|\Phi(\underline{f})\underline{f^n}\|^2 = (n+1)!\; W_2(\underline{f}^*, \underline{f})^{n+1}$$

with

$$W_2(\underline{f}^*, \underline{f}) = \sum_{\kappa_1=1}^{N_c} \sum_{\kappa_2=1}^{N_c} W_2(f^*_{\kappa_1}\, f_{\kappa_2})_{\kappa_1\kappa_2}$$

from (27). Then

$$\frac{\|\Phi(\underline{f})\underline{f^n}\|^2}{\|\underline{f^n}\|^2} = (n+1)\; W_2(\underline{f}^*, \underline{f}) > C^2$$

for any finite real constant C, some $n \in \mathbb{N}$ and some $\underline{f^n} \in \mathbf{H}_{\mathcal{P}}$. The constructed quantum fields $\Phi(\underline{f})$ are unbounded.

Do realizations with VEV such as (17) exhibit interaction "at a point" or are they string theories? Of course, such geometric characterizations are not intrinsic to quantum mechanics but are motivated by a predisposition for classical description. Neither functions of point support nor functions of string support are included among the state describing functions in the constructions. The constructed Hamiltonians (102) generate temporal translations in the Hilbert space realizations of the Poincaré group. The Hamiltonian is determined by realization of the Poincaré group rather than a correspondence with the classical dynamics of points or strings. These Hamiltonians coincide with what is identified in the canonical formalism as free field Hamiltonians even though interaction is manifest, sections 4.5 and 5, and [31, 33, 35]. The constructed VEV are solutions to the Klein-Gordon equation yet exhibit interaction. Mass shell singularities of the VEV together with a lack of constraints that set momenta equal in pairs result in nontrivial scattering. Canonical formalism-compliant solutions to the Klein-Gordon equation necessarily exhibit trivial physics [10, 56]. These demonstrations do not apply with the revised axioms: interacting quantum fields are not necessarily densely defined Hermitian Hilbert space operators. The constructed scattering likelihoods that coincide with Feynman series at weak coupling demonstrates that solutions to the Klein-Gordon equation are of interest. For VEV that are solutions to the Klein-Gordon equation, interaction is inconsistent with Hermitian field operators. This suggests that interacting relativistic fields are precluded by canonical quantization's quantum-classical correspondence. The constructions suggest a strengthening of the Haag (Haag-Hall-Wightman-Greenberg) theorem to that Poincaré covariance, local commutativity and positive energies together with an exhibition of interaction preclude satisfaction of formal Hermiticity and involutivity, W.a-b.

> The pressure for conformity is enormous. I have experienced it in editors' rejection of submitted papers, based on venomous criticism of anonymous referees. The replacement of impartial reviewing by censorship will be the death of science. – Julian Schwinger.

4.5 Scattering likelihoods

Interaction is exhibited in changes to momenta and particle numbers. A lack of orthogonality of the descriptions of states with differing particle numbers and momenta implements interaction. From Born's rule, scalar products (27) provide the state transition likelihoods and in relativistic physics, these likelihoods are Poincaré invariant. The state descriptions are covariant. In the constructions, the VEV rather than a Hamiltonian describe interaction. The scalar products of classical particle-like states, those with the support of each argument localized and widely space-like separated, unentangled, and well described by a single momentum, approximate the scalar products of free fields, section 4.3. Free fields have established interpretations as classical particles [8, 10, 24, 52, 61, 64]. If two classical particle-like state describing functions are separated by a great time difference,

$$\lim_{\lambda\to\infty} \langle \underline{f} | e^{-iH\lambda} \underline{g} \rangle,$$

the separations of the supports of $\underline{f}$ and $\underline{g}$ become time-like and the transition likelihood is no longer provided by free field VEV. In this case and for non-particle-like state describing functions, the transition likelihoods follow from the physically nontrivial VEV and exhibition of interaction manifests in the state transition likelihoods. Observations of distinct momenta and particle numbers become likely.

Scattering likelihoods are proportional to large time difference transition likelihoods evaluated with incoming and outgoing states described by their momenta [8, 24, 52, 61]. For the example of a single neutral scalar field and states described by product functions

$$f_n((x)_n) = \prod_j \ell(x_j; \lambda_j, \mathbf{q}_j),$$

scattering amplitudes are

$$S_{n,m} = \lim_{\lambda\to\infty} \langle U(\lambda)\ell(\lambda, \mathbf{q}_{n+1}) \dots \ell(\lambda, \mathbf{q}_{n+m}) | U(-\lambda)\ell(-\lambda, \mathbf{q}_1) \dots \ell(-\lambda, \mathbf{q}_n) \rangle$$

with $U(\lambda)$ the unitary operator (103) that translates the states in time. The parameters $\lambda_j, \mathbf{q}_j$ in the functions control a phase and the momentum support, respectively. The designation $\ell(\lambda_j, \mathbf{q}_j)$ indicates the values of the parameters of the state describing functions $\ell(x_j; \lambda_j, \mathbf{q}_j)$. The LSZ (Lehmann-Symanzik-Zimmermann) expressions [10] for scattering amplitudes use functions with Fourier transforms

$$\tilde{\ell}(p_j; \lambda_j, \mathbf{q}_j) = e^{ip_{j0}\lambda_j}(\omega_j + p_{j0})\tilde{f}(\mathbf{p}_j - \mathbf{q}_j) \tag{104}$$

with λ_j a real parameter, $\mathbf{q}_j$ a momentum vector and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ is a Schwartz tempered test function. From section 3.4, $\ell(x_j; \lambda, \mathbf{q}_j)$ is a function in the constructed Hilbert space $\mathbf{H}_\mathcal{P}$. A

convenient choice of test function are the most classical-like state describing functions, Gaussian functions,

$$\tilde{f}(\mathbf{p}) = \left(\frac{L^2}{\pi}\right)^{3/2} e^{-L^2\mathbf{p}^2} > 0. \tag{105}$$

These $\tilde{f}(\mathbf{p})$ are point-wise nonnegative delta sequences heavily weighted near zero momentum when nearing the plane wave limit $L \to \infty$ and

$$\int d\mathbf{p}\ \tilde{f}(\mathbf{p}-\mathbf{q}) = 1.$$

The LSZ scattering amplitudes are VEV of products of fields

$$\Phi(\ell(\lambda,\mathbf{q})) = \int dp\ (\omega + p_0) e^{ip_0\lambda} \tilde{f}(\mathbf{p}-\mathbf{q})\ \tilde{\Phi}(p)$$

in this scalar field example. The VEV functions $\tilde{\mathcal{W}}_{k,n-k}((p)_n)$ are (17) with (32). Temporal translations of the field evaluated with the state describing functions (104) are independent of time if the parameter λ_j is set to λ.

$$\begin{aligned} U(\lambda)\Phi(\ell(\lambda,\mathbf{q}))U(\lambda)^{-1} &= \int dp\ (\omega + p_0) e^{-i(\omega-p_0)\lambda} \tilde{f}(\mathbf{p}-\mathbf{q})\ \tilde{\Phi}(p) \\ &= \Phi(\ell(0,\mathbf{q})) \end{aligned}$$

due to the limitation of the spectral support of the constructed VEV to mass shells.

In a more familiar notation,

$$U(\lambda)\Phi(\ell(\lambda,\mathbf{q}))U(\lambda)^{-1} = i\int d\mathbf{x}\ \hat{u}(\lambda,\mathbf{x}) \overset{\leftrightarrow}{\partial}_o \Phi(\lambda,\mathbf{x})$$

with

$$f(x) \overset{\leftrightarrow}{\partial}_o g(x) = f(x)\dot{g}(x) - \dot{f}(x)g(x),$$

$\dot{f}(x)$ the first time derivative of $f(x)$ and

$$\hat{u}(\lambda,\mathbf{x}) = \frac{1}{2\pi}\int d\mathbf{p}\ e^{i\omega\lambda} e^{-i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}-\mathbf{q})$$

is a smooth solution of the Klein-Gordon equation.

For the Gaussian functions (105), the plane wave scattering amplitudes are the limits

$$\begin{aligned} S_{n,m} &= \lim_{\substack{L\to\infty \\ \lambda\to\infty}} \langle U(\lambda)\ell(\lambda,\mathbf{q}_{n+1})\dots\ell(\lambda,\mathbf{q}_{n+m})|U(-\lambda)\ell(-\lambda,\mathbf{q}_1)\dots\ell(-\lambda,\mathbf{q}_n)\rangle \\ &= \lim_{L\to\infty} \langle \ell(0,\mathbf{q}_{n+1})\dots\ell(0,\mathbf{q}_{n+m})|\ell(0,\mathbf{q}_1)\dots\ell(0,\mathbf{q}_n)\rangle. \end{aligned}$$

Evaluation of the mass shell deltas in the VEV from section 3 simplify the expression and the resulting quadrature is readily evaluated in the plane wave limit [33]. From the scalar product (27) for scalar field VEV,

$$\begin{aligned} S_{n,m} &= \lim_{L\to\infty} \left(\frac{L}{\sqrt{\pi}}\right)^{3(n+m)} \int d(\mathbf{p})_{n+m} \prod_{j=1}^{n} e^{-L^2(-\mathbf{p}_j-\mathbf{q}_j)^2} \prod_{\ell=n+1}^{n+m} e^{-L^2(\mathbf{p}_\ell-\mathbf{q}_\ell)^2} \\ &\qquad \times \widetilde{\mathcal{W}}_{n,m}((-\omega(\mathbf{p}),\mathbf{p})_n, (\omega(\mathbf{p}),\mathbf{p})_{n+1,n+m}) \\ &= \widetilde{\mathcal{W}}_{n,m}((-\omega(\mathbf{q}),-\mathbf{q})_n, (\omega(\mathbf{q}),\mathbf{q})_{n+1,n+m}) \end{aligned} \tag{106}$$

with $\omega(\mathbf{q}_j) = \omega(-\mathbf{q}_j)$ from (10). Each

$$(\omega_j \pm p_{k0})\,\delta(p_k^2 - \lambda_c^2) = \delta(p_{k0} \pm \omega_k)$$

and the sign is determined by whether p_k is the argument of a function or a $*$-dual function (14). Each energy is evaluated on the appropriate mass shell. Non-forward amplitudes result if only the connected contribution ${}^C\widetilde{\mathcal{W}}_{n,m}$ of the VEV is considered. Then for the example connected function (17), the forward amplitudes are

$$S_{n,m} = c_{n+m}\,\delta(\omega(\mathbf{q}_1)\ldots+\omega(\mathbf{q}_n)-\omega(\mathbf{q}_{n+1})\ldots-\omega(\mathbf{q}_{n+m}))\delta(\mathbf{q}_1\ldots+\mathbf{q}_n-\mathbf{q}_{n+1}\ldots-\mathbf{q}_{n+m}).$$

Up to a factor of i, this amplitude equals the first order expansion from a Feynman-Dyson series with an interaction Hamiltonian density $\mathcal{H}_{int}(x) = \sum_\ell a_\ell : \Phi(x)^\ell :$ with $\ell \geq 4$ and $a_\ell = c_\ell\,(2\pi)^{2\ell-4}/\ell!$, appendix 8.9. c_ℓ is from (52). For $n = 4$, the non-forward, elastic scattering amplitude is

$$S_{2,2} = c_4\delta(\omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) - \omega(\mathbf{q}_3) - \omega(\mathbf{q}_4))\delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \tag{107}$$

that corresponds to a Yukawa-like equivalent potential in first Born approximation [35]. For this scalar field example, the scattering cross sections of the realizations of relativistic quantum mechanics equal first order contributions from Feynman series. The phase difference of forward and non-forward amplitudes is unobservable in the scattering limit applicable to Feynman-Dyson series. In the example of Compton scattering, the cross sections deviate from first order Feynman series results for extremely relativistic exchange momenta (small distances) [35].

The scattering likelihoods are independent of time in the plane wave limit [33]. In this development, the propagation of the support of localized states can be followed through intermediate times. The lack of time dependence in finite time, plane wave limits is understood from a Hamiltonian that contributes only a phase in the plane wave limit and that plane waves uniformly cover all space.

5 Quantum-classical correspondences

> We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. At least I do, because I'm an old enough man that I haven't got to the point that this stuff is obvious to me. Okay, I still get nervous with it... You know how it always is, every new idea, it takes a generation or two until it becomes obvious that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm not sure there's no real problem. – Richard Feynman, 1982, p. 471 in [18].

Understanding quantum mechanics as the description of nature has been a persistently controversial topic [6, 9, 17, 44, 51, 57, 65]. Quantum mechanics is often regarded as "strange," perhaps referring to the incompatibility of the quantum description of nature with well-established classical understandings. Emphasized by the Einstein-Podolosky-Rosen (EPR) paradox, [17] and appendix 8.6, the quantum description of nature contradicts classical concepts. In the classical concept, every particle is distinguishable and described by determined classical dynamical variables. In the quantum description, particles are indistinguishable, their locations and momenta are never simultaneously described with arbitrary precision, and state descriptions are elements of Hilbert spaces that linearly expand in terms of other state descriptions. Although quantum effects like the Heisenberg uncertainty bounds are difficult to observe with massive bodies, Schrödinger's cat paradox [51] illustrates that quantum mechanics can not be relegated to a "strange" microscopic world.

Understanding approximate descriptions of states in quantum mechanics with classical dynamical variables is a quantum-classical correspondence. A quantum-classical correspondence applies when the classical description suffices. When sufficient, classical dynamical variables represent the support of appropriate state describing functions. Classical description is applied to perceptions of the quantum state.

Widely studied quantum-classical correspondences include the scattering likelihoods. Initial states and scattered products are perceived as classical particles. Scattering likelihoods describe the rates that classical particle-like states will transition to other classical particle-like states. Cluster decomposition A.6 provides that isolated initial and final scattered states are readily interpreted as freely propagating classical particles. The quantum-classical correspondences from scattering likelihoods include massless particles that are inherently relativistic. Scattering likelihoods enable comparison of the constructions with Feynman series. Scattering likelihoods, including approximation of Feynman series likelihoods, are provided in [31, 33, 35] and section 4.5.

The discussion in section 2 illustrated that unrealizable constraints on relativistic quantum physics follow from the presumed, exacting quantum-classical correspondence imposed by canonical quantization. In this section, realizable correspondences of classical descriptions and appropriate state descriptions $|\underline{\varphi}\rangle \in \mathbf{H}_{\mathcal{P}}$ (4) are developed. Rather than assert canonical quan-

tization, the quantum-classical correspondences derive from analysis of the support of state describing functions. Correspondences are conditioned on properties of the state describing functions. There is a quantum-classical correspondence if the state description is "macroscopic," but if there is significant spatial overlap of supports, or entanglements, or particle creation, or support is widely distributed over space, then a quantum-classical correspondence need not apply.

The richness of quantum mechanics manifests in the quantum-classical correspondences and a significant effort to characterize these correspondences remains. The relaxed, conditional correspondences vary with instance: no one quantum-classical correspondence necessarily applies universally. One relativistic quantum dynamical realization is perceived with multiple scenario-dependent classical descriptions. In addition to scattering and demonstrated at the first level of approximation for non-relativistic, brief duration state transition likelihoods, the corresponding classical interactions are $-g/r$ potentials.

Brief interval, non-relativistic state transition amplitudes are the primary focus of this section. For states that are well-represented by finite mass, point-like particles with non-relativistic relative velocities over sufficiently brief intervals, evolution of the support of appropriate state describing functions is approximated by classical Newtonian mechanics with $-g/r$ pair potentials. Presumably, classical geometrodynamics and electrodynamics emerge from more precise analysis.

In the designations of appendix 8.1, quantum mechanics is described by Dirac-von Neumann axioms I-III. Dirac-von Neumann axioms IV and V describe the canonical quantization of non-relativistic physics and are not applied to the constructions. The quantum-classical correspondence described in axioms IV and V is replaced by correspondences appropriate for relativistic physics.

Nature is described by equivalence classes of functions and the physically significant features of these functions are Lebesgue summations over measurable subsets of $\mathbb{R}^3$. A correspondence of Lebesgue measurable subsets with the classical dynamical variables $\mathbf{x}$ or $\mathbf{p}$, points in $\mathbb{R}^3$, is necessarily inexact. This adopted quantum-classical correspondence is suited to relativity and results in decisive revision to the mathematical development of RQP. The correspondence, relaxed from canonical quantization, generalizes Erwin Schrödinger's 1926 study of the linear harmonic oscillator [49] and Paul Ehrenfest's extensions [15, 42]: classical dynamical variables are representatives for the support of appropriate state describing functions. For the most classical particle-like descriptions, the quantum mechanical description is as well-represented by classical dynamical variables as possible: Heisenberg uncertainty lower bounds are achieved. The properties of appropriate state describing functions are developed in this section.

The quantum-classical correspondence that is the primary topic of this section appears in Schrödinger's 1926 study of non-relativistic linear harmonic oscillators [49]. The supports of selected solutions to the Schrödinger equation are well-represented by corresponding classical dynamical variables. In 1+1 spacetime, particular Gaussian functions $\psi(x, \lambda)$ satisfy the

Schrödinger equation.

$$\psi(x, ct) = \exp\left(-\frac{(x - A\cos wt)^2}{4\sigma^2} - i\frac{\beta x \sin wt}{\hbar} - i\phi(t)\right)$$

satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\psi = i\hbar\frac{d\psi}{dt}$$

and the solution is characterized by a mass m, spring constant k, and oscillation amplitude A,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}}, \qquad w = \sqrt{\frac{k}{m}}$$
$$\beta = \sqrt{mk}\,A, \qquad \phi(t) = \frac{w}{2}t - \frac{kA^2}{4\hbar w}\sin 2wt.$$

The breadth of support of $\psi(x, ct)$ over x is described by σ and w is the oscillation frequency. Selected as the peak likelihood, the representative for the support of this function is the trajectory

$$x(\lambda) = A\cos w\lambda/c$$

of classical linear harmonic motion. The quantum-classical correspondence is most evident when the spread of support is small with respect to the amplitude of the motion, $\sigma^2 \ll A^2$. As $\sigma^2/A^2 \to 0$, the classical representative becomes essentially indistinguishable from the quantum description. As $m \to \infty$,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}} \ll A^2 = \frac{2E}{k} \tag{108}$$

with the total classical energy $E = \frac{1}{2}kA^2$ a constant of the motion. The support of the function $\psi(x, \lambda)$:

1. persists along classical trajectories with the evolution of time λ:
2. the support of the state that exhibits the classical correspondence has a particular spread σ determined by k and m;
3. these $\psi(x, \lambda)$ are not eigenfunctions of the Hamiltonian, but are among the most-classical particle-like descriptions: the state describing functions meet the Heisenberg lower bound on the breadths of support over location and momentum;
4. and any classical energy E can be matched by a quantum mechanical description, a function $\psi(x, ct)$.

This quantum-classical correspondence does not apply for all state describing functions. For example, the supports of the energy eigenfunctions of a linear harmonic oscillator,

$$e_n(x,ct) = H_n(\alpha x)\,\exp\left(-i(n+\frac{1}{2})wt - \frac{1}{2}\alpha^2 x^2\right)$$

with H_n the nth Hermite polynomial [1] and

$$\alpha^2 = \frac{\sqrt{mk}}{\hbar},$$

do not follow classical trajectories of linear harmonic motion, except for $n = 0$ and $A = 0$. The energy eigenfunctions exhibit quantized energies

$$E = (n+\frac{1}{2})\hbar w.$$

For the energy eigenfunctions, only a phase, and not the support of the state describing functions, evolves with time.

To simplify development and notation, discussions within this section are often limited to a single, neutral scalar field, $N_c = 1$.

5.1 The support of state describing functions

The physically relevant support of states is determined by likelihoods of observation. Likelihoods are determined from Born's rule to be the squared magnitudes of scalar products (27) for state describing functions $\varphi_n \in \mathbf{H}_\mathcal{P}$. From A.3, Born's rule likelihoods are invariant to Poincaré transformations and state descriptions are covariant. The spatial support of $\varphi_n((x)_n)$ follows from the real-valued function over $(\mathbf{y})_k \in \mathbb{R}^{3k}$,

$$|\langle(\mathbf{y})_k|\varphi_n\rangle|^2 \tag{109}$$

with $|(\mathbf{y})_k\rangle$ described by a k-argument state describing function within $\mathbf{H}_\mathcal{P}$ from a delta sequence over space centered on the $\mathbf{y}_j$ and of point support in time. The support of $|(\mathbf{y})_k\rangle$ is concentrated very near $(\mathbf{y})_k$ and is at time zero in one selected reference frame. Due to the very localized support of $|(\mathbf{y})_k\rangle$, the momentum domain support includes very relativistic momenta. For the scalar products constructed in section 3, state describing functions $\varphi_n((x)_n)$ include descriptions of $k \neq n$ particles. And, $|(\mathbf{y})_k\rangle$ will not have strictly limited spatial support: $\mathbf{H}_\mathcal{P}$ includes only anti-local functions [33]. Similarly, the momentum support of $\varphi_n((x)_n)$ derives from

$$|\langle(\mathbf{q})_k|\tilde{\varphi}_n\rangle|^2 \tag{110}$$

with $|(\mathbf{q})_k\rangle$ described by a k-argument function within the Fourier transform domain of $\mathbf{H}_\mathcal{P}$ with momentum support concentrated very near $(\mathbf{q})_k \in \mathbb{R}^{3k}$. The spatial support of the delta sequence $|(\mathbf{q}_k)\rangle$ approaches $\mathbb{R}^3$. If $\mathbf{q}_j^2 \ll \lambda_{cj}^{-2}$, the dominant momentum support is non-relativistic.

Of particular interest for non-relativistic approximations are state describing functions within the Hilbert space completion $\mathbf{H}_{\mathcal{P}}$ of $\underline{\mathcal{P}}$ described at a single time in one selected reference frame. For every $\tilde{f}_n((\mathbf{p})_n) \in \mathcal{S}(\mathbb{R}^{3n})$ that is a function with dominant support centered on $(\mathbf{p})_n = 0$ that has an inverse Fourier transform with dominant support centered on $(\mathbf{x})_n = 0$, there is state describing function

$$\tilde{\varphi}_n((p)_n) = \prod_{j=1}^{n} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{q})_n). \tag{111}$$

The inverse Fourier transform of $\tilde{\varphi}_n$ is

$$\begin{aligned}
\varphi_n((x)_n) &= \int \frac{d(p)_n}{(2\pi)^{2n}} \prod_{j=1}^{n} e^{ip_jx_j} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{q})_n) \\
&= \int \frac{d(p_0)_n}{(2\pi)^{\frac{n}{2}}} \int \frac{d(\mathbf{p})_n}{(2\pi)^{\frac{3n}{2}}} \prod_{j=1}^{n} \left(-i\frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2}-\Delta_j}\right) e^{ip_{j0}x_{j0}} e^{-i\mathbf{p}_j\cdot(\mathbf{x}_j-\mathbf{y}_j)} \tilde{f}_n((\mathbf{p}-\mathbf{q})_n) \\
&= (2\pi)^{\frac{n}{2}} \prod_{j=1}^{n} \left(-i\frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2}-\Delta_j}\right) e^{-i\mathbf{q}_j\cdot(\mathbf{x}_j-\mathbf{y}_j)} \delta(x_{j0}) f_n((\mathbf{x}-\mathbf{y})_n)
\end{aligned} \tag{112}$$

with Δ_j the Laplacian for argument $\mathbf{x}_j \in \mathbf{R}^3$, and the derivative of $\delta(t)$ is a generalized function [20]. $\varphi_n((x)_n)$ is a function within $\mathbf{H}_{\mathcal{P}}$. The spatial support of $\varphi_n((x)_n)$ is centered on the $\mathbf{y}_j$ and the momentum support is centered on the $\mathbf{q}_j$. The completion of $\mathcal{P}(\mathbb{R}^{4n})$ to $\mathbf{H}_{\mathcal{P}}$ includes functions such as (112) that are generalized functions of point support over time and test functions over space. These are the functions of interest in non-relativistic physics, descriptions of functions over space at a specified time. Time generally is kept externally to the observed and perceptions are assigned a time in a particular coordinate frame.

For illustration and of particular utility, from (111) and (112), delta sequences within $\mathbf{H}_{\mathcal{P}}$ include normalizations of

$$\tilde{g}_k((p)_k) = \prod_{j=1}^{k} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\; e^{-L^2(\mathbf{p}_j-\mathbf{q}_j)^2}. \tag{113}$$

The $\tilde{g}_k((p)_k)$ in (113) describe $|(\mathbf{y})_k\rangle$ for $L > 0$ as $L \to 0$. $\tilde{g}_k((p)_k)$ describes a $|(\mathbf{q})_k\rangle$ for $L \to \infty$.

The energy-momentum support of the VEV constructed in section 3 is limited to mass shells. As a consequence, definition of the time and spatial support of functions $\varphi_n((x)_n)_{(\kappa)_n}$ derived from the inverse Fourier transform of a $\tilde{\varphi}_n((p)_n)_{(\kappa)_n}$ is ambiguous. Energy and momentum variables can be freely substituted

$$p_{j0} \longleftrightarrow \sqrt{\lambda_{cj}^{-2}+\mathbf{p}_j^2}.$$

The inverse Fourier transform of a momentum domain state describing function depends on whether one considers any p_{j0} an independent energy variable or a function of the momentum. In this section, a convention for the spacetime support of φ_n is selected.

From (9), function sequences $\underline{\varphi}, \underline{\psi} \in \underline{\mathcal{P}}$ are functions

$$\tilde{\varphi}_n((p)_n) = \prod_{j=1}^{n}(\omega_j + p_{j0})\tilde{f}_n((p)_n)_{\kappa_1 \ldots \kappa_n}$$
$$\tilde{\psi}_n((p)_n) = \prod_{j=1}^{n}(\omega_j + p_{j0})\tilde{g}_n((p)_n)_{\kappa_1 \ldots \kappa_n}$$

with $f_n, g_n \in \mathcal{S}(\mathbb{R}^{4n})$. Due to the form of the VEV constructed in section 3, the scalar products (27) expressed in the momentum domain are

$$\begin{aligned}\langle \underline{\varphi} | \underline{\psi} \rangle &= \sum \int d(p)_{n+m}\, \langle \tilde{\Phi}(p_1)_{\kappa_1} \ldots \tilde{\Phi}(p_n)_{\kappa_n}\Omega | \tilde{\Phi}(p_{n+1})_{\kappa_{n+1}} \ldots \tilde{\Phi}(p_{n+m})_{\kappa_{n+m}}\Omega \rangle \\ &\qquad \times \overline{\tilde{\varphi}_n}(-p_1, \ldots -p_n)_{\kappa_1 \ldots \kappa_n}\, \tilde{\psi}_m(p_{n+1}, \ldots p_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \\ &= \sum \int d(p)_{n+m}\, \overline{\tilde{f}_n}(p_1, \ldots p_n)_{\kappa_1 \ldots \kappa_n} \tilde{g}_m(p_{n+1}, \ldots p_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \prod_{j=1}^{n+m} \delta(p_{j0} - \omega_j) \\ &\qquad \times T_{n,m}((-\mathbf{p})_n, (\mathbf{p})_{n+1,n+m})_{(\kappa)_{n+m}} \delta(p_1 \ldots + p_n - p_{n+1} \ldots - p_{n+m})\end{aligned} \tag{114}$$

from (8), (14), (22), (32), relabeling summation variables and the Fourier transform of generalized functions [20]. The summations are over all $n, m \in \mathbb{N}_0$, $1 \leq \kappa_j \leq N_c$ with $1 \leq j \leq n+m$, and $(p)_{n+m} \in \mathbb{R}^{4n+4m}$. The VEV functions $T_{n,m}((\mathbf{p})_{n+m})_{(\kappa)_{n+m}}$ are composed in a cluster expansion (70) from factors of $1/(2\omega_j)$ from the mass shell delta functions and $c_n Q_{k,n-k}((p)_n)_{(\kappa)_n}$ in (46) with the energy-momentum conservation delta function factored out. From (10),

$$2\omega_j \delta(p_2^2 - \lambda_{cj}^{-2}) = \delta(p_{j0} - \omega_j).$$

$\mathbf{H}_{\mathcal{P}}$ includes limits of $\underline{\mathcal{P}}$ and these limits include $f_n, g_n \in \mathcal{S}(\mathbb{R}^{3n})$ with no energy dependence. In (114), all energies p_{j0} are constrained to positive mass shells

$$p_{j0} = \omega_j$$

after reflection of the n four-dimensional summation variables $(p)_n \mapsto (-p)_n$ for the $*$-dual functions $\tilde{\varphi}_n$. An ambiguity in the spacetime support estimated from the momentum domain state describing functions originates from this energy support constraint.

Isolating consideration to one function argument and one component field, let

$$\tilde{\varphi}(p_j) = (\omega_j + p_{j0})\tilde{f}(p_j)$$

describe argument j for component field κ_j. Setting every $p_{j0} = \omega_j$, the inverse Fourier transform of the state describing function appearing in the scalar product (114) is

$$\begin{aligned}\int \frac{dp}{(2\pi)^2}\, e^{ipx} \tilde{f}(\omega(\mathbf{p}), \mathbf{p}) &= \int \frac{dp_0}{(2\pi)^{\frac{1}{2}}}\, e^{ip_0(x_0-\lambda)} \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega(\mathbf{p})\lambda} \tilde{h}(\mathbf{p}) \\ &= (2\pi)^{\frac{1}{2}} \delta(x_0 - \lambda) h(\lambda, \mathbf{x})\end{aligned} \tag{115}$$

with designations

$$\tilde{h}(\mathbf{p}) = \tilde{f}(\omega(\mathbf{p}), \mathbf{p})$$

and

$$h(\lambda, \mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega(\mathbf{p})\lambda} \tilde{h}(\mathbf{p}). \tag{116}$$

$e^{i\omega(\mathbf{p})}$ is the time translation operator (103) that applies to argument j. The transform (116) is evaluated for convenient selections of $\tilde{h}(\mathbf{p})$ in appendix 8.10. (115) illustrates a dependence on the selection of a time λ. Substitutions of p_0 for $\omega(\mathbf{p})$ in $\tilde{f}(\omega(\mathbf{p}), \mathbf{p})$ lead ambiguously to state describing functions that are functions over time and not of temporal point support.

5.2 States with a quantum-classical correspondence

Verified daily, classical dynamical variables provide accurate representatives for the perceptions of quantum state descriptions when the support of states is "macroscopic," that is, classical particle-like. A quantum state description is classical particle-like if the spatial support of each argument is isolated and well-represented by a single location, the support of the Fourier transform of each argument is well-represented by a single momentum, and supports are not entangled. The Michelson interferometer illustrates that localization is necessary, and an atom or relativistic collision illustrate the necessity of support isolation to apply an effective quantum-classical correspondence, appendix 8.5. The EPR paradox [17] illustrates that descriptions must not be entangled for a quantum-classical correspondence to apply. Nevertheless, if the dominant support of states is classical particle-like, an approximate correspondence of classical and quantum state descriptions substitutes for the canonical quantization-conjectured elevations of classical dynamical variables to densely defined Hermitian operators.

Dominant support, localization, representative momenta, isolation and non-relativistic support are described below for the instance of one neutral scalar field and more generally, the definitions apply for each constituent function labeled by field component κ_j.

Dominant support: The physically relevant dominant support of a state describing function $\varphi_n((x)_n)$ is evaluated by comparison of the supports within distinct, common, finite volumes within $\mathbb{R}^3$. A convenient, but generally loose, characterization for the dominant support

of the jth argument of $\varphi_n((x)_n)$ is to find the spherical volume $V(\mathbf{u}_j)$ centered on the location $\mathbf{u}_j$ with

$$\int_{V(\mathbf{u}_j)} d\mathbf{y}_j \, |\langle (\mathbf{y})_k | \varphi_n \rangle|^2 = (1-\epsilon) \int_{\mathbb{R}^3} d\mathbf{y}_j \, |\langle (\mathbf{y})_k | \varphi_n \rangle|^2 \tag{117}$$

with $(\mathbf{y})_k$ the delta function-like element (113) of $\mathbf{H}_{\mathcal{P}}$ in (109). In the understanding of section 5.1, this is the support at time zero. $1-\epsilon$ is the relative likelihood that a position measurement for the jth argument will lie within the volume $V(\mathbf{u}_j)$. Designate by R_j the radius of this sphere $V(\mathbf{u}_j)$. The dominant volume of support $V(\mathbf{u}_j)$ and representative location $\mathbf{u}_j$ achieve a specified likelihood $1-\epsilon$ with the minimal R_j.

Similarly, the physically relevant dominant momentum support is characterized by the volume $U(\mathbf{w}_j)$ centered on a momentum $\mathbf{w}_j$ in the Fourier transform domain such that

$$\int_{U(\mathbf{w}_j)} d\mathbf{q}_j \, |\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2 = (1-\epsilon) \int_{\mathbb{R}^3} d\mathbf{q}_j \, |\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2 \tag{118}$$

with $(\mathbf{q})_k$ the delta function-like element (113) of $\mathbf{H}_{\mathcal{P}}$ in (110). Designate the radius of this sphere $U(\mathbf{w}_j)$ by Q_j. The dominant volume of momentum support $U(\mathbf{w}_j)$ and representative momentum $\mathbf{w}_j$ achieve a specified likelihood $1-\epsilon$ with the minimal Q_j.

Localization: The jth argument of a state describing function $\varphi_n((x)_n)$ is localized near $\mathbf{y}_j$ if the radius R_j of the dominant support is within a tolerance L_j set by a consideration of interest.

$$L_j > R_j.$$

From the development of non-relativistic support (125) below, to be non-relativistic,

$$R_j \gg \lambda_c.$$

If the jth argument of $\varphi_n((x)_n)$ is localized near $\mathbf{u}_j$, then

$$\langle \psi_n | g(\mathbf{x}_j) \varphi_n \rangle \approx g(\mathbf{u}_j) \langle \psi_n | \varphi_n \rangle \tag{119}$$

for multiplier functions $g(\mathbf{x})$ of slow variation within the dominant support of $\varphi_n((x)_n)$. $\mathbf{u}_j$ is the representative value of the argument $\mathbf{x}_j$, the jth argument of $\varphi_n((x)_n)$. $g(\mathbf{x}_j)\varphi_n$ designates the n-argument function

$$g(\mathbf{x}_j)\varphi_n(x_1, x_2, \ldots x_j, \ldots x_n).$$

One example consideration of interest setting the scale is an R_j small with respect to the separation of slits in Young's double slit: the observation is propagation through the slits followed by localized detection far behind the slits. If the radius of dominant support R_j is small with respect to the slit separation, then no interference is observed. Conversely, if R_j is much larger than the slit separation, then the quantum state description behaves as a wave rather than a classical particle.

From the connectedness of constructed VEV (45), if the spatial support of a state describing function φ_n is localized, then the localized support is indicative of the physically relevant spatial support (109). Although the dominant support of the state describing function $\varphi_n((x)_n)$ does not equal the physically relevant support, it does characterize the physically relevant dominant support. The connectedness of VEV in the evaluation of the scalar product (27) provides that the summation is dominated by points within a limited distance from the $(\mathbf{y})_k$. Then from (109) and if the support of φ_n considered at times $x_{j0} = 0$ (115) is localized, the support of $\langle(\mathbf{y})_k|\varphi_n\rangle$ is localized (119) about the $(\mathbf{y})_k$ within a tolerance set by the breadth of the VEV.

Representative momenta: Similarly, the momentum support of the Fourier transform $\tilde{\varphi}_n((p)_n)$ of a state describing function may be "localized." That is, the description of the momenta may be well-characterized by one representative momentum $\mathbf{w}_j$. If the jth argument of a state description $\tilde{\varphi}_n((p)_n)$ is descriptively supported near a representative momentum $\mathbf{w}_j$, then

$$\langle\tilde{\psi}_n|\tilde{g}(\mathbf{p}_j)\tilde{\varphi}_n\rangle \approx \tilde{g}(\mathbf{w}_j)\langle\tilde{\psi}_n|\tilde{\varphi}_n\rangle \tag{120}$$

for multiplier functions $\tilde{g}(\mathbf{p})$ of slow variation within the dominant support over $\mathbf{p}_j$ of $\tilde{\varphi}_n((p)_n)$.

Isolated: A localized argument $\mathbf{x}_j$ is isolated if there is little likelihood that any neighborhood that lies within the dominant support is included within a dominant support of any other argument of the state describing function. The support of a localized argument $\mathbf{x}_j$ is isolated if

$$(\mathbf{u}_j - \mathbf{u}_\ell)^2 > (R_j + R_\ell)^2 \tag{121}$$

for all $\ell \in \{1, n\}$ with $\ell \neq j$. Two arguments are isolated from each other if the spatial separation of the two representative locations exceeds the sum of the two radii of dominant support. If isolation is satisfied, only the support of the jth argument associates to great likelihood with the volume of space near the representative $\mathbf{u}_j$.

Cluster decomposition A.6 provides that if localized supports for every argument of the state describing functions are sufficiently isolated, then only two-point VEV contribute in the scalar product (27). In this case, scalar products (27) approximately vanish unless particle number is conserved.

$$\begin{aligned}\langle\underline{\varphi}|\underline{\psi}\rangle &= \sum_{n,m}\langle\varphi_n|\psi_m\rangle \\ &\approx \sum_{n}\langle\varphi_n|\psi_n\rangle.\end{aligned}$$

The most classical particle-like descriptions meet the Heisenberg uncertainty bound for the breadth of localization and representative momenta. The association of spatial volumes with non-relativistic classical dynamical variables occurs on relatively large spatial scales, for spatial supports with extents large with respect to Compton wavelengths.

The quantum-classical correspondence discussed in this section applies only as long as the support of a state describing function remains isolated and well-represented by a single location and momentum, and particle number is conserved. The non-relativistic quantum-classical

correspondences are limited to finite mass elementary particles, $m > 0$. Massless particles, included by the constructions in section 3, are inherently relativistic and this discussion of non-relativistic quantum-classical correspondences does not apply.

Non-relativistic classical trajectories $\mathbf{u}_j(\lambda)$ are spatial vectors,

$$\mathbf{u}_j(\lambda) = (u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)),$$

defined in a particular reference frame. Associated spacetime vectors $u_j(\lambda)$ are designated

$$u_j(\lambda) = (0, u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)). \tag{122}$$

Derivatives with respect to the temporal parameter $\lambda = ct$ are designated

$$\dot{\mathbf{u}}_j(\lambda) = \frac{d\mathbf{u}_j(\lambda)}{d\lambda}. \tag{123}$$

Trajectories are twice differentiable curves in $\mathbb{R}^3$ that specify one classical body's history of locations $\mathbf{u}_j(\lambda)$ and velocities $\dot{\mathbf{u}}_j(\lambda)$. Momenta $\hbar\mathbf{w}_j(\lambda)$ are Euclidean three-vectors associated with the particle trajectories $\mathbf{u}_j(\lambda)$.

$$\begin{aligned} \lambda_c \mathbf{w}_j(\lambda) &= \gamma_j \, \dot{\mathbf{u}}_j(\lambda) \\ &\approx \dot{\mathbf{u}}_j(\lambda) \end{aligned} \tag{124}$$

with

$$\gamma_j = \frac{1}{\sqrt{1 - \dot{\mathbf{u}}_j^2}}$$

and the approximation applies in non-relativistic ($\dot{\mathbf{u}}_j^2 \ll 1$) instances. The energy-momentum Lorentz vector of a finite mass particle is $p_j = (\omega(\mathbf{w}_j), \mathbf{w}_j)$ with $\omega(\mathbf{p})$ from (10).

Non-relativistic: For finite mass particles, non-relativistic physics applies if the momentum domain supports of the state describing functions φ_n are sufficiently limited. The support of an argument $\mathbf{p}_j$ is non-relativistic if there exists a boost to a reference frame such that the dominant momentum support satisfies $\hbar^2\mathbf{p}_j^2 \ll (mc)^2$ for any momenta $\mathbf{p}_j$ within the dominant support. In this frame,

$$\begin{aligned} \mathbf{p}_j^2 &\ll \lambda_c^{-2} \\ \langle \varphi_n | \omega_j \varphi_n \rangle &\approx \langle \tilde{\varphi}_n | (\lambda_c^{-1} + \tfrac{1}{2}\lambda_c \mathbf{p}_j^2) \tilde{\varphi}_n \rangle \\ \mathbf{w}_j(\lambda) &\approx \lambda_c^{-1} \dot{\mathbf{u}}_j(\lambda). \end{aligned} \tag{125}$$

A state describing function φ_n is non-relativistic if there is a single reference frame that sets the support of all arguments of interest to non-relativistic.

Appropriate functions: If $\tilde{f}_n((\mathbf{p})_n, \lambda) \in \mathcal{S}(\mathbb{R}^{3n})$ is a function descriptively supported near $(\mathbf{p})_n = 0$ with inverse Fourier transforms localized near $(\mathbf{x})_n = 0$, then state describing functions

$$\tilde{\varphi}_n((p)_n; \lambda) = \prod_{j=1}^{n} e^{i\mathbf{p}_j \cdot \mathbf{u}_j(\lambda)} (p_{j0} + \omega_j) \tilde{f}_n((\mathbf{p} - \mathbf{w}(\lambda))_n; \lambda) \tag{126}$$

have spatial supports centered on $\mathbf{u}_j(\lambda)$ and momentum supports centered on $\mathbf{w}_j(\lambda)$. The more closely the breadth of the supports of $\tilde{f}_n((\mathbf{p})_n, \lambda)$ satisfy the Heisenberg uncertainty bound, the more classical particle-like are the state describing functions (126). Additional requirements are that the description lacks entanglement and the $\mathbf{u}_j(\lambda)$ are sufficiently space-like separated to satisfy isolation. With appropriate selections for $\mathbf{u}_j(\lambda)$, $\mathbf{w}_j(\lambda)$ and $\tilde{f}_n((\mathbf{p})_n, \lambda)$, the support of (126) is non-relativistic (125), isolated (121), localized (119) and has representative momenta (120). ω_j is from (10). From the Fourier transform pair (81) with (82), (126) has an inverse Fourier transform (22),

$$\varphi_n((x)_n; \lambda) = \prod_{j=1}^{n} \left(-i \frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2} - \Delta_j} \right) e^{-i\mathbf{w}_j(\lambda) \cdot (\mathbf{x}_j - \mathbf{u}_j(\lambda))} f_n((x - u(\lambda))_n; \lambda) \tag{127}$$

with Δ_j the Laplacian in $\mathbb{R}^3$ for argument $\mathbf{x}_j$. With the understanding of support from section 5.1, the inverse Fourier transforms (22) of (126) have point support at time zero, each $x_{j0} = 0$. In (127), classical trajectories $u_j(\lambda) = (0, \mathbf{u}_j(\lambda))$ are representatives for the volumes of dominant spatial support, and the $\hbar \mathbf{w}_j(\lambda)$ are representatives for the dominant momentum supports. The temporal parameter λ appears in f_n in (127) to include a continuous deformation of $f_n((x)_n; \lambda)$ with time evolution (80). Typically the unitary time evolution of $\varphi_n((x)_n; 0)$ results in a growth in the breadth of support of $f_n((x)_n; \lambda)$: classically, a spreading of the spatial support over time follows from uncertainty in the initial momenta. Example classical particle-like state describing functions include Gaussians with spreads determined by a complex λ-dependent parameter. The functions $\varphi_n((\mathbf{x})_n; \lambda)$ are infinitely differentiable with respect to $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$.

Position and momentum eigenfunctions are not suitable for quantum-classical correspondences. These eigenfunctions are among the least classical particle-like by the consideration that location and momentum are both determined in a classical description. While $|\langle (\mathbf{y})_k | \varphi_n \rangle|^2$ provides the likelihood that the state described by φ_n will be jointly perceived near $(\mathbf{y})_k$, $(\mathbf{y})_k$ does not describe a state with an evident classical interpretation: there is no indication of a momentum and k is not necessarily equal to n since $(\mathbf{y})_k$ is relativistic when sufficiently localized. Similarly, there is no indication of location in the likelihoods $|\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2$. Nevertheless, before a plane wave limit, likelihoods $|\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2$ are useful in scattering instances since asymptotically the supports of the state describing functions are sufficiently spatially separated that cluster decomposition A.6 provides that localization (119) and isolation (121) apply to associate φ_n with corresponding classical particles. This correspondence in scattering does not apply for brief intervals λ. In the plane wave limit, the $\tilde{\varphi}_n$ become eigenfunctions of the Hamiltonian

(102), and like in Schrödinger's analysis of the linear harmonic oscillator, are not appropriate state describing functions for a quantum-classical correspondence.

5.3 A non-relativistic quantum-classical correspondence

Evident in the scattering likelihoods, including (107) and [31, 33, 35], the VEV constructed in section 3.3 exhibit interaction. Interaction is also manifest in the relativistically invariant, finite interval transition likelihoods. In this section, a non-relativistic correspondence of classical particle trajectories with the time evolution of state describing functions is developed.

Our perceptions of "macroscopic" quantum state descriptions satisfy classical dynamics. The quantum-classical correspondence introduced in sections 2 and 5.2 associates isolated volumes of support of the state describing functions $\varphi_n(\lambda)$ with the classical dynamical variables, location and momenta. These classical dynamical variables follow Newtonian dynamics in non-relativistic instances (125). More generally, with consideration limited to observation of features associated with classical bodies, if we calculate that the bodies move guided by classical fields $A(x)_\kappa$ with sources on the bodies (e.g., the Lorentz force and Maxwell's equations in electrodynamics, or geometrodynamical gravity) and this classical dynamics produces an accurate approximation to the quantum dynamics when the supports of state describing functions are well-represented by classical bodies, then we would say that the quantum model corresponds with the classical field theory. This correspondence is valid regardless of whether there is a "quantization" of the classical fields. A correspondence of classical field theory with quantum dynamics is established by the approximation of the motions of observed bodies.

In this relaxed quantum-classical correspondence, time evolution of the state describing function is represented by the evolution of classical dynamical variables. A correspondence of classical particle trajectories $\mathbf{u}_j(\lambda)$ with the evolution of a state describing function is realized if

$$|U(-\lambda)\hat{\varphi}_n(0)\rangle \approx e^{i\phi_I(\lambda)}|\hat{\varphi}_n(\lambda)\rangle \tag{128}$$

for times $\lambda > 0$. The quantum mechanical evolution of states is a unitary mapping $U(-\lambda)$ (80) of the state describing functions $\hat{\varphi}_n(0)$. For state describing functions (126), the Fourier transform of the quantum evolution is

$$U(-\lambda)\tilde{\varphi}_n(0) = \prod_{j=1}^{n} e^{i\omega_j\lambda} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(0)}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{w}(0))_n;0)$$

with the substitution (103) for the time translation operator. The correspondence (128) asserts that the non-relativistic evolution of the support of $\varphi_n(0)$ is perceived to follow trajectories $\mathbf{u}_j(\lambda)$ that satisfy Newtonian dynamics,

$$\tilde{\varphi}_n(\lambda) = \prod_{j=1}^{n} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{w}(\lambda))_n;\lambda)$$

with $\dot{\mathbf{u}}(\lambda) \approx \lambda_c \mathbf{w}(\lambda)$. A quantum-classical correspondence is justified for appropriate state describing functions, those with spatial and momentum support that is well-represented by one location (119) and one momentum (120), and that is identifiable due to isolation (121). Newtonian mechanics applies if the support is also non-relativistic (125). Time is advanced in the state describing function to match the advancement of time in the description of trajectories:

$$U(-\lambda)\varphi_n((0,\mathbf{x})_n;0) = \varphi_n((\lambda,\mathbf{x})_n;0)$$

from (80). From Born's rule, the squared magnitude of the scalar product

$$|\langle U(-\lambda)\hat{\varphi}_n(0)\rangle|\hat{\varphi}_n(\lambda)\rangle|^2$$

is the likelihood that the state described by $U(-\lambda)\hat{\varphi}_n(0)$ is perceived as the state described by $\hat{\varphi}_n(\lambda)$. The phase $\phi_I(\lambda)$ in the correspondence (128) is a purely quantum mechanical consideration following from the descriptions of states as rays in a complex Hilbert space. Justified by Poincaré invariance of the scalar product (76), the reference time is taken as zero without loss of generality. In (128), the state describing functions are normalized in the Hilbert space norm (26),

$$\hat{\varphi}_n(\lambda) = \frac{\varphi_n(\lambda)}{\|\varphi_n(\lambda)\|}. \tag{129}$$

The trajectories $\mathbf{u}_j(\lambda)$ that are most likely to be observed optimize the correspondence (128). Conversely, given a trajectory $\mathbf{u}_j(\lambda)$, (128) defines a matching distorted function,

$$\varphi_n(\lambda) = e^{-i\phi_I(\lambda)} U(-\lambda)\hat{\varphi}_n(0)\rangle, \tag{130}$$

whether there is a quantum-classical correspondence or not. These distorted functions (130) are not of interest to identify the corresponding trajectories. The solution of interest to (128) identifies trajectories determined by the quantum dynamics. The most classical particle-like state describing functions $\varphi_n(0)$ are selected to identify corresponding $\mathbf{u}_j(\lambda)$. The most classical particle-like state describing functions are Gaussians and the evolution of Gaussians remain Gaussian with complex spreading parameters as long as non-relativistic approximations apply. Gaussian functions initially exhibit the strongest quantum-classical correspondence: for real spreading parameters, the Heisenberg lower bound on location and momentum support spreads is met. With inclusion of the phase $\phi_I(\lambda)$ and free particle-like spreading of the support of Gaussian state describing functions, the unitary evolution of the most classical particle-like state describing function $\varphi_n(0)$ determines corresponding classical dynamical variables $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$. However, the motion of the center-of-momentum is free, and consequently irrelevant to determination of relative trajectories. Interaction is exhibited in the relative motions of the corresponding classical particles.

The approximate equality of two state describing functions (128) is established in the Hilbert space norm (26). The phase $\phi_I(\lambda)$ is determined to minimize the error in the approximation

(128).

$$\begin{aligned} 0 &\approx \left\| U(-\lambda)\hat{\varphi}_n(0) - e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda) \right\|^2 \\ &= 2 - 2\Re e\left(e^{i\phi_I(\lambda)} \langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle \right). \end{aligned}$$

The error is minimized by maximization of the real component of the product of the phase factor and the indicated scalar product. Polar decomposition determines the phase $\phi_I(\lambda)$ that maximizes $\Re e\left(e^{i\phi_I(\lambda)} \langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle \right)$.

$$\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle = |\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle| \, e^{-i\phi_I(\lambda)} \tag{131}$$

The phase is an additive and homogeneous function over time.

$$\begin{aligned} \phi_I(a+b) &= \phi_I(a) + \phi_I(b) \\ \phi_I(0) &= 0. \end{aligned}$$

The Cauchy-Schwarz-Bunyakovsky inequality provides that

$$|\langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle| \leq 1 \tag{132}$$

with the normalization (129).

It is convenient to designate a scalar product of normalized state describing functions

$$I(\lambda) = \frac{\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|} \tag{133}$$

and from Poincaré invariance of the scalar product,

$$\|U(-\lambda)\varphi_n(0)\| = \|\varphi_n(0)\|.$$

From the selection of phase (131),

$$|I(\lambda)| = e^{i\phi_I(\lambda)} I(\lambda).$$

This scalar product (133) is a function of the time interval λ, and unknown trajectories $\mathbf{u}_j(\lambda)$ and momenta $\mathbf{w}_j(\lambda)$. $\mathbf{u}_j(0), \dot{\mathbf{u}}_j(0)$ are considered initial conditions for the corresponding classical trajectories. The phase factor $e^{i\phi_I(\lambda)}$ in (128) does not contribute to likelihood $|I(\lambda)|^2$: $z\overline{z} = |z|^2$.

The condition (128) establishes a quantum-classical correspondence if three considerations are satisfied:

C.1) the likelihood $|I(\lambda)|^2$ is near unity

C.2) the dominant supports of the evolved functions (127) remain well-represented by one location (119) and one momentum (120), identifiable due to isolation (121), and non-relativistic (125)

C.3) the corresponding trajectories $\mathbf{u}_j(\lambda)$ satisfy classical dynamics.

The state describing functions $\tilde{\varphi}_n(\lambda)$ (127) are composite functions over $6n$ functions $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$. The center-of-momentum of the n interacting classical particles representative of $\varphi_n(\lambda)$ evolves as a single free body independently of the relative motion of the bodies. Independence applies in non-relativistic approximation (125). Poincaré invariance (76) of the scalar product (27) and covariance (77) of state descriptions $\varphi_n(\lambda)$ provide that a Poincaré transformation equates any scalar product to the scalar product in a center-of-momentum reference frame determined by the n trajectories $\mathbf{u}_j(\lambda)$ with momenta $\mathbf{w}_j(\lambda)$. In non-relativistic approximation $\dot{\mathbf{u}}_j(\lambda) = \lambda_c \mathbf{w}_j(\lambda)$. In this center-of-momentum frame and for non-relativistic momenta,

$$\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda) = 0, \quad \text{and} \qquad \sum_{j=1}^{n} m_{\kappa_j} \mathbf{w}_j(\lambda) = 0.$$

Transformation to this frame is a boost described in appendix 8.11, and a translation to collocate the center-of-mass

$$\frac{\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda)}{\sum_{j=1}^{n} m_{\kappa_j}}$$

with the origin of coordinates. As a consequence, only $n-1$ of the $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$ are independent variables determining $I(\lambda)$.

Continuous differentiability of $|I(\lambda)|^2$ suffices to satisfy C.1 for sufficiently brief intervals λ. A Taylor theorem polynomial approximates the likelihood.

$$|I(\lambda)|^2 = |I(0)|^2 + \lambda \frac{d|I(0)|^2}{d\lambda} + \frac{1}{2}\lambda^2 \frac{d^2|I(0)|^2}{d\lambda^2} + \dots \tag{134}$$

and from the Cauchy-Schwarz-Bunyakovsky inequality (132), $\lambda = 0$ is recognized as a maxima. Then

$$\frac{d|I(0)|^2}{d\lambda} = 0.$$

From (133), $|I(0)|^2 = 1$ and for sufficiently brief λ,

$$|I(\lambda)|^2 \approx 1 - \frac{1}{2}\lambda^2 \left| \frac{d^2|I(0)|^2}{d\lambda^2} \right|. \tag{135}$$

A selection of trajectories that minimize the first temporal derivative of $|I(\lambda)|^2$,

$$\frac{d|I(\lambda)|^2}{d\lambda} \approx 0, \tag{136}$$

provides that $|I(\lambda)| \approx |I(0)| = 1$.

For the most likely trajectory, satisfaction of the quantum-classical correspondence (128) is not improved by any modification to the trajectory. For the optimal trajectory,

$$\begin{aligned} \frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} &= 0 \\ \frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} &= 0 \end{aligned} \tag{137}$$

individually for each component of the $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda) = \lambda_c \mathbf{w}_j(\lambda)$. Here, partial derivatives with respect to a spatial vector designates a gradient vector

$$\frac{\partial F}{\partial \mathbf{u}_j} = \frac{\partial F}{\partial u_{jx}}, \frac{\partial F}{\partial u_{jy}}, \frac{\partial F}{\partial u_{jz}}.$$

Chain rule expansion of the derivative of $|I(\lambda)|^2$ includes partial derivatives with respect to the $n-1$ independent $\mathbf{u}_j(\lambda), \dot{\mathbf{u}}_j(\lambda)$.

$$\frac{d|I(\lambda)|^2}{d\lambda} = \sum_{j-1}^{n-1} \frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} \cdot \dot{\mathbf{u}}_j(\lambda) + \sum_{j-1}^{n-1} \frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} \cdot \ddot{\mathbf{u}}_j(\lambda) + \frac{\partial |I(\lambda)|^2}{\partial \lambda} \tag{138}$$

with the $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$ held constant in the evaluation of

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda}.$$

Finally, if the optimality condition (137) is satisfied, then the $\varphi_n(\lambda)$ that minimize the loss of likelihood (182) with increasing λ also solve

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda} \approx 0. \tag{139}$$

The product rule and $|I(\lambda)| \approx 1$ for brief λ provide that (139) is satisfied if and only if

$$\frac{\partial |I(\lambda)|}{\partial \lambda} = 0.$$

Sesquilinearity of the scalar product (27) and $U(-\lambda) = e^{iH\lambda}$ (103) provide that the derivative of $|I(\lambda)|$ evaluated at $\lambda = 0$ is

$$\begin{aligned} 0 &= \frac{\partial |I(\lambda)|}{\partial \lambda} \\ &= \frac{\partial}{\partial \lambda} \frac{\langle e^{iH\lambda} \hat{\varphi}_n(0) | e^{i\phi_I(\lambda)} \varphi_n(\lambda) \rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|} \\ &= -i \frac{\langle H\varphi_n(0) | \varphi_n(0) \rangle}{\|\varphi_n(0)\|^2} + i \frac{\partial \phi_I(\lambda)}{\partial \lambda} + \frac{\partial}{\partial \lambda} \frac{\langle \varphi_n(0) | \varphi_n(\lambda) \rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|}. \end{aligned} \tag{140}$$

H is the Hamiltonian (103) in the n-argument subspace. The first term in the final line of (140) is the expectation of the energy of $\varphi_n(0)$, and (140) relates this energy to the derivative of the phase $\phi_I(\lambda)$ with a third correcting term. At $\lambda = 0$, (140) is satisfied for the most likely trajectories $\mathbf{u}_j(\lambda)$.

Extrapolation of the quantum-classical correspondence (128) from brief to more extended intervals is discussed in section 5.5.

Representations of the evolution of the state describing function by a single classical trajectory per body (128) deteriorate with increased propagation intervals. Considering the classical correspondence for each classical body, nearby initial conditions produce trajectories that diverge with time. With sufficient divergence of corresponding trajectories, association of the support of a state describing function with a classical body, satisfaction of C.2, is lost over time. Another perspective on the limited duration of the quantum-classical correspondence (128) is suggested by the Riemann-Lebesgue lemma [48]. The likelihood

$$|\langle U(-\lambda)\varphi_n(0)|e^{i\phi_I(\lambda)}\varphi_n(\lambda)\rangle|^2$$

will asymptotically converge to zero for $\lambda \to \infty$ unless the phases $\exp(i\omega_k\lambda)$ from the Hamiltonian (102) are compensated by the state describing function $\varphi_n(\lambda)$. The phase $\phi_I(\lambda)$ does not suffice to compensate $\exp(i\omega_k\lambda)$ since $\phi_I(\lambda)$ does not vary with momenta $(\mathbf{p})_n$. An uncompensated phase proportional to λ leads to an asymptotically vanishing scalar product.

Persistent classical correspondences such as observations of planetary motions over great periods typically include recurring localizing observations. The relevant descriptions of state undergo recurring localization from interactions that are effectively observations. Localizing observations are the result of scatter and emission of radiation, and perturbations of the motion of additional bodies. Given approximations for the trajectories, distorted functions $\varphi_n(\lambda)$ centered on those trajectories follow (130). Inclusion of these effects on observed classical correspondences is an additional insight requiring further development.

5.4 Two body correspondence

This section is a substantial digression to evaluate the quantum-classical correspondence (128) for two-argument state describing functions. For two-argument state describing functions, the non-relativistic correspondence is with two classical bodies. The two classical bodies are described by a freely evolving center-of-momentum and one independent trajectory $\mathbf{u}_1(\lambda)$. Non-relativistic, brief interval, limited acceleration approximations for the scalar products (128) and first derivatives are evaluated. For two corresponding classical bodies, classical trajectories are available, for example, appendix 8.13. This example uses the VEV for a single neutral scalar field and appropriate state describing functions (127) from $\mathbf{H}_{\mathcal{P}}$.

A quantum-classical correspondences satisfies C.1-3 in section 5.3. For two-argument state describing functions, satisfaction of the quantum-classical correspondence (128) maximizes the

likelihood

$$|I(\lambda)|^2 = \frac{|\langle e^{iH\lambda}\varphi_2(0)|\varphi_2(\lambda)\rangle|^2}{\langle\varphi_2(0)|\varphi_2(0)\rangle\langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle} \tag{141}$$

and satisfies the energy relation (140). Following the development of the state describing functions with classical correspondences (126) in section 5.2, appropriate two-argument state describing functions

$$\varphi_2(x_1, x_2; \lambda)$$

have Fourier transforms

$$\tilde{\varphi}_2(p_1, p_2; \lambda) = \prod_{j=1}^{2} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0} + \omega_j)\ \tilde{f}_2(\mathbf{p}_1 - \mathbf{w}_1(\lambda), \mathbf{p}_2 - \mathbf{w}_2(\lambda); \lambda). \tag{142}$$

Poincaré invariance is exploited to express scalar products in the center-of-momentum coordinate frame determined by the corresponding classical locations $\mathbf{u}_j(\lambda)$ and momenta $\mathbf{w}_j(\lambda)$. In the resulting center-of-momentum coordinate frame with the center-of-mass collocated with the origin,

$$\mathbf{u}_2(\lambda) = -\mathbf{u}_1(\lambda)$$

and $\mathbf{w}_2(\lambda) = -\mathbf{w}_1(\lambda)$. For non-relativistic (125) momenta, $\lambda_c\mathbf{w}_j(\lambda) \approx \dot{\mathbf{u}}_j(\lambda)$. In this reference frame, an abbreviated notation,

$$\mathbf{u}(\lambda) = \mathbf{u}_1(\lambda), \tag{143}$$

is substituted. Similarly, $\mathbf{w}(\lambda) = \mathbf{w}_1(\lambda)$.

5.4.1 Non-relativistic approximation

In this section, example state describing and VEV functions are selected, and non-relativistic approximations for the scalar products of the state describing functions $U(-\lambda)\varphi_2(0)$ and $\varphi_2(\lambda)$ are developed as quadratures. These quadratures evaluate the scalar products and their derivatives. In the non-relativistic approximation and for the selected state describing functions, it is demonstrated that the description of the center-of-momentum motion factors from the relative motion of the two corresponding classical particles in the scalar products: length contraction and time dilation are negligible.

The form for $\tilde{f}_2$ in (142) is selected to remove consideration of the motion of the center-of-momentum from evaluation of the likelihoods $|I(\lambda)|^2$. With the selected form and in the evaluation of scalar products of interest, description of the motion of the center-of-momentum is factored from description of the relative motion of the two bodies. The selected $\tilde{f}_2$ separates in Jacobi coordinates.

$$\tilde{f}_2(\mathbf{p}_1 - \mathbf{w}(\lambda), \mathbf{p}_2 + \mathbf{w}(\lambda); \lambda) = \tilde{f}_M(\mathbf{p}_1 + \mathbf{p}_2; \lambda)\tilde{f}_I(\mathbf{p}_1 - \mathbf{p}_2 - 2\mathbf{w}(\lambda); \lambda). \tag{144}$$

f_M describes the center-of-momentum, and f_I describes the relative motion of two corresponding classical bodies designated 1 and 2. Reliable identification of classical bodies with arguments requires isolation (121). (144) is expressed in the center-of-momentum reference frame with the abbreviated notation (143). The support of $\tilde{f}_M(\mathbf{p};\lambda)$ is centered on the origins in both the spatial and momentum domains, and the support of $\tilde{f}_I(\mathbf{p};\lambda)$ is centered on $\mathbf{u}(\lambda)$ in space and on the origin in momentum. Jacobi coordinates are

$$\mathbf{p}'_1 = \mathbf{p}_1 + \mathbf{p}_2 \qquad \text{and} \qquad \mathbf{p}'_2 = \mathbf{p}_1 - \mathbf{p}_2 \tag{145}$$

and then

$$\mathbf{p}_1 = \frac{\mathbf{p}'_1 + \mathbf{p}'_2}{2} \quad \text{and} \qquad \mathbf{p}_2 = \frac{\mathbf{p}'_1 - \mathbf{p}'_2}{2}.$$

With similar substitutions for $\mathbf{p}_3, \mathbf{p}_4$, the determinant of the Jacobian matrix for the coordinate transformation $(\mathbf{p})_4 \mapsto (\mathbf{p}')_4$ is $\left(\frac{1}{4}\right)^3$ for the four variables in three-dimensional space.

Conservation of momentum separates in the Jacobi coordinates.

$$\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = \delta(\mathbf{p}'_1 + \mathbf{p}'_3)$$

does not include the $\mathbf{p}'_2, \mathbf{p}'_4$ dependence that describes the internal motion. Factors of $\omega_j + p_{j0}$ after evaluation of mass shell delta functions in the state describing functions become approximately constant in non-relativistic approximation, $\omega_j \approx \lambda_c^{-1}$ if $\lambda_c^2 \mathbf{p}_j^2 \ll 1$. The remaining separability considerations are the factorability of the momentum dependence in the unitary time translation $U(-\lambda)$ and the energy conservation delta function.

A separation of variables in the time translation (103) follows from non-relativistic approximation (125) for the Hamiltonian (102). Factors of $\omega_j + p_{j0}$ commute with the Hamiltonian (102) and then the time evolution of the state describing function is

$$U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) = e^{i(\omega_1+\omega_2)\lambda}\ \tilde{f}_M(\mathbf{p}_1+\mathbf{p}_2;0)\tilde{f}_I(\mathbf{p}_1-\mathbf{p}_2-2\mathbf{w}(0);0).$$

The Hamiltonian in the two-argument subspace is $\omega_1 + \omega_2$. Taylor theorem polynomial approximation results in

$$\begin{aligned} \omega_1 + \omega_2 &= \omega(\tfrac{\mathbf{p}'_1+\mathbf{p}'_2}{2}) + \omega(\tfrac{\mathbf{p}'_1-\mathbf{p}'_2}{2}) \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}'_2) + \frac{\mathbf{p}_1'^2}{4\omega(\frac{1}{2}\mathbf{p}'_2)} \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}'_2) + \tfrac{1}{4}\lambda_c\mathbf{p}_1'^2 \end{aligned} \tag{146}$$

from (10) and with the non-relativistic approximation (278) from appendix 8.14. $\lambda_c^2 {\mathbf{p}'_1}^2 \ll 1$. Then, for the state describing function (142) in non-relativistic instances, time translation factors in Jacobi coordinates,

$$U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) \approx \left(e^{i\frac{1}{4}\lambda_c\mathbf{p}_1'^2\lambda}\tilde{f}_M(\mathbf{p}'_1;0)\right)\left(e^{2i\omega(\frac{\mathbf{p}'_2}{2})\lambda}\tilde{f}_I(\mathbf{p}'_2-2\mathbf{w}(0);0)\right). \tag{147}$$

Non-relativistic approximation of the Hamiltonian limits the duration of the interval λ that a non-relativistic approximation of time translation applies. For negligible error,

$$\left(\omega_1 + \omega_2 - 2\omega(\frac{1}{2}\mathbf{p}'_2) - \frac{1}{4}\lambda_c \mathbf{p}'^2_1\right)\lambda \ll \pi.$$

Indeed, the non-relativistic approximations to ω_j are polynomials in $\mathbf{p}_j$ and are qualitatively distinct from the Hamiltonian. $\sqrt{\lambda_c^{-2} - \Delta}$ is an anti-local operator [53] while powers of the Laplacian Δ are local.

Separation of variables in the energy conservation delta function also follows from non-relativistic approximation, (279) in appendix 8.14.

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 \approx 2\omega(\frac{1}{2}\mathbf{p}'_2) - 2\omega(\frac{1}{2}\mathbf{p}'_4).$$

Conservation of momentum provides that $\mathbf{p}'_1 = \mathbf{p}'_3$ and then only the momenta in the description of the internal motion appear in the energy conservation delta function.

With time evolution applied to define the deformed center-of-mass state describing function,

$$\tilde{f}_M(\mathbf{p};\lambda) = e^{i\frac{1}{4}\lambda_c \mathbf{p}^2 \lambda} \tilde{f}_M(\mathbf{p};0), \tag{148}$$

non-relativistic approximation of the Hamiltonians (147) and (148) provide that the state describing functions include a common factor.

$$\begin{aligned} \tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;\lambda) &\approx \tilde{f}_M(\mathbf{p}'_1;\lambda)\, \tilde{f}_I(\mathbf{p}'_2 - 2\mathbf{w}(\lambda);\lambda) \\ U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) &\approx \tilde{f}_M(\mathbf{p}'_1;\lambda)\, e^{2i\omega(\frac{\mathbf{p}'_2}{2})\lambda} \tilde{f}_I(\mathbf{p}'_2 - 2\mathbf{w}(0);0) \end{aligned} \tag{149}$$

provide the non-relativistic descriptions of the two state describing functions in the likelihood (141).

Cluster expansion (61) with (65) for the VEV of a single neutral scalar field provides that the VEV in the scalar product for the two-argument function subspace are

$$\tilde{\mathcal{W}}_{2,2}((p)_4) = \tilde{\mathcal{U}}_{2,2}(((p)_4) + \tilde{W}_2(p_1,p_3)\tilde{W}_2(p_2,p_4) + \tilde{W}_2(p_1,p_4)\tilde{W}_2(p_2,p_3) \tag{150}$$

with connected functions

$$\begin{aligned} \tilde{W}_2((p)_2) &= 2\sqrt{\omega_1\omega_2}\, \delta(\mathbf{p}_1 + \mathbf{p}_2) \prod_{j=1}^{2} \delta(p_j^2 - \lambda_c^{-2}) \\ \tilde{\mathcal{U}}_{2,2}((p)_4) &= c_4\, \delta(p_1 + p_2 + p_3 + p_4) \prod_{j=1}^{4} \delta(p_j^2 - \lambda_c^{-2}) \end{aligned}$$

from (35) and (53).

The frequency domain representation (114) of the scalar product with state describing functions (142) and the VEV (150) is

$$\begin{aligned}\langle\varphi_2(\lambda')|\varphi_2(\lambda)\rangle &\approx \int d(\mathbf{p})_4\ \overline{\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda'))_2;\lambda')}\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_{3,4};\lambda)\\ &\quad\times\left\{\frac{4}{\lambda_c^2}\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)+c_4\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\right\}\\ &\approx \left(\tfrac{1}{4}\right)^3\int d(\mathbf{p}')_4\ \overline{\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda'))_2;\lambda')}\ \tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_{3,4};\lambda)\\ &\quad\times\left\{\frac{32}{\lambda_c^2}\delta(\mathbf{p}_2'-\mathbf{p}_4')+c_4\delta(2\omega(\tfrac{\mathbf{p}_2'}{2})-2\omega(\tfrac{\mathbf{p}_4'}{2}))\right\}\delta(\mathbf{p}_1'-\mathbf{p}_3')\end{aligned} \tag{151}$$

with the change to Jacobi coordinates (145) and in non-relativistic approximation. The

$$\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$$

"cross" term from the free field contribution to the VEV is negligible for state describing functions with a reliable quantum-classical correspondence. State describing functions that have a reliable quantum-classical correspondence satisfy isolation (121). The Pauli-Jordan two-point function has an exponential spatial decline in space-like directions characterized by $\|\mathbf{u}(\lambda)\|/\lambda_c$ and if separations $\|\mathbf{u}(\lambda)\|$ are large compared to λ_c, the contribution of $\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$ is negligible. From $\delta(f(x))=\delta(x-x_o)/|f'(x_o)|$ if $f(x)$ has a single zero at x_o [20], and in three dimensions,

$$\begin{aligned}\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4) &= \delta(\tfrac{1}{2}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'-\mathbf{p}_4'))\delta(\tfrac{1}{2}(\mathbf{p}_1'-\mathbf{p}_2'-\mathbf{p}_3'+\mathbf{p}_4'))\\ &= 2^3\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'-\mathbf{p}_4')\delta(\tfrac{1}{2}((\mathbf{p}_3'+\mathbf{p}_4'-\mathbf{p}_2')-\mathbf{p}_2'-\mathbf{p}_3'+\mathbf{p}_4'))\\ &= 8\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'-\mathbf{p}_4')\delta(\mathbf{p}_2'-\mathbf{p}_4')\\ &= 8\delta(\mathbf{p}_1'-\mathbf{p}_3')\delta(\mathbf{p}_2'-\mathbf{p}_4')\end{aligned}$$

and

$$\begin{aligned}\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3) &= \delta(\tfrac{1}{2}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'+\mathbf{p}_4'))\delta(\tfrac{1}{2}(\mathbf{p}_1'-\mathbf{p}_2'-\mathbf{p}_3'-\mathbf{p}_4'))\\ &= 2^3\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'+\mathbf{p}_4')\delta(\tfrac{1}{2}((\mathbf{p}_3'-\mathbf{p}_4'-\mathbf{p}_2')-\mathbf{p}_2'-\mathbf{p}_3'-\mathbf{p}_4'))\\ &= 8\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_3'+\mathbf{p}_4')\delta(\mathbf{p}_2'+\mathbf{p}_4')\\ &= 8\delta(\mathbf{p}_1'-\mathbf{p}_3')\delta(\mathbf{p}_2'+\mathbf{p}_4').\end{aligned}$$

Three scalar products contribute to $|I(\lambda)|^2$ in (141). Designate the generalized function in the scalar product (151) as

$$\begin{aligned}T_4(\mathbf{p}_2',\mathbf{p}_4') &= \frac{1}{\lambda_c^2}\delta(\mathbf{p}_2'-\mathbf{p}_4')+\frac{c_4}{32}\delta(2\omega(\tfrac{\mathbf{p}_2'}{2})-2\omega(\tfrac{\mathbf{p}_4'}{2}))\\ &= \frac{1}{\lambda_c^2}\delta(\mathbf{p}_2'-\mathbf{p}_4')+\frac{c_4}{8\lambda_c}\delta(\mathbf{p}_2'^2-\mathbf{p}_4'^2)\end{aligned} \tag{152}$$

after the substitution

$$\delta(2\omega(\frac{\mathbf{p}_2'}{2}) - 2\omega(\frac{\mathbf{p}_4'}{2})) = 4\omega(\frac{\mathbf{p}_2'}{2})\delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2)$$

with the non-relativistic approximation $\omega(\frac{\mathbf{p}_2'}{2}) \approx \lambda_c^{-1}$. Then, substitution of the state description (149) and VEV (152) into the scalar product (151) results in

$$\begin{aligned}\langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \;\approx\; & \tfrac{1}{2}\int d\mathbf{p}\; |\tilde{f}_M(\mathbf{p};\lambda)|^2 \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ & \times e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}\; \overline{\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(0);0)}\; \tilde{f}_I(\mathbf{p}_4' - 2\mathbf{w}(\lambda);\lambda)\end{aligned}$$

denoted the mixed scalar product and

$$\begin{aligned}\langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \;\approx\; & \tfrac{1}{2}\int d\mathbf{p}\; |\tilde{f}_M(\mathbf{p};\lambda)|^2 \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ & \times \overline{\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(\lambda);\lambda)}\; \tilde{f}_I(\mathbf{p}_4' - 2\mathbf{w}(\lambda);\lambda)\end{aligned}$$

provides the square of both norms, $\|\varphi_2(0)\|$ and $\|\varphi_2(\lambda)\|$. Due to the unimodular phase in (148), the summation

$$\int d\mathbf{p}\; |\tilde{f}_M(\mathbf{p};\lambda)|^2 = \int d\mathbf{p}\; |\tilde{f}_M(\mathbf{p};0)|^2$$

does not vary with time λ and becomes a common factor of the three scalar products of interest. Designate the common factor in the three scalar products composing the likelihood (141)

$$a_g = \frac{1}{2}\int d\mathbf{p}\; |\tilde{f}_M(\mathbf{p};\lambda)|^2. \tag{153}$$

Common factors do not contribute to the likelihood (141). Also, since likelihood depends only on the magnitude of the mixed scalar product, unimodular factors $e^{i\phi_I(\lambda)}$ do not contribute.

Time translations of a Gaussian $\tilde{f}_I$ remains "in family" with non-relativistic approximation. A Taylor theorem polynomial expansion for $2\omega(\frac{\mathbf{p}_2'}{2})$ is

$$2\omega(\frac{\mathbf{p}_2'}{2}) \approx 2\omega(\mathbf{w}(\lambda)) + \frac{\mathbf{w}(\lambda)\cdot(\mathbf{p}_2' - 2\mathbf{w}(\lambda))}{\omega(\mathbf{w}(\lambda))} + \frac{(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}{4\omega(\mathbf{w}(\lambda))}$$

from (278) in appendix 8.14. Expansion is about the representative momentum $2\mathbf{w}(\lambda)$ in the support over $\mathbf{p}_2'$. The support of the description of the relative motion is non-relativistic if

$$\|\mathbf{p}_2' - 2\mathbf{w}(\lambda)\| \ll \lambda_c^{-1} \leq \omega(\mathbf{w}(\lambda))$$

within the dominant support of $\tilde{f}_I$ and the corresponding classical trajectory is non-relativistic if $\|\mathbf{w}(\lambda)\| \ll \lambda_c^{-1}$. Then, for non-relativistic momenta, the Hamiltonian that describes the

evolution of the relative motion is

$$\begin{aligned} 2\omega(\tfrac{\mathbf{p}_2'}{2}) &\approx 2\lambda_c^{-1} + \lambda_c \mathbf{w}(\lambda)^2 + \lambda_c \mathbf{w}(\lambda)\cdot(\mathbf{p}_2' - 2\mathbf{w}(\lambda)) + \frac{\lambda_c}{4}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2 \\ &= 2\lambda_c^{-1} + \frac{\lambda_c}{4}\mathbf{p}_2'^2 \end{aligned} \tag{154}$$

to second order in small quantities $\lambda_c\|\mathbf{p}_2' - 2\mathbf{w}(\lambda)\| \ll 1$ and $\lambda_c\|\mathbf{w}(\lambda)\| \ll 1$.

With non-relativistic approximation for the Hamiltonian, time translations of a Gaussian relative motion describing function remain Gaussian. The selection of the description of the relative motion of the two bodies,

$$\tilde{f}_I(\mathbf{p};\lambda) = e^{i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}\exp(-L(\lambda)^2\mathbf{p}^2), \tag{155}$$

is a Gaussian. Appropriate selections of $\mathbf{u}(\lambda)$ and $\mathbf{w}(\lambda)$ satisfy isolation (121) and are non-relativistic (125). $L(\lambda)$ is a complex length characterizing the breadth of the spatial support. In (155),

$$\mathbb{R}e(L(\lambda)^2) > 0.$$

Real $L(0)^2$ are the most classical particle-like state describing functions, meeting the Heisenberg uncertainty lower bound for simultaneous knowledge of location and momentum. Complex $L(\lambda)^2$ exceed the lower uncertainty bound with additional spreading in location for a constant spread in momentum. The translated functions

$$\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(\lambda);\lambda) = \exp\left(-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 + i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)\right)$$

have inverse Fourier transforms

$$\left(\frac{1}{2L(\lambda)^2}\right)^{\frac{3}{2}} \exp\left(-\frac{(\mathbf{x}-\mathbf{u}(\lambda))^2}{4L(\lambda)^2} - 2i\mathbf{w}(\lambda)\cdot\mathbf{x}\right).$$

To satisfy a non-relativistic quantum-classical correspondence, the support spread parameter $L(\lambda)^2$ is constrained by both upper and lower bounds. An upper bound

$$\frac{|L(\lambda)^2|^2}{\mathbb{R}e(L(\lambda)^2)} \ll \mathbf{u}(\lambda)^2$$

limits spatial extent with respect to separation to isolate (121) the support of the two corresponding classical particles. A lower bound suppresses support on relativistic momenta (125),

$$\mathbb{R}e(L_0(\lambda)^2) \gg \lambda_c^2. \tag{156}$$

The bounds preclude both plane wave and point-like limits.

From (118), the dominant support of the argument of the relative motion state describing function (155) $\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(0); 0)$ satisfies

$$\|\mathbf{p}_2 - 2\mathbf{w}(0)\| \leq Q_2.$$

If

$$\lambda_c \|\mathbf{p}_2 - 2\mathbf{w}(0)\| \ll 1$$

and if the classical trajectory is non-relativistic, $\lambda_c \|\mathbf{w}(0)\| \ll 1$, then the dominant support is non-relativistic. For a non-relativistically supported function, the dominant momentum support satisfies

$$\frac{1}{Q_2} \gg \lambda_c.$$

For dominant supports, the Heisenberg uncertainty bound provides that

$$Q_2 R_2 > \frac{1}{2}$$

for the radii of the dominant supports R_2. Then,

$$R_2 > \frac{1}{2Q_2} \gg \lambda_c$$

provides (156).

Substitution of the Gaussian relative motion describing function (155) into the scalar product results in

$$\begin{aligned} \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \end{aligned} \tag{157}$$

for the mixed scalar product and

$$\begin{aligned} \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \;\; \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \end{aligned} \tag{158}$$

for the squared norms.

With $\mathbb{R}e(L(\lambda)^2) > 0$ and $\mathbf{u}(\lambda), \mathbf{w}(\lambda) \in \mathbb{R}$, after evaluation of the delta functions, the rapid decline of Gaussian functions ensures convergence of the scalar products (157) and (158). From the dominated convergence theorem, derivatives of the scalar products are summations of the derivatives of state describing functions. With the designations

$$\begin{aligned} S_m &= \langle U(\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle \\ S_0 &= \langle \varphi_n(0)|\varphi_n(0)\rangle \\ S_\lambda &= \langle \varphi_n(\lambda)|\varphi_n(\lambda)\rangle, \end{aligned} \tag{159}$$

the derivatives of the scalar products with respect to the components of $\mathbf{u}(\lambda)$ are

$$\frac{\partial S_m}{\partial \mathbf{u}(\lambda)} \approx i a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; (\mathbf{p}_4' - 2\mathbf{w}(\lambda)) \\ \times e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{160}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \mathbf{u}(\lambda)} \approx -i a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; (\mathbf{p}_2' - \mathbf{p}_4') \\ \times e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{161}$$

for the $\lambda > 0$ squared norms. The $\lambda = 0$ squared norms are independent of $\mathbf{u}(\lambda)$, $\dot{\mathbf{u}}(\lambda)$ and λ.

Similarly, the partial derivatives of the scalar products with respect to the components of $\dot{\mathbf{u}}(\lambda) = \lambda_c \mathbf{w}(\lambda)$ in the non-relativistic approximation (125) are

$$\frac{\partial S_m}{\partial \dot{\mathbf{u}}(\lambda)} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; \left(-2i\lambda_c^{-1}\mathbf{u}(\lambda) + 4\lambda_c^{-1} L(\lambda)^2 (\mathbf{p}_4' - 2\mathbf{w}(\lambda))\right) \\ \times e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} \; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \; e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \tag{162}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \dot{\mathbf{u}}(\lambda)} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; 4\lambda_c^{-1} \left(\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda)) + L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\right) \\ \times e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{163}$$

for the $\lambda > 0$ squared norms.

Finally, the partial derivatives of the scalar products with respect to λ with $\mathbf{u}(\lambda)$ and $\dot{\mathbf{u}}(\lambda)$ held constant are

$$\frac{\partial S_m}{\partial \lambda} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; \left(-2i\omega(\tfrac{\mathbf{p}_2'}{2}) - \frac{dL(\lambda)^2}{d\lambda}(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2\right) \\ \times e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{164}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \lambda} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; \left(-\frac{d\overline{L(\lambda)^2}}{d\lambda}((\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2 - \frac{dL(\lambda)^2}{d\lambda}(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2)\right) \\ \times e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{165}$$

for the $\lambda > 0$ squared norms. In non-relativistic approximation, $\omega(\frac{\mathbf{p}_2'}{2}) \approx \lambda_c^{-1}$.

5.4.2 Brief interval and limited acceleration approximations

If the duration of the interval λ in the correspondence (128) is sufficiently limited that linear approximations of the classical dynamics suffice, then simplified approximations of the scalar products and their first derivatives (157)-(165) follow. This constraint is in addition to the limitation on λ to apply non-relativistic approximation of the time translation. In this section, non-relativistic, brief interval λ, limited acceleration approximate relationships between the state describing functions $U(\lambda)\varphi_2(0)$ and $\varphi_2(\lambda)$ are developed. With these approximations, the phase $\phi_I(\lambda)$ from the quantum-classical correspondence condition (128) and the support spread parameter $L(\lambda)^2$ in the description of the relative motion (155) are determined.

If the quantum-classical correspondence is examined for sufficiently brief intervals $\lambda > 0$, then Taylor theorem polynomial approximation to first order in λ suffices.

$$\begin{aligned} \mathbf{u}(\lambda) &\approx \mathbf{u}(0) + \lambda\dot{\mathbf{u}}(0) \\ \mathbf{w}(\lambda) &\approx \mathbf{w}(0) + \lambda\dot{\mathbf{w}}(0) \end{aligned} \tag{166}$$

and for non-relativistic momenta, $\lambda_c \mathbf{w}(\lambda) \approx \dot{\mathbf{u}}(\lambda)$. The two Gaussian functions (155) that provide the classical and quantum descriptions (128) of the relative motion are simply related to $\mathcal{O}(\lambda^2)$ for non-relativistic momenta (125). The classical description of the evolution of the support $e^{i\phi_I(\lambda)}\varphi_2(\lambda)$ equals

$$\begin{aligned} e^{i\phi_I(\lambda)}\tilde{f}_I(\mathbf{p}-2\mathbf{w}(\lambda);\lambda) &= e^{i\phi_I(\lambda)} e^{i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2} \\ &\approx e^{2i\omega(\frac{\mathbf{p}}{2})\lambda}\ e^{i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2}\ e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} \\ &= e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}\ e^{2i\omega(\frac{\mathbf{p}}{2})\lambda}\ \tilde{f}_I(\mathbf{p}-2\mathbf{w}(0);0) \end{aligned} \tag{167}$$

an envelope evolution correction factor times the quantum mechanical description of the internal motion $U(-\lambda)\varphi_2(0)$. The relationship (167) includes the envelope evolution correction factor

$$e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}.$$

The demonstration of (167) follows from the linear approximation (166) of the classical dynamics and non-relativistic approximation (154) of the Hamiltonian. Linear expansion (166) and neglecting $\mathcal{O}(\lambda^2)$ contributions,

$$\begin{aligned} (\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda) &= (\mathbf{p}-2\mathbf{w}(0)-2\lambda\dot{\mathbf{w}}(0))\cdot(\mathbf{u}(0)+\lambda\dot{\mathbf{u}}(0)) \\ &\approx (\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0) + \lambda(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{u}}(0) - 2\lambda\dot{\mathbf{w}}(0)\cdot\mathbf{u}(0). \end{aligned}$$

The linear expansion (166) and setting

$$L(\lambda)^2 = L(0)^2 - i\frac{\lambda_c}{4}\lambda \tag{168}$$

results in an expansion for the Gaussian function in the first line of (167). Neglecting $\mathcal{O}(\lambda^2)$,

$$\begin{aligned} L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 &= (L(0)^2 - i\frac{\lambda_c}{4}\lambda)\,(\mathbf{p}-2\mathbf{w}(0)-2\lambda\dot{\mathbf{w}}(0))^2 \\ &\approx L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 - 4\lambda L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0) - i\frac{\lambda_c}{4}\lambda(\mathbf{p}-2\mathbf{w}(0))^2. \end{aligned}$$

This $L(\lambda)^2$ describes the support spread from non-relativistic free propagation of a Gaussian function [25] in Jacobi coordinates (145).

The Taylor polynomial approximation (154) of the Hamiltonian with the momentum $\mathbf{p}$ centered on $2\mathbf{w}(\lambda)$ is

$$2\omega(\frac{\mathbf{p}}{2}) \approx 2\lambda_c^{-1} + \lambda_c\mathbf{w}(\lambda)^2 + \lambda_c\mathbf{w}(\lambda)\cdot(\mathbf{p}-2\mathbf{w}(\lambda)) + \frac{\lambda_c}{4}(\mathbf{p}-2\mathbf{w}(\lambda))^2.$$

Substitution of the expansions into the phase in the first line of (167) with subtraction of the phase contributed by time translation results in

$$\begin{aligned} &i\phi_I(\lambda) + i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda) - L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 - 2i\omega(\tfrac{\mathbf{p}}{2})\lambda \\ &\quad\approx i\phi_I(\lambda) + i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0) + i\lambda(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{u}}(0) - 2i\lambda\dot{\mathbf{w}}(0)\cdot\mathbf{u}(0)) \\ &\qquad -L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 + 4\lambda L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0) + i\frac{\lambda_c}{4}\lambda(\mathbf{p}-2\mathbf{w}(0))^2 \\ &\qquad -i\lambda\left(2\lambda_c^{-1} + \lambda_c\mathbf{w}(\lambda)^2 + \lambda_c\mathbf{w}(\lambda)\cdot(\mathbf{p}-2\mathbf{w}(\lambda)) + \frac{\lambda_c}{4}(\mathbf{p}-2\mathbf{w}(\lambda))^2\right). \end{aligned}$$

Substitution of $\lambda\mathbf{w}(\lambda) = \lambda\mathbf{w}(0)$ to $\mathcal{O}(\lambda^2)$ and the non-relativistic approximation $\lambda_c\mathbf{w}(\lambda) = \dot{\mathbf{u}}(\lambda)$, and collecting equal terms results in

$$\begin{aligned} &\approx i\phi_I(\lambda) + i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0) - 2i\lambda\lambda_c^{-1}\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0)) \\ &\quad -L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 + 4\lambda L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0) \\ &\quad -i\lambda\left(2\lambda_c^{-1} + \lambda_c^{-1}\dot{\mathbf{u}}(\lambda)^2\right). \end{aligned}$$

Identification of the phase $\phi_I(\lambda)$,

$$\phi_I(\lambda) = (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0))\frac{\lambda}{\lambda_c}, \tag{169}$$

provides the demonstration of (167).

$$\begin{aligned} &i\phi_I(\lambda) + i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda) - L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 - 2i\omega(\tfrac{\mathbf{p}}{2})\lambda \\ &\quad\approx i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0) - L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 + 4\lambda L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0). \end{aligned}$$

Identification of the support spread parameter (168) and phase (169) demonstrate the quantum-classical correspondence (128) over brief intervals λ.

The phase (169) corresponds to the classical energy of two non-relativistic particles.

$$2mc^2 + \text{K.E.} + \text{V} = mc^2\ (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0)).$$

$\lambda_c mc^2 = \hbar c$. The rest mass energy is 2, the kinetic energy of the two particles is $\dot{\mathbf{u}}(0)^2$, and the pair potential V is identified in more familiar form in section 5.4.5 below. If the trajectories satisfy Newton's equation of motion, this energy is independent of λ and $2\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0))$ is a $-g/r$ pair potential.

Neglecting terms of $\mathcal{O}(\lambda^2)$ in the brief interval expansion, the envelope evolution correction factor does not vary with time.

$$e^{-4\lambda L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)} = e^{-4\lambda L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\dot{\mathbf{w}}(\lambda)}. \tag{170}$$

An acceleration limitation is convenient to analyze the quantum-classical correspondence (128) by justifying neglect of the envelope evolution correction factor. The envelope evolution correction in the approximation (167) is negligible either with an acceleration limit, or after application of representative momentum (120). From the Cauchy-Schwarz-Bunyakovsky inequality,

$$|(\mathbf{p}-2\mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)| \leq \|\mathbf{p}-2\mathbf{w}(0)\|\ \|\dot{\mathbf{w}}(0)\|.$$

If the initial acceleration in the brief interval approximation is limited,

$$\|\dot{\mathbf{w}}(0)\| \leq \frac{\epsilon}{\mathbf{u}(0)^2}, \tag{171}$$

then

$$|(\mathbf{p}-2\mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)| \leq \|\mathbf{p}-2\mathbf{w}(0)\|\frac{\epsilon}{\mathbf{u}(0)^2}$$

and the envelope evolution correction factor $e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}$ is negligible compared to the envelope,

$$4\mathbb{R}e(L(0)^2)\|\mathbf{p}-2\mathbf{w}(0)\|\frac{\epsilon}{\mathbf{u}(0)^2}\lambda \ll \mathbb{R}e(L(0)^2)(\mathbf{p}-2\mathbf{w}(0))^2,$$

as long as

$$\frac{4\epsilon\lambda}{\mathbf{u}(0)^2} \ll \|\mathbf{p}-2\mathbf{w}(0)\|.$$

The support of the summation evaluating a scalar product includes $\mathbf{p} \approx 2\mathbf{w}(0)$ but the envelope evolution correction is small in the neighborhood of these points. To approximate scalar products, the bound need only apply for $\mathbf{p}$ where the envelope deviates significantly from unity, $\mathbf{p}$ that satisfy

$$\sqrt{\mathbb{R}e(L(0)^2)}\ \|\mathbf{p}-2\mathbf{w}(0)\| \geq \epsilon.$$

An accurate approximation for the scalar product results if $\epsilon \ll 1$. Then, the envelope evolution factor is well-approximated by unity if the interval λ is bounded,

$$\lambda \ll \frac{\mathbf{u}(0)^2}{4\sqrt{\mathbb{R}e(L(0)^2)}}.$$

This upper bound on λ increases with greater initial separations $\|\mathbf{u}(0)\|$. With the acceleration limit (171) and for the sufficiently brief intervals λ, the envelope evolution correction factor (170) contributed by substitutions of (172) are negligible,

$$e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} \approx 1.$$

Finally, the non-relativistic, brief interval, limited acceleration approximation is that the state describing functions $U(\lambda)\varphi_2(0)$ and $\varphi_2(\lambda)$ are related

$$e^{i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}e^{-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2} \approx e^{-i\phi_I(\lambda)}e^{-2i\omega(\frac{\mathbf{p}}{2})\lambda}\ e^{i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2} \tag{172}$$

with $L(\lambda)^2$ from (168) and $\phi_I(\lambda)$ from (169).

The state describing functions $\varphi_2(\lambda)$ in this demonstration are of the form (142) with the product description (144) in Jacobi coordinates and with a Gaussian function description (155) of the relative motion. Approximations for the scalar products and their derivatives follow from the approximate relation (172) of the quantum and classical descriptions for the relative motion, the factoring of time evolution, the independence of the descriptions of relative from center-of-momentum motions, and the selection (148) for the distorted description of the center-of-momentum for $\lambda > 0$ developed in section 5.4.1. Substitution of the brief interval, limited acceleration approximation (172) into the scalar products (157) and (158) result in

$$\begin{aligned}
S_m &= \langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\ T_4(\mathbf{p}_2', \mathbf{p}_4')\ e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}e^{-i(\mathbf{p}_2'-2\mathbf{w}(0))\cdot\mathbf{u}(0)}\ e^{-\overline{L(0)^2}(\mathbf{p}_2'-2\mathbf{w}(0))^2} \\
&\qquad \times e^{i(\mathbf{p}_4'-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}\ e^{-L(\lambda)^2(\mathbf{p}_4'-2\mathbf{w}(\lambda))^2} \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\ T_4(\mathbf{p}_2', \mathbf{p}_4')\ e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}e^{-i(\mathbf{p}_2'-2\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-\overline{L(0)^2}(\mathbf{p}_2'-2\mathbf{w}(0))^2} \\
&\qquad \times e^{-i\phi_I(\lambda)}e^{2i\omega(\frac{\mathbf{p}_4'}{2})\lambda}\ e^{i(\mathbf{p}_4'-2\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{p}_4'-2\mathbf{w}(0))^2}
\end{aligned}$$

for the mixed scalar product and

$$\begin{aligned} S_\lambda &= \langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle \\ &\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{-i(\mathbf{p}_2' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \\ &\quad \times e^{i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \\ &\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{i\phi_I(\lambda)} e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} \; e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \\ &\quad \times e^{-i\phi_I(\lambda)} e^{-2i\omega(\frac{\mathbf{p}_4'}{2})\lambda} \; e^{i(\mathbf{p}_4' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2} \end{aligned}$$

for the squared norms. The Hamiltonian derived factors $e^{-2i(\omega(\frac{\mathbf{p}_2'}{2}) - \omega(\frac{\mathbf{p}_4'}{2}))\lambda} = 1$ due either to conservation of momentum, $\mathbf{p}_2' = \mathbf{p}_4'$, from the free field VEV or conservation of energy,

$$2\omega(\frac{\mathbf{p}_2'}{2}) = 2\omega(\frac{\mathbf{p}_4'}{2})$$

from the connected VEV. The unimodular phase factor $e^{-i\phi_i(\lambda)}$ contributed by substitution of (172) distributes out of the summation and therefore does not contribute to likelihood (141). The approximations for the scalar products neglect terms of $\mathcal{O}(\lambda^2)$, are to second order in the non-relativistic approximation (125), and apply for limited accelerations (171). The approximations are to second order in the small quantities

$$\lambda_c \|\mathbf{p}_2' - 2\mathbf{w}(0)\|, \lambda_c \|\mathbf{p}_4' - 2\mathbf{w}(0)\| \text{ and } \lambda_c \|\mathbf{w}(0)\| \ll 1.$$

Factors

$$e^{-i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{i(\mathbf{p}_4' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} = e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)}$$

from the common time of the brief interval approximation in (172). The simplifications result in

$$\begin{aligned} S_m &\approx a_g e^{-i\phi_I(\lambda)} \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2} \\ S_\lambda &\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{-i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2} \end{aligned} \tag{173}$$

5.4.3 A functional $Q(F(\mathbf{q}_2, \mathbf{q}_4))$

Inspection of the brief interval, non-relativistic approximations (173) for the scalar products (157) and (158) identifies

$$\begin{aligned} S_m &= \langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \\ &\approx e^{-i\phi_I(\lambda)}Q(1) \\ S_\lambda &= \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \\ &\approx Q(1) \end{aligned} \tag{174}$$

with definition of a functional

$$\begin{aligned} Q(F(\mathbf{q}_2, \mathbf{q}_4)) = a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; F(\mathbf{p}_2' - 2\mathbf{w}, \mathbf{p}_4' - 2\mathbf{w}) \\ \times e^{\mathbf{b}_2 \cdot \mathbf{p}_2' + \mathbf{b}_4 \cdot \mathbf{p}_4'} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}. \end{aligned} \tag{175}$$

The generalized function $T_4(\mathbf{p}_2', \mathbf{p}_4')$ is from the VEV (152) and a_g is from (153). σ^2, $\mathbf{b}_2$ and $\mathbf{b}_4$ are complex parameters. These parameters equal

$$\begin{aligned} \sigma^2 &= L(0)^2 \\ \mathbf{b}_2 &= -i\mathbf{u}(0) \\ \mathbf{b}_4 &= i\mathbf{u}(0) \end{aligned} \tag{176}$$

in the approximations for the scalar products (174) and their first derivatives (157)-(165). $\mathbf{u}(0)$ is the initial location of particle one and $L(0)^2$ is the initial complex support spread parameter in the description of relative motion (155). Relations similar to (174) with multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ determined in (160)-(165) follow for the approximations for first derivatives. $Q(1)$ does not vary with λ; the multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ from (160)-(165) include variation with λ.

The functional Q maps multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ of two spatial vectors $\mathbf{q}_2, \mathbf{q}_4 \in \mathbb{R}^3$ to $\mathbb{C}$. $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is a linear functional.

$$Q(\alpha F_1 + \beta F_2) = \alpha Q(F_1) + \beta Q(F_2)$$

for $\alpha, \beta \in \mathbb{C}$ and multinomials $F_1(\mathbf{q}_2, \mathbf{q}_4), F_2(\mathbf{q}_2, \mathbf{q}_4)$. $\mathbf{q}_j$ indicates a factor of $\mathbf{p}_j' - 2\mathbf{w}$ within the summation (175). After evaluation of the momentum and energy conservation generalized functions in T_4 (152), the summation (175) is absolutely convergent. The dominated convergence theorem justifies interchange of summation and differentiation. Then,

$$\begin{aligned} Q(F(\mathbf{q}_2, \mathbf{q}_4)) &= a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ &\quad \times F(\frac{\partial}{\partial \mathbf{b}_2} - 2\mathbf{w}, \frac{\partial}{\partial \mathbf{b}_4} - 2\mathbf{w}) \; e^{\mathbf{b}_2 \cdot \mathbf{p}_2' + \mathbf{b}_4 \cdot \mathbf{p}_4'} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2} \\ &= F(\frac{\partial}{\partial \mathbf{b}_2} - 2\mathbf{w}, \frac{\partial}{\partial \mathbf{b}_4} - 2\mathbf{w}) \, Q(1). \end{aligned} \tag{177}$$

$Q(1)$ is approximated by elementary functions in appendix 8.15.

If $\mathbf{b}_2 = \overline{\mathbf{b}_4}$, then $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is transpose conjugate symmetric. Transpose symmetry of the real $T_4(\mathbf{p}'_2, \mathbf{p}'_4)$ (152) and relabeling of the summation variables $\mathbf{p}'_2 \leftrightarrow \mathbf{p}'_4$ demonstrates that

$$Q(F(\mathbf{q}_2, \mathbf{q}_4)) = \overline{Q(\,\overline{F(\mathbf{q}_4, \mathbf{q}_2)}\,)}. \tag{178}$$

From (178), $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is real for real, transpose symmetric multinomials,

$$F(\mathbf{q}_2, \mathbf{q}_4) = F(\mathbf{q}_4, \mathbf{q}_2) = \overline{F(\mathbf{q}_2, \mathbf{q}_4)}, \tag{179}$$

and $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is imaginary for real, transpose antisymmetric multinomials,

$$F(\mathbf{q}_2, \mathbf{q}_4) = -F(\mathbf{q}_4, \mathbf{q}_2) = \overline{F(\mathbf{q}_2, \mathbf{q}_4)}.$$

5.4.4 Zeros of the derivatives of likelihood $|I(\lambda)|^2$

From the development in section 5.3, the corresponding classical trajectory $\mathbf{u}(\lambda)$ results in zeros (137) and (139) of the first derivatives of the likelihood $|I(\lambda)|^2$. The brief interval, non-relativistic, limited acceleration approximations of the first derivatives are evaluated in this section. It follows that the approximations for the derivatives

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = 0$$

without constraints on the classical trajectory $\mathbf{u}(\lambda)$. β is one of three components of $\mathbf{u}(\lambda)$, one of three components of $\dot{\mathbf{u}}(\lambda)$, or λ. The non-relativistic, brief interval, limited acceleration approximations of the derivatives of likelihood $|I(\lambda)|^2$ provide no insight into the corresponding classical trajectories but justifies application of condition (140). The condition (140) relates the quantum and classical descriptions of energy and identifies a corresponding classical pair potential. This energy correspondence and the corresponding classical pair potential are developed in section 5.4.5.

Derivatives of the likelihood $|I(\lambda)|^2$ follow from derivatives of the scalar products for the unnormalized state describing functions. With the designation (133) for the scalar product of normalized state describing functions and the chain rule,

$$\frac{\partial I(\lambda)}{\partial \beta} = \frac{1}{\sqrt{S_0 S_\lambda}} \frac{\partial S_m}{\partial \beta} - \frac{S_m}{2 S_\lambda \sqrt{S_0 S_\lambda}} \frac{\partial S_\lambda}{\partial \beta}$$

with the designations for scalar products

$$\begin{aligned} S_m &= \langle U(-\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle \\ &\approx e^{-i\phi_I(\lambda)}Q(1) \\ S_0 &= \langle \varphi_n(0)|\varphi_n(0)\rangle \\ &\approx e^{-i\phi_I(\lambda)}Q(1) \\ S_\lambda &= \langle \varphi_n(\lambda)|\varphi_n(\lambda)\rangle \\ &\approx e^{-i\phi_I(\lambda)}Q(1) \end{aligned}$$

with the notation (159) and from the approximate relation (174) of scalar products and the functional $Q(1)$. The first derivatives follow as derivatives of $Q(1)$ from (177) after substitution of (172) into (160)-(165). Since $I(\lambda) \neq 0$, the optimality conditions (136) and (139) are satisfied if

$$\begin{aligned} \frac{\partial |I(\lambda)|^2}{\partial \beta} &= 2\mathbb{R}e\left(\overline{I(\lambda)}\,\frac{\partial I(\lambda)}{\partial \beta}\right) \\ &= 2\mathbb{R}e\left(\frac{\overline{S_m}}{S_0 S_\lambda}\,\frac{\partial S_m}{\partial \beta} - \frac{|S_m|^2}{2S_0 S_\lambda^2}\,\frac{\partial S_\lambda}{\partial \beta}\right) \\ &= 0. \end{aligned}$$

$S_0, S_\lambda, S_m \neq 0$ and $S_0, S_\lambda, |S_m|^2 \in \mathbb{R}$. Then

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = 0$$

if and only if

$$\mathbb{R}e\left(\frac{2}{S_m}\,\frac{\partial S_m}{\partial \beta}\right) - \frac{1}{S_\lambda}\mathbb{R}e\left(\frac{\partial S_\lambda}{\partial \beta}\right) = 0. \tag{180}$$

Expansions of the derivatives of the scalar products in real and imaginary components to first order in λ are designated

$$\begin{aligned} \frac{\partial S_m}{\partial \beta} &= e^{-i\phi_I(\lambda)}\left(c_m + id_m + \lambda\alpha_m + i\lambda\epsilon_m\right) \\ \frac{\partial S_\lambda}{\partial \beta} &= c_\lambda + id_\lambda + \lambda\alpha_\lambda + i\lambda\epsilon_\lambda \end{aligned} \tag{181}$$

with $c_m, d_m, \alpha_m, \epsilon_m, c_\lambda, d_\lambda, \alpha_\lambda, \epsilon_\lambda \in \mathbb{R}$. In this notation,

$$\begin{aligned}\mathbb{R}e\left(\frac{2}{S_m}\frac{\partial S_m}{\partial \beta}\right) - \frac{1}{S_\lambda}\mathbb{R}e\left(\frac{\partial S_\lambda}{\partial \beta}\right) &= \mathbb{R}e\left(2\,\frac{c_m + id_m + \lambda\alpha_m + i\lambda\epsilon_m}{Q(1)}\right) \\ &\quad -\mathbb{R}e\left(\frac{c_\lambda + id_\lambda + \lambda\alpha_\lambda + i\lambda\epsilon_\lambda}{Q(1)}\right) \\ &= \frac{2c_m - c_\lambda + \lambda(2\alpha_m - \alpha_\lambda)}{Q(1)}\end{aligned}$$

to $\mathcal{O}(\lambda^2)$. The imaginary terms $d_m, d_\lambda, \epsilon_m, \epsilon_\lambda$ do not contribute. Since $Q(1) \neq 0$, the brief interval optimality conditions are satisfied if

$$2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) = 0 \tag{182}$$

for each selection of β from the three components of $\mathbf{u}(\lambda)$, the three components of $\dot{\mathbf{u}}(\lambda)$ and λ.

To further abbreviate notation here and in section 5.4.5, the designations (143) for the initial conditions of the classical trajectories are designated

$$\mathbf{u} = \mathbf{u}(0), \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0), \quad \mathbf{w} = \mathbf{w}(0), \quad \text{and} \quad \dot{\mathbf{w}} = \dot{\mathbf{w}}(0). \tag{183}$$

Like the approximations to scalar products (174), substitution of (172) provides that the first derivatives of the scalar products are conveniently approximated with $Q(F)$. With the abbreviated notations (173), (183) and (175), the first derivatives of S_m with respect to components of $\mathbf{u}(\lambda)$ are from (160),

$$\begin{aligned}e^{i\phi_I(\lambda)}\frac{\partial S_m}{\partial \mathbf{u}(\lambda)} &= Q(i(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})) \\ &= iQ(\mathbf{q}_4) - 2i\lambda\dot{\mathbf{w}}Q(1).\end{aligned}$$

There is an offset by $2\mathbf{w}$ of the summation variables $\mathbf{p}'_j$ from $\mathbf{q}_j$ in the definition (175) of $Q(F)$, and $\mathbf{w}(\lambda)$ is approximated as $\mathbf{w} + \lambda\dot{\mathbf{w}}$. In the designations (181), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned}c_m &= \mathbb{I}m(Q(\mathbf{q}_4)) \\ \alpha_m &= 0.\end{aligned}$$

Derivatives of the squared norm S_λ follow similarly from (161),

$$\begin{aligned}\frac{\partial S_\lambda}{\partial \mathbf{u}(\lambda)} &= Q(-i(\mathbf{q}_2 - \mathbf{q}_4)) \\ &= -iQ(\mathbf{q}_2 - \mathbf{q}_4).\end{aligned}$$

From transpose symmetry (178) of $Q(F)$,

$$-iQ(\mathbf{q}_2 - \mathbf{q}_4) = 2\mathbb{I}m(Q(\mathbf{q}_4))$$

In the designations of (181),

$$\begin{aligned} c_\lambda &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \\ \alpha_\lambda &= 0. \end{aligned}$$

Substitution into (182) with β a component of $\mathbf{u}(\lambda)$ produces

$$\begin{aligned} 0 &= 2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) \\ &= 2\mathbb{I}m(Q(\mathbf{q}_4)) - 2\mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

providing that

$$\frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}(\lambda)} = 0 \tag{184}$$

without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

The derivatives of the mixed scalar product S_m with respect to the components of $\dot{\mathbf{u}}(\lambda)$ are from (162) in section 5.4.1.

$$\begin{aligned} \lambda_c e^{i\phi_I(\lambda)} \frac{\partial S_m}{\partial \dot{\mathbf{u}}(\lambda)} &= Q(\big(-2i(\mathbf{u} + \lambda\dot{\mathbf{u}}) + (4L(0)^2 - i\lambda_c\lambda)(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})\big)) \\ &= Q(\big(-2i\mathbf{u} + 4L(0)^2\mathbf{q}_4 + \lambda\left\{-2i\dot{\mathbf{u}} - i\lambda_c\mathbf{q}_4 - 8L(0)^2\dot{\mathbf{w}}\right\}\big)) \\ &= 2i\mathbf{u}Q(1) + 4L(0)^2Q(\mathbf{q}_4) + \lambda\left\{(2i\dot{\mathbf{u}} - 8L(0)^2\dot{\mathbf{w}})Q(1) - i\lambda_c Q(\mathbf{q}_4)\right\} \end{aligned}$$

from substitution of (168) for $L(\lambda)^2$, and linear expansion (166) of $\mathbf{u}(\lambda)$ and $\mathbf{w}(\lambda)$. Then, in the notation (181), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned} c_m &= 4\mathbb{R}e(L(0)^2Q(\mathbf{q}_4)) \\ \alpha_m &= -8\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + \lambda_c\mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

with the common scaling by λ_c^{-1} neglected. The derivatives of the squared norm S_λ are from (163),

$$\begin{aligned} \lambda_c \frac{\partial S_\lambda}{\partial \dot{\mathbf{u}}(\lambda)} &= Q\left((4L(0)^2 + i\lambda_c\lambda)(\mathbf{q}_2 - 2\lambda\dot{\mathbf{w}}) + (4L(0)^2 - i\lambda_c\lambda)(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})\right) \\ &= Q(4L(0)^2\,(\mathbf{q}_2 + \mathbf{q}_4 - 4\lambda\dot{\mathbf{w}}) + i\lambda_c\lambda\,(\mathbf{q}_2 - \mathbf{q}_4)) \\ &= 4L(0)^2Q(\mathbf{q}_2 + \mathbf{q}_4) + \lambda\left\{-16L(0)^2\dot{\mathbf{w}}Q(1) + i\lambda_c Q(\mathbf{q}_2 - \mathbf{q}_4)\right\} \end{aligned}$$

with substitution of (168) for $L(\lambda)^2$. $Q(\mathbf{q}_2+\mathbf{q}_4)$ is real from (179) and

$$\begin{aligned} Q(\mathbf{q}_2+\mathbf{q}_4) &= 2\mathbb{R}e(Q(\mathbf{q}_4)) \\ iQ(\mathbf{q}_2-\mathbf{q}_4) &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

from (178). In the designations of (181),

$$\begin{aligned} c_\lambda &= 8\mathbb{R}e(L(0)^2Q(\mathbf{q}_4)) \\ \alpha_\lambda &= -16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + 2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

with the common scaling by λ_c^{-1} neglected.

Substitution into (182) with β a component of $\dot{\mathbf{u}}(\lambda)$ produces

$$\begin{aligned} 0 &= 2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) \\ &= 8\mathbb{R}e(L(0)^2Q(\mathbf{q}_4)) - 8\mathbb{R}e(L(0)^2Q(\mathbf{q}_4)) \\ &\quad +\lambda(-16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + 2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4)) + 16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) - 2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4))) \end{aligned}$$

providing that

$$\frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}(\lambda)} = 0 \tag{185}$$

also without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

Finally, the partial derivative of the mixed scalar product S_m with respect to λ with $\mathbf{u}(\lambda)$ and $\dot{\mathbf{u}}(\lambda)$ held constant is from (164),

$$\begin{aligned} e^{i\phi_I(\lambda)}\frac{\partial S_m}{\partial \lambda} &= Q((-2i\lambda_c^{-1} - i\tfrac{\lambda_c}{4}(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})^2)\,) \\ &= Q((-2i\lambda_c^{-1} - i\tfrac{\lambda_c}{4}\mathbf{q}_4^2 + i\lambda\lambda_c(\mathbf{q}_4\cdot\dot{\mathbf{w}}))) \\ &= -2i\lambda_c^{-1}Q(1) - i\tfrac{\lambda_c}{4}Q(\mathbf{q}_4^2) + i\lambda\,\{\lambda_c Q(\mathbf{q}_4)\cdot\dot{\mathbf{w}}\} \end{aligned}$$

from non-relativistic approximation, substitution of the expression (168) for $L(\lambda)^2$, and linear expansion (166) of $\mathbf{w}(\lambda)$. The substitution (172) that introduces the factor $e^{-i\phi_I(\lambda)}$ is subsequent to the differentiation by λ. Then, in the notation (181), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned} c_m &= \tfrac{\lambda_c^2}{4}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ \alpha_m &= -\lambda_c\mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}}. \end{aligned}$$

The squared norm is from (165),

$$\begin{aligned}\frac{\partial S_\lambda}{\partial \lambda} &= Q(i\tfrac{\lambda_c}{4}\left((\mathbf{q}_2 - 2\lambda\dot{\mathbf{w}})^2 - (\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})^2\right)) \\ &= i\tfrac{\lambda_c}{4}Q(\mathbf{q}_2^2 - \mathbf{q}_4^2) + \lambda\left\{-i\lambda_c Q((\mathbf{q}_2 - \mathbf{q}_4)\cdot\dot{\mathbf{w}})\right\}.\end{aligned}$$

In the designations of (181),

$$\begin{aligned}c_\lambda &= \tfrac{\lambda_c^2}{2}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ \alpha_\lambda &= -2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}}.\end{aligned}$$

$Q(\mathbf{q}_2 - \mathbf{q}_4)$ and $Q(\mathbf{q}_2^2 - \mathbf{q}_4^2)$ are imaginary from (179) and

$$\begin{aligned}iQ(\mathbf{q}_2 - \mathbf{q}_4) &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \\ iQ(\mathbf{q}_2^2 - \mathbf{q}_4^2) &= 2\mathbb{I}m(Q(\mathbf{q}_4^2))\end{aligned}$$

from (178).

Substitution into (182) with $\beta = \lambda$ produces

$$\begin{aligned}0 &= 2c_m - c_\lambda + \lambda\left(2\alpha_m - \alpha_\lambda\right) \\ &= 2\tfrac{\lambda_c^2}{4}\mathbb{I}m(Q(\mathbf{q}_4^2)) - \tfrac{\lambda_c^2}{2}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ &\qquad + \lambda(-2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}} + 2\lambda_c\mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}})\end{aligned}$$

providing that

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda} = 0 \tag{186}$$

again without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

A most likely trajectory $\mathbf{u}(\lambda)$ is not resolved by maximization of the likelihood (141) within the non-relativistic (125), brief interval λ, and limited acceleration (171) approximations. The derivatives

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = 0$$

for any $\mathbf{u}(\lambda), \dot{\mathbf{u}}(\lambda)$ and β a component of $\mathbf{u}(\lambda)$, a component of $\dot{\mathbf{u}}(\lambda)$, or λ. In the next section, the optimal trajectory condition (140) that follows if the first derivatives of the likelihood $|I(\lambda)|^2$ vanish provides a correspondence of the quantum and classical descriptions of energy and provides corresponding trajectories.

5.4.5 Energy correspondence and $-g/r$ potentials

In this section, the quantum-classical correspondence (140) is exploited to identify classical trajectories that correspond with the constructed, single neutral scalar field realization of relativistic quantum physics. These classical particle approximations apply when states are described by functions with isolated (121) concentrations of support well-represented by a single location (119) and momentum (120), and the momentum support is non-relativistic (125).

The condition (140) in section 5.3 follows if the likelihood optimizing conditions (136) and (137) are satisfied. Conditions (136) and (137) were demonstrated in section 5.4.4 to $\mathcal{O}(\lambda^2)$ if limited acceleration and non-relativistic approximations apply. Then (140) provides a correspondence of quantum and classical expressions for energy. Evaluated at $\lambda = 0$, (140) provides that

$$\begin{aligned} 0 &= \frac{\partial |I(\lambda)|}{\partial \lambda} \\ &= -i \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\|\varphi_2(0)\|^2} + i\frac{\partial \phi_I(\lambda)}{\partial \lambda} + \frac{\partial}{\partial \lambda}\frac{\langle \varphi_2(0)|\varphi_2(\lambda)\rangle}{\|\varphi_2(0)\|\,\|\varphi_2(\lambda)\|} \end{aligned}$$

for a likelihood maximizing trajectory $\mathbf{u}(\lambda)$.

The third term from (140) is zero. Only one coefficient d_m from (181) in the evaluation of

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda}$$

differs from the evaluation of

$$\frac{\partial}{\partial \lambda}\left(\frac{\langle \varphi_2(0)|\varphi_2(\lambda)\rangle}{\|\varphi_2(0)\|\,\|\varphi_2(\lambda)\|}\right)^2. \tag{187}$$

In the condition (180) equivalent to the vanishing of the partial derivative of likelihood $|I(\lambda)|^2$ with respect to λ,

$$S_m = \langle\, U(-\lambda)\,\varphi_2(0)|\varphi_2(\lambda)\rangle$$

and

$$S_m = \langle \varphi_2(0)|\varphi_2(\lambda)\rangle$$

for evaluation of (187). Unitarity provides that $\|\varphi_2(\lambda)\|$ is common to both evaluations. The only coefficient (181) that distinguishes the two evaluations is a removal of $\langle H\varphi_2(0)|\varphi_2(0)\rangle$ from d_m. The Hermiticity of H provides that $\langle H\varphi_2(0)|\varphi_2(0)\rangle$ is real and the contribution of the removed term is imaginary. The imaginary terms d_m do not contribute and the other coefficients in (181) are unchanged. As a consequence, the calculation (186) of the first derivative of $|I(\lambda)|^2$ provides that

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\langle \varphi_2(0)|\varphi_2(\lambda)\rangle}{\|\varphi_2(0)\|\,\|\varphi_2(\lambda)\|}\right)^2 = 0$$

at $\lambda = 0$.

Finally, the vanishing of the third term in (140) and the expression (169) for $\phi_I(\lambda)$ provide that

$$\begin{aligned} \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\langle\varphi_2(0)|\varphi_2(0)\rangle} &= \frac{d\phi_I(\lambda)}{d\lambda} \\ &= (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0))\frac{1}{\lambda_c}. \end{aligned} \tag{188}$$

The expectation value of the energy H approximates the classical energy (169) of two corresponding classical particles. Equality applies for state describing functions $\varphi_2(0)$ of the form (144) with (148) and (155) within the fidelity of the non-relativistic, brief interval and limited acceleration approximations.

Substitution of the two-argument state descriptions (149) with (148) and (155), and VEV (152) into the scalar product $\langle H\varphi_2(0)|\varphi_2(0)\rangle$ provides

$$\begin{aligned} \langle H\varphi_2(0)|\varphi_2(0)\rangle \approx \tfrac{1}{2}\int d\mathbf{p}_1' \, |\tilde{f}_M(\mathbf{p}_1';\lambda)|^2 \int d\mathbf{p}_2' d\mathbf{p}_4' \, T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times H \; e^{-i(\mathbf{p}_2'-\mathbf{p}_4')\cdot\mathbf{u}} \; \overline{\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w};0)} \; \tilde{f}_I(\mathbf{p}_4' - 2\mathbf{w};0) \end{aligned}$$

from the frequency domain representation of the scalar product (151) and with the abbreviated notation (183) for initial trajectory parameters. The evaluation of $Q(1)$ in appendix 8.15 and of the expectation value of the energy in appendix 8.16 provides

$$\begin{aligned} \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\langle\varphi_2(0)|\varphi_2(0)\rangle} &\approx \frac{2}{\lambda_c} + \left(\frac{3\lambda_c^2}{16\sigma_R^2} + \dot{\mathbf{u}}^2\right)\frac{Q_F(1)}{\lambda_c Q(1)} + \left(\frac{\lambda_c(r_2+r_4)}{8\sigma_R^2}\right)^2 \frac{Q_C(1)}{\lambda_c Q(1)} \\ &\approx (2 + \dot{\mathbf{u}}^2 + 2\ddot{\mathbf{u}}\cdot\mathbf{u})\,\frac{1}{\lambda_c}. \end{aligned} \tag{189}$$

The two-point VEV $Q_F(1)$ and four-point connected VEV $Q_C(1)$ contributions to $Q(1)$ are from (292) in appendix 8.15. The r_j are

$$\frac{\lambda_c(r_2+r_4)}{8\sigma_R^2} = \mathbb{R}e\left(\left(i\frac{\lambda_c\mathbf{u}}{4\sigma_R^2} + \dot{\mathbf{u}}\right)^2\right)^{\frac{1}{2}}$$

from (289) in appendix 8.15, $\sigma_R^2 = \mathbb{R}e(L(0)^2)$, and the energy expectation estimate in (189) applies for a large r_j approximation. The energy consists of the rest mass and kinetic energy of two particles

$$(2 + \dot{\mathbf{u}}^2)\,\frac{1}{\lambda_c},$$

a small zero-point energy contributed by the free field VEV

$$\frac{3\lambda_c^2}{16\sigma_R^2}\frac{Q_F(1)}{\lambda_c Q(1)},$$

and the remainder vanishes if $Q_C(1)$ vanishes

$$\left(\left(\frac{\lambda_c (r_2 + r_4)}{8\sigma_R^2} \right)^2 - \dot{\mathbf{u}}^2 \right) \frac{Q_C(1)}{\lambda_c Q(1)} \approx 2 \, \frac{\ddot{\mathbf{u}} \cdot \mathbf{u}}{\lambda_c}. \tag{190}$$

From (292), $Q_C(1) > 0$, $Q(1) > 0$ and the contribution of the potential to the energy is negative,

$$-\frac{\lambda_c^2 \mathbf{u}^2}{(4\sigma_R^2)^2} \leq \left(\frac{\lambda_c (r_2 + r_4)}{8\sigma_R^2} \right)^2 - \dot{\mathbf{u}}^2 \leq 0$$

and the large r_j approximation constrains $\mathbf{u}, \dot{\mathbf{u}}$ to

$$\mathbb{R}e(\left((i \frac{\lambda_c \mathbf{u}}{4\sigma_R^2} + \dot{\mathbf{u}})^2 \right)^{\frac{1}{2}}) \gg \frac{\lambda_c}{\sigma_R}$$

from (287) in appendix 8.15. The zero-point energy is small for non-relativistic states (156), $\lambda_c^2 \ll \sigma_R^2$. The expected value of the energy of the state described by $\varphi_2(0)$ with an interaction characterized by the single neutral scalar field VEV (150) equals a zero-point energy plus the classical energy of two bodies with a relative motion described by $\mathbf{u}(\lambda)$ and an interaction characterized by the pair potential $2\ddot{\mathbf{u}} \cdot \mathbf{u}$.

This correspondence (189) exhibits several properties in common with Schrödinger's linear harmonic oscillator example, [49] and section 5. The quantum dynamics determines a corresponding state description, σ^2 in (108) for the linear harmonic oscillator and $L(\lambda)^2$ in the constructed example. There is a correspondence for any classical energy. The state describing functions that exhibit a quantum-classical correspondence are among the most classical particle-like and generally do not include energy eigenfunctions. For the linear harmonic oscillator, a mass and spring constant characterize the interaction and σ^2 is determined to satisfy Schrödinger's equation for the classical particle-like Gaussian wave functions. For the constructed realization of relativistic quantum physics, the mass and a pair potential parameter g characterize the interaction. $\mathbb{R}e(L(0)^2)$ is determined by the quantum-classical correspondence for classical particle-like Gaussian state describing functions (155). The potential strength g follows from observation of the corresponding classical trajectories, and $L(0)^2$ is determined from g and the quantum dynamics that includes the coupling constant c_4. $c_4 = 0$ implies that $g = 0$ ($Q_C(1) = 0$) but otherwise, a range of g are consistent with the VEV. That only particular $\mathbb{R}e(L(0)^2)$ exhibit a quantum-classical correspondences but for any energy is a puzzlement, both here and in Schrödinger's study of the linear harmonic oscillator. Another puzzlement is how Schrödinger's equation for non-relativistic quantum physics emerges from the realizations of relativistic quantum physics. At the fidelity of the approximations, Schrödinger's equation has the same quantum-classical correspondences as the constructed realization of relativistic quantum physics.

The final development within this section verifies that the $2\ddot{\mathbf{u}}\cdot\mathbf{u}$ term in the classical energy is a $-g/r$ pair potential for trajectories $\mathbf{u}(\lambda)$ that satisfy Newton's equation of motion. $\mathbf{u}(\lambda)$ is half the two body separation and Newton's equation for the trajectory $\mathbf{u}(\lambda)$ is then

$$mc^2\ddot{\mathbf{u}}(\lambda) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial(2\mathbf{u}(\lambda))}$$

from (276) in appendix 8.13 and in the units of this note. The terms in $\phi_I(\lambda)$ from (169) and Newton's equation of motion results in the identification

$$\begin{aligned} 2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda) &= \frac{V(2\|\mathbf{u}(\lambda)\|)}{mc^2} \\ &= -\frac{1}{mc^2}\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda). \end{aligned} \tag{191}$$

The corresponding potential satisfies

$$V(2\|\mathbf{u}(\lambda)\|) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda).$$

The solution is a $-g/r$ pair potential,

$$2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda) = -\frac{g}{\|2\mathbf{u}(\lambda)\|}. \tag{192}$$

The characteristic length g determines the strength of the potential. The chain rule verifies the identification (192).

$$\begin{aligned} -\left(\nabla_{\mathbf{u}}\frac{1}{\|\mathbf{u}\|}\right)\cdot\mathbf{u} &= -\left(\frac{\partial\|\mathbf{u}\|^{-1}}{\partial\|\mathbf{u}\|}\right)\left(\frac{\partial\|\mathbf{u}\|}{\partial u_x}u_x + \frac{\partial\|\mathbf{u}\|}{\partial u_y}u_y + \frac{\partial\|\mathbf{u}\|}{\partial u_z}u_z\right) \\ &= \frac{1}{\|\mathbf{u}\|^2}\left(\frac{u_x^2}{\|\mathbf{u}\|} + \frac{u_y^2}{\|\mathbf{u}\|} + \frac{u_z^2}{\|\mathbf{u}\|}\right) \\ &= \frac{1}{\|\mathbf{u}\|}. \end{aligned}$$

While maximization of the likelihood provides no insight into the corresponding classical trajectory, the trajectory optimizing condition (140) that follows with the vanishing of the derivatives of $|I(\lambda)|^2$ results in an identification of the corresponding trajectories $\mathbf{u}(\lambda)$ as solutions of Newton's equation for a $-g/r$ pair potential. Then (189) relates quantum and classical parameters.

5.4.6 The Schrödinger equation

In non-relativistic quantum mechanics, the Schrödinger equation [52] describes the evolution of quantum state descriptions. In nonrelativistic instances, the Schrödinger equation and the constructed realizations of relativistic quantum physics share a common classical correspondence.

In the center-of-momentum reference frame and with non-relativistic, brief interval, limited acceleration approximations, the expected value of the Hamiltonian (102) equals the classical energy of two corresponding classical particles. From (189) in section 5.4.5,

$$\frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\langle\varphi_2(0)|\varphi_2(0)\rangle} \approx (2 + \dot{\mathbf{u}}^2 + 2\ddot{\mathbf{u}}\cdot\mathbf{u})\ \frac{1}{\lambda_c}.$$

This relation suggests that the Schrödinger equation describes the quantum mechanical evolution of state. For two-argument state describing functions $\varphi_2(0)$ with (142) and (144), (189) demonstrates that the expectation of the "trivial" Hamiltonian realized in section 3 includes contributions from interaction. Interaction is contributed by the VEV of the field. From (190), an $n \geq 4$, n-point connected contribution in the VEV is necessary to a contribution to the energy from interaction. From sections 5.4.2 and 5.4.5, the interaction is a $-g/r$ potential with a strength g determined by correspondence with classical dynamics. Non-relativistic instances apply in selected reference frames.

A second indication that the Schrödinger equation describes the quantum mechanical evolution of state follows from the classical description of the evolution of a quantum state describing function in (167) from section 5.4.2.

The description (142) of two-argument state describing functions corresponding with two classically described particles is developed in sections 5.4.1 and 5.4.2. The appropriate state describing functions $\varphi_2(\lambda)$ have Fourier transforms

$$\tilde{\varphi}_2(p_1, p_2; \lambda) = \prod_{j=1}^{2}(p_{j0} + \omega_j)\ \tilde{f}_M(\mathbf{p}_1 + \mathbf{p}_2; \lambda) e^{i((\mathbf{p}_1 - \mathbf{p}_2) - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-L(\lambda)^2((\mathbf{p}_1 - \mathbf{p}_2) - 2\mathbf{w}(\lambda))^2}$$

and

$$\tilde{f}_M(\mathbf{p}; \lambda) = e^{i\frac{1}{4}\lambda_c \mathbf{p}^2 \lambda} \tilde{f}_M(\mathbf{p}; 0)$$

from (148). The Gaussian description of the separation $\mathbf{x}_1 - \mathbf{x}_2$ is the most classical-like description: the location and momenta are determined with minimal standard errors consistent with the Heisenberg uncertainty principle in quantum mechanics. The classical trajectory $\mathbf{u}(\lambda)$ with $\lambda_c \mathbf{w}_j(\lambda) \approx \dot{\mathbf{u}}_j(\lambda)$ for non-relativistic (125) momenta provides the classical trajectory that approximates the quantum evolution of $\varphi_2(0)$. The trajectory $\mathbf{u}(\lambda)$ satisfies Newton's equation of motion. If limited acceleration, brief interval, and non-relativistic approximations apply, then (167) follows.

$$\begin{aligned} e^{2i\omega(\frac{\mathbf{p}}{2})\lambda}\ e^{-L(0)^2(\mathbf{p} - 2\mathbf{w}(0))^2}\ e^{4L(0)^2(\mathbf{p} - 2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} \\ \approx e^{i\phi_I(\lambda)} e^{i(\mathbf{p} - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-L(\lambda)^2(\mathbf{p} - 2\mathbf{w}(\lambda))^2} \end{aligned} \tag{193}$$

with $e^{2i\omega(\frac{\mathbf{p}}{2})\lambda}$ the contribution of the factored time translation operator $U(\lambda)$, section 5.4.1. The phase $\phi_I(\lambda)$ is independent of the arguments $\mathbf{p}_1, \mathbf{p}_2$ of $\tilde{\varphi}_2(\lambda)$ and varies with the parameters of $\tilde{\varphi}_2(\lambda)$. The approximation (193) applies if the acceleration is sufficiently small that an envelope evolution correction factor is negligible. $\tilde{f}_M(\mathbf{p};\lambda)$ describes the center-of-momentum, and the Gaussian function describes the separation of two corresponding classical bodies designated 1 and 2. Reliable association of classical bodies with arguments applies for states with spatial and momentum support that is well-represented by one location (119) and one momentum (120), and that is identifiable due to isolation (121). The dominant support of $\tilde{f}_M(\mathbf{p};\lambda)$ is centered on the origins in both the spatial and momentum domains, and the dominant support of the Gaussian is centered on the origin in the momentum domain and on $\mathbf{x}_1 - \mathbf{x}_2 \approx 2\mathbf{u}(\lambda)$ in space. The unimodular factor $e^{i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}$ translates the spatial support of $\varphi_2(\mathbf{x}_1, \mathbf{x}_2; \lambda)$,

If the dominant support of the state describing functions satisfies localization (119) with a representative momentum (120), then substitution of the location and momentum domain arguments of the state describing functions approximate the representatives $\mathbf{u}(\lambda)$ and $\mathbf{w}(\lambda)$ in the phase $\phi_I(\lambda)$. From the non-relativistic, brief interval and limited acceleration approximation (169) for the phase (131) in the instance of two-argument state describing functions,

$$\begin{aligned}\phi_I(\lambda) &= (2 + \lambda_c^2 \mathbf{w}(0)^2 - \frac{g}{\|2\mathbf{u}(0)\|})\frac{\lambda}{\lambda_c} \\ &= (2 + \lambda_c^2 \mathbf{w}(\lambda)^2 - \frac{g}{\|2\mathbf{u}(\lambda)\|})\frac{\lambda}{\lambda_c}\end{aligned}$$

since the total energy is a constant of the motion for Newton's equation. Then substitution of the arguments of the state describing functions (142) in the center-of-momentum frame produces

$$\begin{aligned}\phi_I(\lambda) &= (2 + \lambda_c^2 \mathbf{w}(\lambda)^2 - \frac{g}{\|2\mathbf{u}(\lambda)\|})\frac{\lambda}{\lambda_c} \\ &\approx (2 + \lambda_c^2 \left(\frac{\mathbf{p}_1 + \mathbf{p}_2}{2}\right)^2 + \lambda_c^2 \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2}\right)^2 - \frac{g}{\|\mathbf{x}_1 - \mathbf{x}_2\|})\frac{\lambda}{\lambda_c} \\ &= (2 + \frac{\lambda_c^2 \mathbf{p}_1^2}{2} + \frac{\lambda_c^2 \mathbf{p}_2^2}{2} - \frac{g}{\|\mathbf{x}_1 - \mathbf{x}_2\|})\frac{\lambda}{\lambda_c} \\ &= (2 - \frac{\lambda_c^2}{2}\frac{\partial^2}{\partial \mathbf{x}_1^2} - \frac{\lambda_c^2}{2}\frac{\partial^2}{\partial \mathbf{x}_2^2} - \frac{g}{\|\mathbf{x}_1 - \mathbf{x}_2\|})\frac{\lambda}{\lambda_c}\end{aligned}$$

after simplification and expression in the spacetime domain. A representative momentum (120) applied for the dominant support of f_M provides that

$$\mathbf{p}_1 + \mathbf{p}_2 \approx 0.$$

The final result follows from

$$\mathbf{p}_k = i\frac{\partial}{\partial \mathbf{x}_k}$$

in the spacetime domain and expressed in the units of this note. Then, the time evolution of the state describing function $\varphi_2((\mathbf{x})_2;\lambda)$ in (142) follows from substitution of the time evolution (148) of f_M, substitution of the approximation (193) for the time evolution of the description of the relative position of the two corresponding classical particles, and substitution of the phase $\phi_I(\lambda)$ expressed over $\mathbf{x}_1, \mathbf{x}_2$. From (147) and for the state describing functions $\varphi_2((\mathbf{x})_2;\lambda)$ from (142) and with

$$\tilde{f}_I(\mathbf{p}-2\mathbf{w}(\lambda);\lambda) = e^{i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}e^{-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2},$$

the evolution of the state describing functions is approximated by

$$\begin{aligned} U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) &\approx \left(e^{i\frac{1}{4}\lambda_c\mathbf{p}_1'^2\lambda}\tilde{f}_M(\mathbf{p}_1';0)\right)\left(e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}\tilde{f}_I(\mathbf{p}_2'-2\mathbf{w}(0);0)\right) \\ &\approx \left(\tilde{f}_M(\mathbf{p}_1';\lambda)\right)\left(e^{i\phi_I(\lambda)}\tilde{f}_I(\mathbf{p}_2'-2\mathbf{w}(\lambda);\lambda)\right) \\ &\approx e^{iH_S\lambda}\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;\lambda) \end{aligned}$$

using Jacobi coordinates (145). The Schrödinger Hamiltonian expressed in the spacetime domain is

$$H_S = (2 - \frac{\lambda_c^2}{2}\frac{\partial^2}{\partial\mathbf{x}_1^2} - \frac{\lambda_c^2}{2}\frac{\partial^2}{\partial\mathbf{x}_2^2} - \frac{g}{\|\mathbf{x}_1-\mathbf{x}_2\|})\frac{1}{\lambda_c}.$$

This Hamiltonian applies in the two-argument subspace and corresponds with the classical description of two bodies interacting with a $-g/r$ potential. This Hamiltonian follows from the expression for phase $\phi_I(\lambda)$ and is independent of time.

Finally, it follows that

$$e^{iH_S\lambda}\varphi_2(\mathbf{x}_1,\mathbf{x}_2;\lambda) \approx e^{i\phi_i(\lambda)}\varphi_2(\mathbf{x}_1,\mathbf{x}_2;\lambda) \approx e^{iH\lambda}\varphi_2(\mathbf{x}_1,\mathbf{x}_2;0)$$

with H_S the Schrödinger Hamiltonian and H the non-relativistic approximations (146) and (154) of the two-argument subspace Hamiltonian $\omega_1+\omega_2$ of the constructions. The approximation applies to representatives of Hilbert space norm-equivalence classes. This demonstration is limited to the nonrelativistic, brief interval, limited acceleration instances with state describing functions with the dominant support of a Gaussian description of separation that satisfies localization (119), and the dominant support over both $\mathbf{p}_1, \mathbf{p}_2$ has representative momenta (120).

To $\mathcal{O}(\lambda^2)$ with the nonrelativistic, limited interval and limited acceleration approximations, and for two-argument state describing functions $\varphi_2(\lambda)$ that reliably correspond with two classically described particles, for states $|\underline{g}\rangle \in \mathbf{H}_\mathcal{P}$,

$$|\langle\underline{g}|e^{iH_S\lambda}\varphi_2(\lambda)\rangle|^2 \approx |\langle\underline{g}|\varphi_2(\lambda)\rangle|^2 \approx |\langle\underline{g}|e^{iH\lambda}\varphi_2(0)\rangle|^2.$$

The classical description of state evolution $\varphi_2(\lambda)$ is a good approximation to the quantum mechanical evolved state $e^{iH\lambda}\varphi_2(0)$, and this $\varphi_2(\lambda)$ is an approximate eigenfunction of the Schrödinger Hamiltonian H_S,

$$e^{iH_S\lambda}\varphi_2(\lambda) \approx e^{i\phi_I(\lambda)}\varphi_2(\lambda).$$

Significantly, the trajectory (128) that defines the classical evolution $\varphi(\lambda)$ satisfies Newton's equation for a $-g/r$ potential, section 5.4.5. The Schrödinger Hamiltonian corresponds to the classical dynamics. The Schrödinger equation and the constructed realizations of relativistic quantum physics have a common classical correspondence in nonrelativistic instances.

5.5 Extended interval propagation

In this section, satisfaction of the quantum-classical correspondence (128) over brief intervals is extended to longer intervals.

The quantum-classical correspondence (128)

$$U(-\ell\lambda)\hat{\varphi}_n(\lambda_o;0) \approx e^{i\phi_I(\ell\lambda)}\hat{\varphi}_n(\lambda_o;\ell\lambda)$$

is expressed as a sequence of briefer interval approximations,

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) \approx \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o;\ell\lambda)$$

using the group property of time translation and additivity of the phase $\phi_I(\lambda)$. Here, referring to section 5.3, the notation is augmented to explicitly display the time argument λ_o of the state describing function as well as the temporal parameter for the corresponding classical trajectories.

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) = \hat{\varphi}_n(\lambda_o-\ell\lambda;0)$$

from (80). The error in the quantum-classical correspondence (128) for an interval $\ell\lambda$ follows from the approximation errors for each subinterval within $\ell\lambda$. The error at step ℓ is described by state describing functions ϵ_ℓ.

$$U(-\lambda)\hat{\varphi}_n(\lambda_o-\ell\lambda;\ell\lambda) = e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o-\ell\lambda;(\ell+1)\lambda) + \epsilon_{\ell+1}. \tag{194}$$

The error after accumulation of the ℓ steps of duration λ is described by ϵ.

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) = \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o;n\lambda) + \epsilon.$$

From successive substitution of (194) it follows that

$$\begin{aligned}
U(-\lambda)^{\ell}\hat{\varphi}_n(\lambda_o;0) &= U(-\lambda)^{\ell-1}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o;\lambda)+\epsilon_1\right)\\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-1}\hat{\varphi}_n(\lambda_o;\lambda)+U(-\lambda)^{\ell-1}\epsilon_1\\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-2}\hat{\varphi}_n(\lambda_o-\lambda;\lambda)+U(-\lambda)^{\ell-1}\epsilon_1\\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-3}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o-\lambda;2\lambda)+\epsilon_2\right)+U(-\lambda)^{\ell-1}\epsilon_1\\
&= \left(e^{i\phi_I(\lambda)}\right)^2 U(-\lambda)^{\ell-4}\hat{\varphi}_n(\lambda_o-2\lambda;2\lambda)+e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-3}\epsilon_2+U(-\lambda)^{\ell-1}\epsilon_1\\
&= \ddots\\
&= \left(e^{i\phi_I(\lambda)}\right)^{\ell} U(-\lambda)^{-\ell}\hat{\varphi}_n(\lambda_o-\ell\lambda;\ell\lambda)+\sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1}U(-\lambda)^{\ell+1-2j}\epsilon_j\\
&= \left(e^{i\phi_I(\lambda)}\right)^{\ell}\hat{\varphi}_n(\lambda_o;\ell\lambda)+\sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1}U(-\lambda)^{\ell+1-2j}\epsilon_j.
\end{aligned}$$

Then, the error ϵ after ℓ steps is

$$\epsilon = \sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1}U(-\lambda)^{\ell+1-2j}\epsilon_j$$

from the errors ϵ_j for each subinterval. From the triangle inequality, and unitarity of $U(-\lambda)$ and $e^{i\phi_I(\lambda)}$,

$$\|\epsilon\| \leq \sum_{j=1}^{\ell}\|\epsilon_j\|.$$

The accumulated error grows no faster than the number of subintervals. If the errors $\|\epsilon_j\|$ decline faster than the duration of the subintervals, then the limit of an ever briefer short interval analysis converges and demonstrates a correspondence over an extended interval. For instance, if the brief intervals errors were all $\mathcal{O}(\lambda^2)$, then the error over a fixed interval would converge as $1/\ell$. However, the non-relativistic Hamiltonian and limited acceleration approximations improve linearly with the number of subintervals. Extension of the interval applies only as long C.1-3 remain valid.

Brief interval propagation is also of interest for recurring observations. Recurring observations, for example, massive bodies awash in photons, are common. Then, an observed trajectory results from the accumulation of brief interval likelihoods in a random walk composed of the likely trajectories.

6 Bound states

Interaction is expressed in the vacuum expectation values (VEV) of fields but not in the Hamiltonians (103) of the constructions, section 3. The constructed Hamiltonians satisfy relativity but exhibit only continuous spectra. As a consequence, description of bound states with discrete rest masses is an evident question for the constructions. In this section, the presence of bound states in the constructed realizations of relativistic quantum physics is demonstrated. There are multiple argument state describing functions with centers-of-momentum described as free particles and stably localized descriptions for the internals. These state describing functions evolve as a free particle with a mass that is not necessarily a multiple of the elementary particle masses, m_κ.

One selected example construction suffices to establish existence. The selected construction has a single, neutral scalar field with one elementary particle of finite mass m, and the bound state is described by a function in the two-argument subspace of $\mathbf{H}_\mathcal{P}$. Bound state describing functions are functions with localized support that evolve as single, free particles. The Hamiltonian for evolution of the combined description of the center-of-momentum and bound state internals is (102) in the two-argument subspace, and this evolution is equivalent to a free particle description for the center-of-momentum with a periodic evolution for the internals.

Analogously to the many possibilities for bound states within $\mathcal{L}^2$ Hilbert spaces in nonrelativistic quantum mechanics, the Hilbert spaces $\mathbf{H}_\mathcal{P}$ include many descriptions of bound states. Selection of the bound states of physical interest suggests that considerations in addition to satisfaction of axioms A.1-7 must be applied.

6.1 The evolution of bound state describing functions

If a quantum-classical correspondence applies, a bound state consists of two identifiable bodies stably coupled together by an attractive potential. More generally, a bound state is identified here as a state describing function with an effective mass that is not an integer multiple of the elementary particle masses, and a stably localized description for evolution of the separations of arguments. The center-of-momentum, described as a free particle, exhibits a spreading of support over time typical of quantum mechanics. The support of the differences of spatial arguments is periodic over time, with a period determined by the binding energy.

An energy

$$\omega_b(\mathbf{p}) = \sqrt{\lambda_b^{-2} + \mathbf{p}^2} \tag{195}$$

is defined similarly to the elementary particle energies (10) but with

$$\lambda_b = \frac{\hbar}{m_b c}, \tag{196}$$

the reduced Compton wavelength (11) for a bound state of rest mass m_b.

The time evolution of the two-argument state describing function $\varphi_2((x)_2)$ is described in two equivalent ways. For the single, neutral scalar field example, the unitary temporal translation operator (80) and (103) provides

$$\begin{aligned} U(\lambda)\tilde{\varphi}_2((p)_2) &= e^{-i(p_{10}+p_{20})\lambda}\tilde{\varphi}_2((p)_2) \\ &= e^{-i(\omega_1+\omega_2)\lambda}\tilde{\varphi}_2((p)_2) \\ &= e^{-i(\omega_b(\mathbf{p}_1+\mathbf{p}_2)+\mu_b)\lambda}\tilde{\varphi}_2((p)_2) \end{aligned} \tag{197}$$

in the Fourier transform domain. $\omega(\mathbf{p}_j)$ is the Hamiltonian (102) that applies for each argument $\mathbf{p}_j$. The last line of (197) is the assertion that $\tilde{\varphi}_2$ evolves as a bound state: the center-of-momentum evolves as a free particle of mass m_b with momentum $\mathbf{p}_1 + \mathbf{p}_2$, and the evolution includes a constant energy offset equal to a binding energy μ_b. From conservation of energy, this equivalence of energy descriptions provides that

$$\delta(\sum_{j=1}^{n} p_{j0}) = \delta(\omega_b(\mathbf{p}_1 + \mathbf{p}_2) + \mu_b + \sum_{j=3}^{n} p_{j0})$$

if the bound arguments are labeled $1, 2$. This ensures both conservation of energy and Lorentz covariance of the descriptions.

Due to the unitarity of time translation $U(\lambda)$, (197) is equivalent to

$$\| |U_B(\lambda)\tilde{\varphi}_2\rangle - |\tilde{\varphi}_2\rangle \| = 0 \tag{198}$$

with the operator

$$U_B(\lambda) = e^{-iH_B\lambda}. \tag{199}$$

(199) defines a unitarily implemented, one parameter group $U_B(\lambda)$ generated by a densely defined Hermitian H_B, Stone's theorem [25]. The generator of this group is

$$\begin{aligned} H_B &= \omega_1 + \omega_2 - \omega_b(\mathbf{p}_1 + \mathbf{p}_2) - \mu_b \\ &= \sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12})} + \sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\phi_{12})} \\ &\quad -\sqrt{\lambda_b^{-2} + \rho_1^2} + \lambda_b^{-1} - 2\lambda_c^{-1} \end{aligned} \tag{200}$$

with a change to Jacobi coordinates,

$$\begin{aligned} &\mathbf{q}_1 = \mathbf{p}_1 + \mathbf{p}_2, \qquad \mathbf{q}_2 = \mathbf{p}_1 - \mathbf{p}_2 \\ &\rho_j = \|\mathbf{q}_j\|, \end{aligned} \tag{201}$$

$\omega_j = \omega(\mathbf{p}_j)$ from (10), $j \in \{1, 2\}$ and

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = \rho_1 \rho_2 \cos\phi_{12}.$$

ϕ_{12} is the angular separation of the momentum vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. $H_B = H_B(\mathbf{p}_1, \mathbf{p}_2)$ is a function over rotational invariants ρ_1, ρ_2 and $\mathbf{q}_1 \cdot \mathbf{q}_2$.

Functions within the null space of the Hermitian Hilbert space operator H_B describe bound states. This contrasts with non-relativistic quantum mechanics that describes bound states as eigenfunctions from the discrete spectra of selected Hamiltonian operators. In both instances, the descriptions of internals have localized dominant support.

A determination of the binding energy μ_b follows from a kinematic analysis of scattering events and the cluster decomposition property A.6 of VEV, section 4.1. The cluster decomposition property provides that the bound states and elementary particles have correspondences with classical bodies when the particles are distantly space-like separated. The binding energy μ_b also follows from the observation that the binding energy is a constant independent of the momenta $\mathbf{q}_1, \mathbf{q}_2$ and then

$$\mu_b = 2\lambda_c^{-1} - \lambda_b^{-1} \tag{202}$$

to satisfy the equality (197). The energies in (200) are equal, $H_B = 0$, if $\rho_1 = 0$ and $\rho_2 = 0$. The wavelengths are from (11) and (196). To be stably bound, $\mu_b > 0$ and consequently

$$0 \leq m_b < 2m.$$

The zeros of H_B identify the null space and are used to evaluate the support of state describing functions. From the definition (200) for the generator of $U_B(\lambda)$ and the evaluation (202) of μ_b, $H_B = 0$ follows if

$$\begin{aligned}
\omega_b(\mathbf{p}_1 + \mathbf{p}_2) + \mu_b &= \omega_b(\mathbf{q}_1) - \lambda_b^{-1} + 2\lambda_c^{-1} \\
&= \omega_1 + \omega_2 \\
&= \omega(\tfrac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)) + \omega(\tfrac{1}{2}(\mathbf{q}_1 - \mathbf{q}_2)) \\
&= \sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12})} + \sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\phi_{12})}
\end{aligned}$$

in Jacobi coordinates (201). From (200) and with the designation

$$\begin{aligned}
A_b &= \omega_b(\mathbf{q}_1) + \mu_b \\
&= \omega_b(\rho_1) - \lambda_b^{-1} + 2\lambda_c^{-1},
\end{aligned} \tag{203}$$

values of ρ_2 that set $H_B = 0$ are solutions to

$$\sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12})} + \sqrt{\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\phi_{12})} = A_b.$$

Squaring results in

$$\begin{aligned} A_b^2 &= 2\lambda_c^{-2} + \tfrac{1}{2}(\rho_1^2 + \rho_2^2) \\ &\quad +2\sqrt{(\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12}))(\lambda_c^{-2} + \tfrac{1}{4}(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\phi_{12}))} \\ &= 2\lambda_c^{-2} + \tfrac{1}{2}(\rho_1^2 + \rho_2^2) \\ &\quad +2\sqrt{\lambda_c^{-4} + \tfrac{1}{2}\lambda_c^{-2}(\rho_1^2 + \rho_2^2) + \tfrac{1}{16}((\rho_1^2 + \rho_2^2)^2 - 4\rho_1^2\rho_2^2\cos^2\phi_{12})} \end{aligned}$$

Reorganization and squaring again results in a linear function of ρ_2^2.

$$(\tfrac{1}{2}A_b^2 - \lambda_c^{-2} - \tfrac{1}{4}(\rho_1^2 + \rho_2^2))^2 = \lambda_c^{-4} + \tfrac{1}{2}\lambda_c^{-2}(\rho_1^2 + \rho_2^2) + \tfrac{1}{16}((\rho_1^2 + \rho_2^2)^2 - 4\rho_1^2\rho_2^2\cos^2\phi_{12}).$$

Collecting terms results in

$$\begin{aligned} \rho_2^2 &= \frac{A_b^2(A_b^2 - 4\lambda_c^{-2} - \rho_1^2)}{A_b^2 - \rho_1^2\cos^2\phi_{12}} \\ &\geq 0 \end{aligned}$$

for any $\rho_1, \phi_{12}, \lambda_b^{-1}, \lambda_c^{-1}$. Only the nonnegative root applies. (203) and $\omega_b(\rho_1) \geq \lambda_b^{-1}$ provides that

$$\begin{aligned} A_b^2 &= \lambda_b^{-2} + \rho_1^2 + 2\omega_b(\rho_1)(2\lambda_c^{-1} - \lambda_b^{-1}) + 4\lambda_c^{-2} - 4\lambda_c^{-1}\lambda_b^{-2} + \lambda_b^{-2} \\ &\geq 4\lambda_c^{-2} + \rho_1^2. \end{aligned} \tag{204}$$

Designate the value of ρ_2 that sets $H_B = 0$ as

$$\rho_{2o} = \sqrt{\frac{A_b^2(A_b^2 - 4\lambda_c^{-2} - \rho_1^2)}{A_b^2 - \rho_1^2\cos^2\phi_{12}}}, \tag{205}$$

a function over ρ_1 and ϕ_{12}. From (204), $A_b^2 > 0$, $A_b^2 - 4\lambda_c^2 - \rho_1^2 \geq 0$ and equals zero at $\rho_1 = 0$, and the denominator is never zero: $A_b^2 - \rho_1^2\cos^2\phi_{12} > 0$ for finite mass m elementary particles. For $\rho_1 \ll \lambda_c^{-1}$,

$$\rho_{2o} \approx \rho_1. \tag{206}$$

ρ_{2o} is zero for $\rho_1 = 0$ and is finite otherwise for finite masses m.

Finally, a condition that ensures that a two-argument state describing function (197) behaves in time as a bound state is limitation of the support of bound state describing functions to the null space of H_B. With limitation of the support to the null space of H_B, (198) is satisfied. The suggested two-argument bound state describing functions become

$$\begin{aligned} \tilde{\varphi}_2(p_1, p_2) &= (p_{10} + \omega_1)(p_{20} + \omega_2)\left|\frac{\partial H_B}{\partial \rho_2}\right| \delta(H_B)\tilde{f}_2(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2) \\ &= (p_{10} + \omega_1)(p_{20} + \omega_2)\,\delta(\rho_2 - \rho_{2o})\tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2). \end{aligned} \tag{207}$$

These functions $\tilde{\varphi}_2$ over energy-momenta p_1, p_2 are of the form (9) and follow from functions $\tilde{f}_2\mathbf{q}_1, \mathbf{q}_2)$ with support constrained by the delta function $\delta(H_B)$. ρ_{2o} is from (205). The $\mathbf{p}_1+\mathbf{p}_2$ dependence describes the center-of momentum and $\mathbf{p}_1-\mathbf{p}_2$ describes relative motion within the bound state. These descriptions (144) are independent in the non-relativistic approximation, but relativistic length contraction and time dilation raises the concern that the two descriptions are coupled. Nonetheless, even with relativity, $H = \omega_1+\omega_2 = \omega_b(\mathbf{p}_1+\mathbf{p}_2)+\mu_b$ in the null space of H_B and the Hamiltonian $H = \omega_1 + \omega_2$ decomposes as a sum of functions over $\mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{p}_1 - \mathbf{p}_2$. This description of bound state describing functions, particularly that the description is within one constant time plane, is peculiar to selected frames of reference. The transition likelihoods are Poincaré covariant, A.3.

Anticipated for energy eigenfunctions of the "trivial" Hamiltonians in the constructions, $\delta(\rho_2 - \rho_{2o})\tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2) \notin \mathcal{S}(\mathbb{R}^6)$ and it must be determined whether the $\varphi_2((x)_2)$ suggested in (207) are elements of $\mathbf{H}_\mathcal{P}$. Arbitrarily accurate, as generalized functions [20], approximations of bound state (207) by elements of $\mathbf{H}_\mathcal{P}$ are developed in section 6.2 below.

6.2 Approximation by elements of $\mathbf{H}_\mathcal{P}$

This section includes a demonstration that the suggested $\varphi_2((x)_2)$ in (207) has a divergent norm and consequently is not an element of $\mathbf{H}_\mathcal{P}$. It is then demonstrated that elements of $\mathbf{H}_\mathcal{P}$ arbitrarily accurately approximate the bound state describing functions (207).

The frequency domain representation (114) of the scalar product with the single, neutral scalar field VEV (150) provides the Hilbert space norm of two-argument state describing functions of the form (207).

$$\begin{aligned}
\|\varphi_2\|^2 \;=\; & \int d(\mathbf{p})_4 \left|\tfrac{\partial H_B(\mathbf{p}_1,\mathbf{p}_2)}{\partial \rho_2}\right| \delta(H_B(\mathbf{p}_1,\mathbf{p}_2))\overline{\tilde{f}_2(\mathbf{p}_1+\mathbf{p}_2, \mathbf{p}_1-\mathbf{p}_2)} \\
& \times \left|\tfrac{\partial H_B(\mathbf{p}_3,\mathbf{p}_4)}{\partial \rho_4}\right| \delta(H_B(\mathbf{p}_3,\mathbf{p}_4))\tilde{f}_2(\mathbf{p}_3+\mathbf{p}_4, \mathbf{p}_3-\mathbf{p}_4) \\
& \times \Bigg(4\,\frac{\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)+\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)}{\sqrt{\omega_1\omega_2\omega_3\omega_4}} \\
& \qquad + c_4\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\Bigg)
\end{aligned} \tag{208}$$

from (151) without the non-relativistic approximations. ρ_{4o} is the evident modification of ρ_{2o} from (205). The contribution from the first term is

$$4\int d(\mathbf{p})_2 \left|\frac{\partial H_B(\mathbf{p}_1,\mathbf{p}_2)}{\partial \rho_2}\right|^2 \delta(H_B(\mathbf{p}_1,\mathbf{p}_2))^2\, \frac{|\tilde{f}_2(\mathbf{p}_1+\mathbf{p}_2, \mathbf{p}_1-\mathbf{p}_2)|^2}{\omega_1\omega_2}$$

and the squared delta function diverges. Without mollification and similarly to plane wave states, bound state describing functions (207) are not elements of $\mathbf{H}_\mathcal{P}$.

A convenient mollification that approximates the bound state describing function (207) arbitrarily well substitutes a test function of $\rho_2 - \rho_{2o}$,

$$\delta(\rho_2 - \rho_{2o}) \mapsto g_L(\rho_2 - \rho_{2o}) = \left(\frac{L^2}{\pi}\right)^{\frac{3}{2}} \; e^{-L^2(\rho_2 - \rho_{2o})^2}, \tag{209}$$

for the support limiting delta function. The delta function is well approximated for finite $L \to \infty$. With the mollification, the bound state describing function (207) is absolutely summable over $\mathbf{q}_1, \mathbf{q}_2$. The $g_L(\rho_2 - \rho_{2o})$ are not test functions over $(\mathbf{q})_2$ since, developed below in section 6.3, ρ_{2o} is not continuously differentiable with respect to the components $q_{j\nu}$ of $\mathbf{q}_1, \mathbf{q}_2$. Similarly, the convenient factor $|\partial H_B/\partial\rho_2|$ in (207) is not infinitely differentiable but is absolutely summable.

$$\frac{\partial H_B}{\partial \rho_2} = \frac{\rho_2 + \rho_1 \cos\phi_{12}}{4\sqrt{\lambda_c^{-2} + \frac{1}{4}(\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12})}} + \frac{\rho_2 - \rho_1 \cos\phi_{12}}{4\sqrt{\lambda_c^{-2} + \frac{1}{4}(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\phi_{12})}} \tag{210}$$

from (200).

With the mollification (209), the norm (208) of $\|\varphi_2\|$ is finite in three or more spacetime dimensions for a finite mass m. The singularity from the energy conservation delta function in the higher order connected functions is summable in three or more spacetime dimensions and then the summation (208) of regular functions of rapid decline at large values over the indicated surfaces within $\mathbb{R}^6$ is finite. Convergence of the summations follows from the demonstration of A.1, section 4.2.5.

6.3 Localized spatial support

In this section, it is demonstrated that the spatial support at time zero of the bound state describing function (209) is localized and the breadth of support of the description of the internals does not grow over time. The breadth of support of the description of the center-of-momentum exhibits free particle growth with time.

The inverse Fourier transform of the mollified function

$$\tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2) = g_L(\rho_2 - \rho_{2o})\tilde{f}_2(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2) \tag{211}$$

approximates the dominant support of the bound state describing function $\varphi_2((x)_2)$ in (207). $g_L(\rho_2 - \rho_{2o})$ designates the mollification (209) of the delta function that restricts the support of φ_2 to the null space of H_B. $\mathbf{q}_1, \mathbf{q}_2$ are the Jacobi coordinates (201). From (262) in appendix 8.10, the inverse Fourier transform u_2 of $\tilde{u}_2$ characterizes the dominant spatial support of φ_2.

$$\varphi_2(x_1, x_2) = 2\pi \prod_{j=1}^{2} \left(\delta(x_{j0})\sqrt{-\Delta_j + \lambda_c^{-2}} - i\delta'(x_{j0})\right) u_2(\mathbf{x}_1, \mathbf{x}_2)$$

with Δ_j the Laplacian for $\mathbf{x}_j \in \mathbb{R}^3$. $\sqrt{-\Delta + \lambda_c^{-2}}$ is an anti-local operator [53] and as a consequence, φ_2 is only essentially localized even when the support of u_2 is strictly local. The physically relevant spatial support $\langle (\mathbf{y})_k | \varphi_2 \rangle$ of the state describing function $\varphi_2((x)_2)$ at time zero (115) is characterized by the support of $\varphi_2(x_1, x_2)$ as a function, section 5.2. The VEV contribute to the breadth of support, but the connectivity of the VEV provide that the support of $\langle (\mathbf{y})_k | \varphi_2 \rangle$ is localized if the support of $u_2(\mathbf{y}_1, \mathbf{y}_2)$ declines with spatial separations from the origin. $\tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2)$ is selected to center the support of $u_2(\mathbf{y}_1, \mathbf{y}_2)$ on the origin.

$$\mathbf{y}_1 = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \qquad \mathbf{y}_2 = \frac{\mathbf{x}_1 - \mathbf{x}_2}{2}$$

are Jacobi spatial coordinates.

The dominant support over $\mathbf{y}_1, \mathbf{y}_2$ of the inverse Fourier transform $u_2(\mathbf{y}_1, \mathbf{y}_2)$ characterizes the physically relevant support $\langle (\mathbf{y})_k | \varphi_2 \rangle$ of the bound state describing function.

$$\begin{aligned} \int d(\mathbf{p})_2 \, \frac{e^{i\mathbf{p}_1 \cdot \mathbf{x}_1} e^{i\mathbf{p}_2 \cdot \mathbf{x}_2}}{(2\pi)^3} \, \left| \tfrac{\partial H_B(\mathbf{p}_1, \mathbf{p}_2)}{\partial \rho_2} \right| \, \delta(H_B(\mathbf{p}_1, \mathbf{p}_2)) \tilde{f}_2(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2) \\ \approx \int d(\mathbf{q})_2 \, \frac{e^{i\mathbf{q}_1 \cdot \mathbf{y}_1} e^{i\mathbf{q}_2 \cdot \mathbf{y}_2}}{(4\pi)^3} \, g_L(\rho_2 - \rho_{2o}) \tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2) \\ = u_2(\mathbf{y}_1, \mathbf{y}_2) \end{aligned} \tag{212}$$

with $\tilde{f}_2$ selected to achieve centered, localized support, ρ_2 is from the Jacobi coordinates (201) and ρ_{2o} is the function of ρ_1 and $\mathbf{q}_1 \cdot \mathbf{q}_2$ from (205). In the Jacobi coordinates,

$$\mathbf{p}_1 \cdot \mathbf{x}_1 + \mathbf{p}_2 \cdot \mathbf{x}_2 = \mathbf{q}_1 \cdot \mathbf{y}_1 + \mathbf{q}_2 \cdot \mathbf{y}_2$$

and the determinant of the Jacobian matrix for the variable change is 2^{-3}. From the Riemann–Lebesgue lemma, $u_2(\mathbf{y}_1, \mathbf{y}_2)$ vanishes for large magnitude $\mathbf{y}_1, \mathbf{y}_2$ and the estimates below bound the rate of decay. The stability of the description of separation $\mathbf{y}_2$ with time follows from the equality (197) for temporal evolution.

Absolute summability of derivatives of the mollified function (211) with respect to the components $q_{j\nu}$ of $\mathbf{q}_1, \mathbf{q}_2$ suffices to bound the dominant support of $u_2(\mathbf{y}_1, \mathbf{y}_2)$. $j \in \{1, 2\}$ and $\nu \in \{x, y, z\}$. For finite L,

$$\begin{aligned} \left| y_{1\nu_1}^{k_1} y_{2\nu_2}^{k_2} \, u_2(\mathbf{y}_1, \mathbf{y}_2) \right| &= \left| \int d(\mathbf{q})_2 \, \left(\frac{\partial^{k_1}}{\partial q_{1\nu_1}^{k_1}} \frac{\partial^{k_2}}{\partial q_{2\nu_1}^{k_2}} \frac{e^{i\mathbf{q}_1 \cdot \mathbf{y}_1} e^{i\mathbf{q}_2 \cdot \mathbf{y}_2}}{(4\pi)^3} \right) \, \tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2) \right| \\ &= \left| \int d(\mathbf{q})_2 \, \frac{e^{i\mathbf{q}_1 \cdot \mathbf{y}_1} e^{i\mathbf{q}_2 \cdot \mathbf{y}_2}}{(4\pi)^3} \, \left(\frac{\partial^{k_1}}{\partial q_{1\nu_1}^{k_1}} \frac{\partial^{k_2}}{\partial q_{2\nu_1}^{k_2}} \tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2) \right) \right| \\ &\leq \int d(\mathbf{q})_2 \, \left| \frac{\partial^{k_1}}{\partial q_{1\nu_1}^{k_1}} \frac{\partial^{k_2}}{\partial q_{2\nu_1}^{k_2}} \tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2) \right| \end{aligned} \tag{213}$$

from integration by parts and the vanishing of the test functions at the limits of summation.

$$r_j = \|\mathbf{y}_j\|$$

are Euclidean norms. The upper bound (213) is independent of $y_{1\nu_1}^{k_1} y_{2\nu_2}^{k_2}$ and provides that $u_2(\mathbf{y}_1, \mathbf{y}_2)$ must decline at least proportionately for large $|y_{1\nu_1}^{k_1} y_{2\nu_2}^{k_2}|$.

The test functions $\tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2) \in \mathcal{S}(\mathbb{R}^6)$ in (211) are infinitely differentiable but $g_L(x) \in \mathcal{S}(\mathbb{R})$ are functions of $\rho_2 - \rho_{2o}$ and develop singularities with differentiations of ρ_{2o} and ρ_2 resulting from the chain rule. The chain rule provides

$$\frac{\partial g_L(\rho_2 - \rho_{2o})}{\partial q_{j\nu}} = -\frac{\partial g_L(\rho_2 - \rho_{2o})}{\partial(\rho_2 - \rho_{2o})} \left(\frac{\partial(\rho_2 - \rho_{2o})}{\partial q_{j\nu}} \right) \tag{214}$$

for $j \in \{1, 2\}$ and $\nu \in \{x, y, z\}$. From the product rule,

$$\begin{aligned} \frac{\partial(\rho_2 - \rho_{2o})}{\partial q_{1\nu}} &= -\frac{1}{2\rho_{2o}} \frac{\partial(\rho_{2o})^2}{\partial q_{1\nu}} \\ \frac{\partial(\rho_2 - \rho_{2o})}{\partial q_{2\nu}} &= \frac{\partial \rho_2}{\partial q_{2\nu}} - \frac{1}{2\rho_{2o}} \frac{\partial(\rho_{2o})^2}{\partial q_{2\nu}}. \end{aligned} \tag{215}$$

From (205), ρ_{2o}^2 is a ratio of multinomials in ρ_1 and $\cos\phi_{12}$ with a denominator that does not vanish. Then the quotient rule provides that ρ_{2o}^2 is infinitely differentiable with respect to ρ_1 and $\cos\phi_{12}$ considered as independent variables. From (206), the factor $1/\rho_{2o}$ is singular, $1/\rho_{2o} = 1/\rho_1$ for $\rho_1 \to 0$. Singularities also develop in derivatives of ρ_1 and $\cos\phi_{12}$ with respect to the $q_{j\nu}$.

The derivatives of $(\rho_{2o})^2$ with respect to the $q_{j\nu}$ are developed using the chain rule.

$$\begin{aligned} \frac{\partial(\rho_{2o})^2}{\partial q_{1\nu}} &= \frac{\partial(\rho_{2o})^2}{\partial \rho_1} \frac{\partial \rho_1}{\partial q_{j\nu}} + \frac{\partial(\rho_{2o})^2}{\partial \cos\phi_{12}} \frac{\partial \cos\phi_{12}}{\partial q_{j\nu}} \\ \frac{\partial(\rho_{2o})^2}{\partial q_{2\nu}} &= \frac{\partial(\rho_{2o})^2}{\partial \cos\phi_{12}} \frac{\partial \cos\phi_{12}}{\partial q_{2\nu}}. \end{aligned} \tag{216}$$

Factors

$$\frac{\partial \rho_j}{\partial q_{j\nu}} = \frac{q_{j\nu}}{\rho_j}$$

satisfy the bound $|q_{j\nu}/\rho_j| \leq 1$ and are regular. Higher derivatives diverge, for example

$$\frac{\partial^2 \rho_2}{\partial q_{2\nu}} = \frac{1}{\rho_2} - \frac{q_{2\nu}^2}{\rho_2^3}$$

diverges as $1/\rho_2$ for $\rho_2 \to 0$.

The derivative of ρ_{2o}^2 with respect to $\cos\phi_{12}$ includes a zero at $\rho_1 = 0$.

$$\begin{aligned}\frac{\partial(\rho_{2o})^2}{\partial\cos\phi_{12}} &= -\frac{1}{2\rho_{2o}}\frac{A_1^2(A_1^2 - 4\lambda_c^{-2} - \rho_1^2)}{(A_1^2 - \rho_1^2\cos^2\phi_{12})^2}\,2\rho_1^2\cos\phi_{12} \\ &= -\frac{\rho_{2o}\,\rho_1^2\cos\phi_{12}}{A_1^2 - \rho_1^2\cos^2\phi_{12}}.\end{aligned}$$

The cosine of the angular separation ϕ_{12} of $\mathbf{q}_1$ and $\mathbf{q}_2$ is

$$\cos\phi_{12} = \frac{q_{1x}q_{2x} + q_{1y}q_{2y} + q_{1z}q_{2z}}{\rho_1\rho_2}. \tag{217}$$

The quotient rule results in

$$\begin{aligned}\frac{\partial\cos\phi_{12}}{\partial q_{j\nu}} &= \frac{q_{\ell\nu}\rho_1\rho_2 - (q_{1x}q_{2x} + q_{1y}q_{2y} + q_{1z}q_{2z})\dfrac{q_{j\nu}}{\rho_j}\rho_\ell}{\rho_1^2\rho_2^2} \\ &= \frac{q_{\ell\nu}}{\rho_1\rho_2} - \cos\phi_{12}\frac{q_{j\nu}}{\rho_j^2}.\end{aligned} \tag{218}$$

ℓ is the other index given j: $\ell = 2$ if $j = 1$ and $\ell = 1$ if $j = 2$. Then, (218) diverges as $1/\rho_j$ for $\rho_j \to 0$. The product

$$\frac{\partial(\rho_{2o})^2}{\partial\cos\phi_{12}}\frac{\partial\cos\phi_{12}}{\partial q_{j\nu}} = -\frac{\rho_{2o}\,\rho_1^2\cos\phi_{12}}{A_1^2 - \rho_1^2\cos^2\phi_{12}}\left(\frac{q_{\ell\nu}}{\rho_1\rho_2} - \cos\phi_{12}\frac{q_{j\nu}}{\rho_j^2}\right)$$

is regular for $j = 1$ but diverges as $1/\rho_2$ for $\rho_2 \to 0$ for $j = 2$.

Collecting results and considering only the most singular term, the singularities of the first derivatives (214) are

$$\frac{\partial g_L(\rho_2 - \rho_{2o})}{\partial q_{1\nu}} = \mathcal{O}\left(\frac{1}{\rho_1}\right), \qquad \frac{\partial g_L(\rho_2 - \rho_{2o})}{\partial q_{2\nu}} = \mathcal{O}\left(\frac{1}{\rho_1\rho_2}\right)$$

for ρ_1, ρ_2 near zero. The measure $d(\mathbf{q})_2$ contributes $\rho_1^2\rho_2^2$ that eliminates these divergences and the first derivatives of the mollified $\tilde{u}_2(\mathbf{q}_1, \mathbf{q}_2)$ are absolutely summable in four or more spacetime dimensions. Summability of the singularity from the energy conservation delta function in the higher order connected functions requires three or more spacetime dimensions, section 4.2.5, and four spacetime dimensions suffices to regularize the bound on the first derivative. Then, the Cauchy-Schwarz-Bunyakovsky inequality demonstrates that

$$|(\alpha_x y_{jx} + \alpha_y y_{jy} + \alpha_z y_{jz})\,u_2(\mathbf{y}_1, \mathbf{y}_2)| < \|\alpha\|\|\beta\|$$

from upper bounds

$$|y_{j\nu}\, u_2(\mathbf{y}_1, \mathbf{y}_2)| \leq \beta_\nu,$$

$\nu = x, y, z$, and all $\alpha_x, \alpha_y, \alpha_z \in \mathbb{R}^3$. $\|\alpha\|, \|\beta\|$ are the Euclidean norms. From the upper bound (213), the β_ν are independent of $y_{j\nu}$. As a consequence, with $r_j = \|\mathbf{y}_j\|$, the decline of $u_2(\mathbf{y}_1, \mathbf{y}_2)$ for $r_j \to \infty$ is $1/r_j$ without constraint on $\tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2)$. The absolute summability of higher derivatives of (211) follow if $\tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2)$ contributes zeros at $\rho_1 = 0$ and $\rho_2 = 0$. Zeros result in demonstrations of more rapid decline of $u_2(\mathbf{y}_1, \mathbf{y}_2)$. From the defining relationship for bound states (197), the localization implying bound for the decline with r_2 is independent of time since the effective Hamiltonian depends only on $\mathbf{q}_1$. The bound on the separation $\mathbf{x}_1 - \mathbf{x}_2$ is consequently constant with time.

7 Technical concerns with canonical quantization

Inquiry into an appropriate mathematical development of quantum mechanics was initiated notably by John von Neumann. This inquiry has been extended by Léon van Hove, Res Jost, Rudolf Haag, Arthur Wightman, Huzihiro Araki, Nikolay Bogolubov, Hans-Jürgen Borchers, Jan Raphael Høegh-Krohn, Franco Strocchi and many others [10, 67]. Their developments use the concept of Hilbert space advanced notably by David Hilbert, Erhard Schmidt, Frigyes Riesz, Marshall Stone and John von Neumann. The discussion here distinguishes a general development of quantum mechanics from a canonical formalism development. The general development includes: quantum mechanics describes the states of nature as elements of rigged Hilbert spaces; the evolution of the observable features of appropriate states is well-approximated by classical mechanics; energies are nonnegative; the temporal evolution of state descriptions is unitary (likelihood preserving) and causal; likelihoods of observation are calculated from Born's rule; and likelihoods, like events, are relativistically invariant. The canonical formalism adds conjecture that the quantum-classical correspondence is established be associating classical dynamical variables with densely defined Hermitian operators [14, 57, 60]. Concerns with the canonical formalism are introduced in section 2. This section extends the discussion of technical concerns with the canonical formalism for quantum mechanics.

Quantum mechanics has provided an expanded and successful description of nature. Nevertheless, and emphasized by von Neumann, issues remain even in the development of nonrelativistic quantum mechanics [60]. The canonical formalism [14, 25, 57, 61] "quantizes" classical dynamics. If there is a "quantization" of the corresponding classical dynamics, is it a concern that while functions of classical dynamical variables are classical dynamical variables (for example, generalized coordinates), products of Hermitian operators are not necessarily Hermitian operators? Are some classical quantities distinguished by having quantizations while other quantities do not? Or is it a concern that the (generalized) eigenfunctions of operators associated with observables by the canonical formalism are not always elements of the Hilbert space, that is, do not describe states of nature? In such cases, what state results from "col-

lapse to an eigenfunction of the observable?" The Schrödinger representation of location X_ν and momentum P_ν operators satisfy the Born-Heisenberg-Jordan relation $[X_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$ and serve as archetypes for a quantization of classical dynamical variables. These X_ν and P_ν apply in the $\mathcal{L}^2$ Hilbert spaces suitable for non-relativistic quantum mechanics. In the following few paragraphs, the discussion includes that: the eigenfunctions of the operators X_ν and P_ν are not elements of $\mathcal{L}^2$ Hilbert spaces; locations and momenta associated with state describing functions are well-defined as observable features even when expectations of the corresponding operators X_ν and P_ν diverge; the quantization of location X_ν fails to be a Hermitian operator in relativistic free field theory; and the quantization of products of locations $\mathbf{x}$ and momenta $\mathbf{p}$ are not necessarily Hermitian operators even in non-relativistic quantum mechanics. Each of these points is a difficulty or ambiguity for canonical quantization. Together, the contradictions motivate a relaxed quantum-classical correspondence.

First, greater detail on Hilbert space operators is introduced. Discussion is limited to the complex Hilbert spaces of interest. Study of Hilbert space operators, particularly unbounded operators in infinite dimensional Hilbert spaces, is a subtle and elegant subject [40] that illustrates many of the "paradoxes of infinity." In appendix 8.2.5, a Hilbert space operator A is introduced as a mapping of elements from a Hilbert space $\mathbf{H}$ back into $\mathbf{H}$, $A : \mathbf{H} \mapsto \mathbf{H}$. If

$$|A\psi\rangle = |g\rangle$$

then $|\psi\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$, the domain of A, and $|g\rangle \in \mathcal{R}_A \subseteq \mathbf{H}$, the range of A. The domains of bounded operators can be extended to the entire $\mathbf{H}$. A is bounded if $\|A\psi\| \leq c\|\psi\|$ for some $c \in \mathbb{R}$ independently of the element $|\psi\rangle \in \mathbf{H}$. The least upper bound c defines an operator norm $\|A\|$. The domain of an unbounded A is necessarily a proper subset of the Hilbert space but the domain may be dense in $\mathbf{H}$. A set of elements $|e_n\rangle$ is *dense* in $\mathbf{H}$ if every $|\psi\rangle \in \mathbf{H}$ is within an arbitrarily small neighborhood of a finite linear combination of the $|e_n\rangle$,

$$\|\psi - \sum_{n=1}^{N} c_n e_n\| < \epsilon$$

for $N > N_\epsilon$ with the $c_n \in \mathbb{C}$ and $\epsilon \to 0$ as N grows without bound. A *separable* Hilbert space has a denumerable (finite or infinite) dense set of elements $|e_n\rangle$. An example is that functions in the Schwartz function space $\underline{\mathcal{S}}$ [21] are dense in the square-summable functions $\mathcal{L}^2$. $\mathcal{L}^2$ has a dense, denumerable basis. For an unbounded operator B, a subsequence from a diverging sequence $|Bu_n\rangle$ can be selected and relabeled such that $\|Bu_n\| > n\|u_n\|$. Then, the neighborhood of each element $|\psi\rangle$ with $|B\psi\rangle \in \mathbf{H}$ contains a sequence constructed below with $\|B(\psi - \psi_n)\| > n\|\psi - \psi_n\|$ for n growing without bound and $\|\psi - \psi_n\| \to 0$. That is, an unbounded operator is not continuous anywhere in the Hilbert space. Indeed, using the divergent sequence $|u_n\rangle$ selected above, set

$$|v_n\rangle = \frac{|u_n\rangle}{n\|u_n\|}, \quad \text{and then} \qquad \|Bv_n\| = \frac{\|Bu_n\|}{n\|u_n\|} > 1.$$

The sequence $|\psi_n\rangle = |\psi\rangle + |v_n\rangle$ is a convergent sequence, $\|\psi - \psi_n\| = \|v_n\| = 1/n \to 0$, but $\|B(\psi - \psi_n)\| = \|Bv_n\| > 1$. Then generally, neighboring states within a Hilbert space do not have nearly the same observables when observables are associated with unbounded Hilbert space operators.

In contrast, states exhibit observable features even when the expectation values of corresponding operators diverge or when the operation is undefined. For the example of location,

$$\psi(x) = e^{-\mathbf{x}^2/(4\sigma^2)} + \frac{\epsilon}{(1+\mathbf{x}^2)^{1-\delta}}$$

is dominantly supported near $\mathbf{x} = 0$ for $0 < \epsilon, \delta \ll 1$ with the length σ determining the size of the neighborhood. The σ, ϵ, δ can be selected to provide an arbitrarily large likelihood that an observation of location is near $\mathbf{x} = 0$. Here, likelihoods are for finite. common volumes. The likelihood that the perceived location will be within a finite volume including the dominant support is much greater that the likelihood the perceived location will be from any other disjoint, equal volume. In this interpretation, the likely location is near $\mathbf{x} = 0$ for the state described by $\psi(x)$ even though $\langle\psi|\mathbf{x}\psi\rangle$ diverges. For location, $x_\nu \mapsto X_\nu$ is the Hilbert space location operator, and the expectation value $\langle\psi|X_\nu\psi\rangle$ diverges even though the likely perception of the body described by $\psi(x)$ is that the body will be observed near the origin. At large $\mathbf{x}^2$ (the unobservable "far side of the moon"), there is a small likelihood of detection within any finite volume but a sufficient volume for the mean value to diverge. There is no actual divergence since we are not capable of detecting location over infinite volumes. Our measurements are always localized; location X_ν and momentum P_ν are idealized observables. Actual observations are limited to finite, localized volumes: without an omniscient classical observer, no knowledge of infinite volumes is available. The likelihoods of physical interest are relative likelihoods of detection comparing finite volumes. The unknowable, distant support of functions only hypothetically affects our considerations. In the example, $\|\psi(x) - e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is small, they are neighboring states, but $\|X_\nu\psi(x) - X_\nu e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is divergent in the $\mathcal{L}^2$ norm.

Correspondences with the classical concepts of location and momentum is provided by the support of the state describing functions. Considering support, the likelihoods of observations of location and momenta are nearly the same for state describing functions that have nearly the same support. The support of state describing functions are more robust and consistent observable features than expected values of unbounded, densely defined Hermitian Hilbert space operators such as X_ν and P_ν.

The observables, location, momentum and field strength, are anticipated to have eigenvalues for every real number. As a consequence, their eigenfunctions cannot be elements of $\mathbf{H}_\mathcal{P}$. Their eigenfunctions are generalized eigenfunctions, [22] and appendix 8.2.1. Eigenfunctions with distinct eigenvalues of a Hermitian operator are orthogonal [46] and there can only be a countable number of mutually orthogonal functions within the separable Hilbert spaces of interest [11]. Observation of location, momentum or field strength can not "collapse" to an eigenfunction of the observable: generalized eigenfunctions are idealizations and not admissible

state describing functions.

Introduced in section 2, extrapolation of the location operator X_ν to relativistic physics does not succeed. With consideration of relativity, the X_ν that result from quantization of the classical dynamical variables x_ν are not Hermitian operators. Illustrated in appendix 8.2.6, operators whose eigenvalues are observable quantities must be Hermitian. Transition likelihoods are determined by the scalar product and these likelihoods describe events independently of inertial observer. As a consequence, the scalar product must be a Lorentz covariant in relativistic physics. The scalar product (27), and the Källén-Lehmann form for the two-point function as a nonnegative summation over masses of the Pauli-Jordan function result in consideration of

$$\begin{aligned}\langle X_\nu^* \psi | g\rangle &= \langle \psi | X_\nu g\rangle \\ &= \int dxdy\ \Delta^+(y-x)\,\overline{\psi(y)}\,x_\nu g(x) \\ &\neq \langle X_\nu \psi | g\rangle \\ &= \int dxdy\ \Delta^+(y-x)\,y_\nu \overline{\psi(y)}\,g(x)\end{aligned}$$

since the Pauli-Jordan function is not of point support over spacetime, $x_\nu\Delta^+(y-x) \neq y_\nu\Delta^+(y-x)$. Then $X_\nu^* \neq X_\nu$ for $\psi, g \in \mathcal{D}_{X_\nu}$. The elevation of $\mathbf{x}_\nu$ to Hilbert space operator X_ν is not Hermitian in relativistic quantum mechanics. This "localization problem" is one of many problems that develop with the canonical formalism [8, 9]. In contrast, the energy-momentum operators,

$$\sum_{\nu=1}^{n} P_\nu$$

in the n-argument subspace, are generators of translations in a unitary realization of the translation group and are densely defined and Hermitian as a consequence of Stone's theorem [25, 40]. In the argument above, $\langle \psi | P_\nu g\rangle = \langle P_\nu \psi | g\rangle$ follows from the point support of the Källén-Lehmann form in the momentum domain,

$$p_\nu \delta(p-q) = \delta(p-q) q_\mu.$$

A unitary realization of the translation group results from translation invariance of the scalar product. Location, an observable of evident importance in physics, does not precisely correspond to the elevation of $\mathbf{x}_\nu$ in relativistic quantum mechanics. Discussed in section 2 and appendix 8.4, the Hermitian operator associated with location is not the quantization of x_ν. Suitable location operators are determined by the relativistically invariant localized functions that describe those states that most closely correspond to a body at a particular location in relativistic quantum physics. These forms [43] are discussed in appendix 8.3.

Quantizations of classical dynamical variables can be excluded as Hermitian operators even in ordinary (non-relativistic) quantum mechanics. The product x^3p can be considered a classical

dynamical quantity and in ordinary quantum mechanics with a single spatial dimension, the corresponding operator in $\mathcal{L}^2$ should be the formally Hermitian

$$i\hbar x^{3/2}\frac{d}{dx}x^{3/2} = i\hbar(\frac{x^3}{2}\frac{d}{dx} + \frac{d}{dx}\frac{x^3}{2}).$$

The classical dynamic variables x and p correspond to unbounded, self-adjoint operators x and $i\hbar d/dx$ in this one spatial dimensional $\mathcal{L}^2$ example. However, this densely defined, formally Hermitian operator that corresponds with x^3p has square-summable eigenfunctions

$$s_\lambda(x) = \sqrt{2\lambda}\ \frac{\exp\left(-\lambda/(2x^2)\right)}{x^{3/2}}$$

with imaginary eigenvalues $-i\hbar\lambda$ [10]. $0 < \lambda \in \mathbb{R}$. As a consequence, $X^{3/2}PX^{3/2}$ is not a densely defined Hermitian operator in the $\mathcal{L}^2$ Hilbert space. This $s_\lambda(x) \in \mathcal{L}^2$ is defined for $x > 0$ and equals zero otherwise, or $s_\lambda(x)$ can be extended to negative x. The formally Hermitian operator $X^{3/2}PX^{3/2}$ that corresponds to x^3p is not Hermitian for $\mathcal{L}^2$ although x^3p is well-defined in classical dynamics. This example illustrates Groenewold's theorem [25]. Nevertheless, for the example of linear harmonic motion and minimum uncertainty (Gaussian) states $s_t(x)$ with small spatial variances, the trajectory of x^3p given by Newtonian mechanics approximates $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ from quantum dynamics [34, 49]. This establishes that there are particular states with real $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ that agree with the classical approximations x^3p even though $X^{3/2}PX^{3/2}$ is not a Hermitian operator. The Hermitian operator most closely corresponding to x^3p need not be the elevation of x^3p. Once again, the support of functions for appropriate state describing functions exhibit quantum-classical correspondences while a correspondence of classical dynamical variables with operators fails. Even when the operator that is the quantization of the classical dynamical variable is not Hermitian, the classical dynamics can approximate the quantum mechanics for appropriate states.

8 Appendices

8.1 The Dirac-von Neumann axioms for quantum mechanics

The Dirac-von Neumann axioms describe non-relativistic quantum mechanics. Here, the statement of the Dirac-von Neumann axioms is adapted from F. Strocchi's discussion [57].

I. *Observables*: The Hermitian operators A corresponding to the observables of a quantum mechanical system are within the algebra of bounded self-adjoint operators $\mathcal{B}(\mathbf{H})$ for a separable Hilbert space $\mathbf{H}$.

II. *States*: The pure states of a quantum mechanical system are described by rays $a|\underline{s}\rangle$, $|s\rangle \in \mathbf{H}$, $a \in \mathbb{C}$ and $a \neq 0$. More generally, a state is described by a nonnegative, unit trace, state density operator $\rho \in \mathcal{B}(\mathbf{H})$.

III. *Expectations*: If a state is represented by the normalized pure state $|s\rangle \in \mathbf{H}$, then, for the observable corresponding to $A \in \mathcal{B}(\mathbf{H})$, the experimental expectation is $\langle s|As\rangle$. More generally, experimental expectations are Trace$(A\rho)$. If A has a complete set of normalized eigenvectors $|e_n\rangle \in \mathbf{H}$, then

$$\langle s|As\rangle = \sum_n \lambda_n |\langle e_n|s\rangle|^2$$

and from Born's rule, the likelihood of observing λ_n is $|\langle e_n|s\rangle|^2$. More generally, A is described by the spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [22], lemma 5.6.7 [40], chapters 7-10 [25]).

Axioms I-III describe a Hilbert space realization of quantum mechanics and are implicit in the development of section 3. Conditions A.1 and A.2 of section 4.1 imply a Hilbert space realization. State density operators $\rho \in \mathcal{B}(\mathbf{H})$ are discussed in section 8.2.6. States are described by elements of the Hilbert space: limits of states such as the eigenfunctions of location and momentum are in the dual to the basis function spaces but depart from the rigged Hilbert spaces of interest [10]. Superselection sectors in the Hilbert spaces of interest illustrate that not all Hilbert space elements represent states of nature. Hilbert spaces of interest are represented as direct sums of superselection sectors and the observables are associated with operators limited to within sectors. The limitation of observables to bounded operators departs from Dirac's development but reflects that only finite values are observable, for example, locations within an accessible, finite volume. Limiting consideration to bounded operators provides several technical conveniences [25, 57] as well as conceding to physical reality.

Axioms I-III describe a general Hilbert space realization of quantum mechanics. Characterization of the observables remains. Dirac-von Neumann axioms IV-V determine properties of particular operators in non-relativistic physics.

IV. *Dirac canonical quantization*: The Hermitian operators that describe canonical coordinates q_i and momenta p_j , $i, j = 1, \ldots N$, obey canonical commutation relations

$$\begin{aligned} [q_i, q_j] &= 0 \\ [p_i, p_j] &= 0 \\ [q_i, p_j] &= -i\hbar\delta_{ij}. \end{aligned}$$

V. *Schrödinger representation*: The canonical commutation relations are realized in the Hilbert space

$$\mathbf{H} = \mathcal{L}^2 = \{\psi(x) | \int d\mathbf{x}\, |\psi(ct, \mathbf{x})|^2 < \infty, x \in \mathbb{R}^4, \mathbf{x} = x_1, x_2, x_3\}$$

by:

$$|q_i\psi\rangle = x_i\,|\psi\rangle$$
$$|p_i\psi\rangle = -i\hbar\,|\frac{\partial\psi}{\partial x_i}\rangle.$$

In the Dirac-von Neumann axioms, canonical variables q_i, p_i are quantizations of classical dynamical variables. If the canonical variables q_i, p_i are location and momentum, respectively, the canonical commutation relations are known as the Born-Heisenberg-Jordan relations. In the $\mathcal{L}^2$ Hilbert space applicable to non-relativistic physics, these quantizations result in unbounded Hermitian Hilbert space operators q_i, p_i, violating I. The correspondence in axiom V is the "elevation of c-number to q-number" for q_i. Unboundedness introduces the consideration that sums of observables are not necessarily observables (if necessarily limited domains are sufficiently disjoint) [25]. And, the (generalized) eigenfunctions of location and momentum are not elements of the Hilbert space. These and additional difficulties with the canonical formalism are discussed in section 7. Axiom V describes a particular, non-relativistic quantum-classical correspondence for location and momentum. The canonical formalism extrapolates axioms IV-V with fields as canonical coordinates [61]. Axiom IV remains valid for location and momentum in the constructions.

> The lack of a clear distinction between the role of the two sets of axioms, I, II, III and IV,V, is at the origin of the widespread point of view, adopted by many textbooks, by which all of them are characteristic of quantum systems. The distinction between classical and quantum systems is rather given by the mathematical structure of [the algebra of observables] and it will have different realizations depending on the particular class of systems. – Franco Strocchi [57].

8.2 Hilbert spaces and quantum mechanics

8.2.1 Hilbert spaces

A Hilbert space **H** is characterized by the number of linearly independent elements. For every two elements in the complex Hilbert spaces of interest here, there is a complex number $\langle \underline{f}|\underline{g}\rangle$,

$$\underline{f}, \underline{g} \in \mathbf{H} \mapsto \langle \underline{f}|\underline{g}\rangle \in \mathbb{C}$$

designated the *scalar product* of the elements. Properties of this scalar product include that $\langle \underline{g}|\underline{f}\rangle$ is the complex conjugate of $\langle \underline{f}|\underline{g}\rangle$ and the scalar product is linear in the second argument, $\langle \underline{f}|\alpha\underline{g} + \beta\underline{h}\rangle = \alpha\langle \underline{f}|\underline{g}\rangle + \beta\langle \underline{f}|\underline{h}\rangle$ for $\alpha, \beta \in \mathbb{C}$, and as a consequence, complex conjugate linear in the first argument. Scalar products are nonnegative, $\langle \underline{f}|\underline{f}\rangle \geq 0$, and this provides satisfaction of the Cauchy-Schwarz-Bunyakovsky inequality,

$$|\langle \underline{g}|\underline{f}\rangle|^2 \leq \langle \underline{g}|\underline{g}\rangle\langle \underline{f}|\underline{f}\rangle.$$

The zero element is unique in a Hilbert space. If $\langle \underline{f}|\underline{f}\rangle = 0$, then $\underline{f} = 0$ and $\langle \underline{g}|\underline{f}\rangle = 0$ for every element $|\underline{g}\rangle$ of the Hilbert space. A *degenerate scalar product* has all the properties of a scalar product except that there is more than one element of zero length. A degenerate scalar product (27) is defined for pairs of function sequences from the basis spaces $\underline{\mathcal{P}}$ considered as a linear vector space. The Hilbert spaces of interest are the completions of linear vector spaces with elements that are equivalence classes of vectors labeled by function sequences [13]. The elements of the Hilbert space may be characterized by any function sequence in an equivalence class. An isometry extends the degenerate scalar product to a scalar product: the elements of the Hilbert space are equivalence classes of functions equal in the norm (26) and $\langle \underline{f}|\underline{f}\rangle = 0$ states that the function sequence $\underline{f}$ is an element of the equivalence class of zero. The separation of two Hilbert space elements $\underline{g}, \underline{f}$ is $\|\underline{g} - \underline{f}\|$ using the norm (26). The distance is zero for two equivalent function sequences, and two is the maximum separation of *normalized* state descriptions ($\|\underline{f}\| = \|\underline{g}\| = 1$). The separation is $\sqrt{2}$ for orthogonal, normalized states. Two states $\underline{f}, \underline{g}$ are *orthogonal* if $\langle \underline{g}|\underline{f}\rangle = 0$. In quantum mechanics, every element on a ray $|a\underline{f}\rangle$ with finite $a \in \mathbb{C}$ describes the same physical state: only relative phase and amplitude within linear combinations are significant. Linearity and completeness are characteristic of Hilbert spaces. A Hilbert space $\mathbf{H}$ is complete: the limit of every Cauchy sequence of elements $|\underline{g}_\nu\rangle \in \mathbf{H}$ is included. That is, if

$$\|\underline{g}_\nu - \underline{g}_n\| \to 0$$

for $\nu, n > N \to \infty$, then there is an element $|\underline{g}\rangle \in \mathbf{H}$ such that $|\underline{g}\rangle = \lim_{\nu\to\infty} |\underline{g}_\nu\rangle$. In a *separable* Hilbert space, every element is arbitrarily well-approximated by a denumerably indexed sum of N linearly independent elements.

$$|\underline{f}\rangle = \sum_\ell c_\ell |e_\ell\rangle$$

with $\ell \in \mathbb{N}$, the natural numbers, and $c_\ell \in \mathbb{C}$. A (closed) subspace of a Hilbert space is the linear span of a subset of elements. If there are only N linearly independent elements, the Hilbert space or subspace is finite dimensional of dimension N, and if the number of linearly independent elements is unbounded but includes a denumerable, dense set of elements, the Hilbert space or subspace is denoted infinite dimensional and separable. The rigged Hilbert spaces of interest here are separable [11].

8.2.2 Rigged Hilbert spaces

Rigged (equipped) Hilbert spaces are appropriate settings for quantum mechanics. VEV that provide a Poincaré invariant scalar product (27) must be generalized functions: the VEV can not be summable functions [10]. The basis function spaces $\underline{\mathcal{P}}$ used in the constructions include only those elements from $\underline{\mathcal{S}}$ with Fourier transforms that vanish on the negative energy mass shells. Three classes of functions, denoted a Gelfand triple after Israel Gelfand, describe a rigged Hilbert space. The elements of a countably normed basis function space are denoted *test*

functions [10, 20, 21]. A particularly useful space of test functions are the Schwartz tempered functions $\underline{\mathcal{S}}$: the tempered functions are smooth (infinitely differentiable) and exhibit rapid decline for large values of their arguments [21]. The space of Fourier transforms of $\underline{\mathcal{S}}$ coincides with $\underline{\mathcal{S}}$. The associated class of generalized functions $\underline{\mathcal{S}}'$ are the linear distributions (generalized functions) dual to $\underline{\mathcal{S}}$. Linear distributions $T(x)$ map functions to complex numbers.

$$T(x) \in \underline{\mathcal{S}}' : \psi(x) \in \underline{\mathcal{S}} \mapsto T(\psi) \in \mathbb{C}.$$

$\underline{\mathcal{S}}'$ is usefully conceived as limits of functions $T(x)$ such that

$$T(\psi) = \int dx\, T(x)\psi(x)$$

is finite when $\psi(x) \in \underline{\mathcal{S}}$ with acknowledgment that this concept includes limits that are not summable using Lebesgue measure.[o] Indeed, the generalized functions dual to the functions of bounded support can be represented

$$T(\psi) = \sum_n \int d\mu_k(x)\, \frac{d^n\psi(x)}{dx^n}$$

with the summation over a finite number of terms using measures μ_k on the real numbers [21]. $\mathbf{H}_{\mathcal{S}}$ consists of $\underline{\mathcal{S}}$ plus the Cauchy sequences convergent in a Hilbert space norm (26). As a consequence, the Gelfand triple $(\mathcal{S}, \mathbf{H}_{\mathcal{S}}, \mathcal{S}')$ satisfies

$$\underline{\mathcal{S}} \subset \mathbf{H}_{\mathcal{S}} \subset \underline{\mathcal{S}}'.$$

The eigenfunctions of location and momentum are not elements of $\mathbf{H}_{\mathcal{S}}$ but are elements of $\underline{\mathcal{S}}'$. *Generalized eigenfunctions* of a linear operator A defined in $\underline{\mathcal{S}}$ are the generalized functions $T_\lambda(x) \in \underline{\mathcal{S}}'$ such that $T_\lambda(Af) = \lambda T_\lambda(f)$ for any $f \in \mathcal{S}$.

The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are a union of nuclear, countably normed spaces [22]. The elements of $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are test functions and their limits within $\mathbf{H}_{\mathcal{P}}$ constructed from the VEV in section 3 include generalized functions.

8.2.3 The Cauchy-Schwarz-Bunyakovsky inequality

The Cauchy-Schwarz-Bunyakovsky [46] inequality is that

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$$

[o] A perspective on the distinction between functions and generalized functions is illustrated by generalized functions with a single point of support. Both $\delta(x)$ and $\delta'(x)$ are limits of test functions and supported solely on the point $x = 0$ but $\delta(f) = f(0)$ and $\delta'(f) = f'(0)$.

for elements u, v within a complex linear vector space $\mathbf{H}$. The Cauchy-Schwarz-Bunyakovsky inequality applies if the vector space has a product $\langle u|v\rangle : u, v \in \mathbf{H} \mapsto \langle u|v\rangle \in \mathbb{C}$ with properties

$$\begin{aligned} \langle v|v\rangle &\geq 0 \\ \langle u|v\rangle &= \overline{\langle v|u\rangle} \\ \langle w|\alpha u + \beta v\rangle &= \alpha\langle w|u\rangle + \beta\langle w|v\rangle \end{aligned} \tag{219}$$

for all u, v, w in the vector space $\mathbf{H}$ and $\alpha, \beta \in \mathbb{C}$. This product is designated a complex scalar product if $\langle v|v\rangle = 0$ implies $v = 0$, and otherwise it is a degenerate scalar product.

For any two elements in the vector space, choose one element to label as v if $\langle v|v\rangle > 0$ and label the remaining element u. This includes all cases except for both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Using the Gram-Schmidt construction, the element

$$z = u - \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v$$

is orthogonal to v.

$$\langle z|v\rangle = 0.$$

From the properties (219) and the construction of z,

$$\begin{aligned} \langle u|u\rangle &= \langle z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v | z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v\rangle \\ &= \langle z|z\rangle + \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle} \\ &\geq 0. \end{aligned}$$

But $\langle z|z\rangle \geq 0$ and it follows that

$$\langle u|u\rangle \geq \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle}.$$

This demonstrates the inequality unless both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. If both are zero, the properties (219) provide that

$$\begin{aligned} \langle u + v|u + v\rangle &= \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle u|u\rangle \\ &= 2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \\ &\geq 0. \end{aligned}$$

Similarly, $\langle u - v|u - v\rangle \geq 0$ provides that

$$2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \leq 0.$$

As a consequence, $\mathbb{R}\mathrm{e}(\langle u|v\rangle) = 0$. Similarly, $\langle u + iv|u + iv\rangle \geq 0$ and $\langle u - iv|u - iv\rangle \geq 0$ provide that $\mathbb{I}\mathrm{m}(\langle u|v\rangle) = 0$. Then,

$$\langle u|v\rangle = 0$$

if both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Summarizing, the Cauchy-Schwarz-Bunyakovsky inequality applies to a linear vector space with elements u, v and product $\langle u|v\rangle$ with properties (219).

If $\langle u|v\rangle$ is a scalar product and neither u nor v are the zero element, then

$$|\langle u|v\rangle|^2 = \langle u|u\rangle\langle v|v\rangle$$

if and only if

$$u = \alpha v$$

for a nonzero $\alpha \in \mathbb{C}$. For nonzero u and v and a scalar product, the equality applies if and only if $z = 0$ for the z constructed above and then

$$u = \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v.$$

8.2.4 Entanglement

Quantum mechanics includes descriptions of *entangled* states. Descriptions of states are elements of Hilbert spaces and elements are expressible as linear combinations of other elements. The linear expansion of states is a property of quantum physics distinct from classical physics.

A simple example of entanglement has four states: |up, up⟩, |up, down⟩, |down, up⟩, and |down, down⟩ that span the spin states of two spin one-half bodies. The states describe the four possibilities for the measurement of spin polarizations along a particular axis. Any linear combination of these four states describes a state. For example,

$$\frac{1}{\sqrt{2}}|\text{up, down}\rangle + \frac{1}{\sqrt{2}}|\text{down, up}\rangle$$

is one possible state. For this example state, the polarizations are entangled: up for the first body occurs only with down for the second body, and down for the first body occurs only with up for the second body. The state of the first body is determined by the state for the second and total spin adds to zero in any observation. However, the states are not determined, the first body can be detected as either spin up or spin down with equal likelihoods in this example. But, knowledge of one of the spins implies knowledge of the other due to an entanglement of states. Entanglement of the spins is established when the particles are causally related; the entanglement persists as the particles become acausally separated. An entanglement of bodies need not be perfect. For example,

$$a\ |\text{up, up, down}\rangle + b\ |\text{up, down, up}\rangle + c\ |\text{down, up, up}\rangle$$

is an example with a particular angular momentum but no perfectly entangled pair. In appendix 8.2.8, the states that result from an observation are described by linear combinations of states of an observer entangled with the possible results of observation, $|\psi\rangle = \sum_k c_k |o_k, s_k\rangle$. The o_k are orthogonal descriptions of the observer's state, and the s_k describe the entangled, observed state. In the example of Schrödinger's cat thought experiment [51], o_1 would be "observed a live cat," o_2 would be "observed a dead cat," s_1 would be "a live cat," s_2 would be "a dead cat" and these descriptions are entangled $c_1|o_1, s_1\rangle + c_2|o_2, s_2\rangle$. An appeal to experience indicates that $|o_1, s_2\rangle$ and $|o_2, s_1\rangle$ do not persist in the evolution of states.

8.2.5 Hilbert space operators

Hilbert space operators are linear maps of elements of a Hilbert space $\mathbf{H}$ to elements within the Hilbert space [3, 40]. A is a *Hilbert space operator* if

$$A : |\underline{f}\rangle \in \mathbf{H} \mapsto |A\underline{f}\rangle \in \mathbf{H}$$

for a subset of elements $|\underline{f}\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$ denoted the domain of A. For a *bounded* operator,

$$\|A\psi\| \leq \|A\| \, \|\psi\|$$

with

$$\|A\| = \sup(\|A\psi\| : \|\psi\| \leq 1) < \infty.$$

A bounded operator is continuous, and the domain can be extended to the entire Hilbert space. Unitary operators are bounded and preserve the scalar product, that is, for unitary U, $\langle Ug|U\psi\rangle = \langle g|\psi\rangle$. Illustrated in section 7, an *unbounded* operator is not continuous, its domain is not a (closed) subspace, and the domain necessarily does not include the entire Hilbert space although the domain may be dense in the Hilbert space. The adjoint operator A^* of a Hilbert space operator A is defined by the property that $\langle A^*h|\psi\rangle = \langle h|A\psi\rangle$ for $|\psi\rangle \in \mathcal{D}_A$ and $|h\rangle \in \mathcal{D}_{A^*}$. The set of $h \in \mathbf{H}$ such that there is a $g \in \mathbf{H}$ with $\langle h|A\psi\rangle = \langle g|\psi\rangle$ is the domain $\mathcal{D}_{A^*}$ of A^*. An operator A with domain $\mathcal{D}_A$ is *Hermitian* if $\langle \underline{u}|A\underline{v}\rangle = \langle A\underline{u}|\underline{v}\rangle$ for every $\underline{u}, \underline{v} \in \mathcal{D}_A$ [3], a Hermitian operator is *symmetric* if $\mathcal{D}_A$ is dense in the Hilbert space, and a symmetric operator is *self-adjoint* if $\mathcal{D}_A = \mathcal{D}_{A^*}$ and $A\underline{u} = A^*\underline{u}$ for every $\underline{u} \in \mathcal{D}_A$. These designations correspond to Hermitian, maximal Hermitian, and hypermaximal respectively in von Neumann's designation [60].

There is an adjoint for every linear Hilbert space operator. Designate a dense set of normalized, linearly independent but not necessarily mutually orthogonal elements of a Hilbert space $\mathbf{H}$ with labels $\underline{e}_j$. An orthonormal basis for $\mathbf{H}$ follows from the Gram-Schmidt construction [46] applied to the $\underline{e}_j$. For unbounded operators on infinite dimensional separable Hilbert spaces, the domain of the adjoint is limited.

A convenient set of vectors $\underline{e}_j$ for analysis of the constructed Hilbert spaces $\mathbf{H}_\mathcal{P}$ and the field (5) consists of a union of bases for each of the n-argument subspaces. Due to interaction,

n and k-argument subspaces with $n \neq k$ are generally not orthogonal. For the dense set $\{\underline{e}_j\}$, every $\underline{v} \in \mathbf{H}$ has a representation

$$\underline{v} = \sum_k \alpha_k \underline{e}_k$$

with $\alpha_k \in \mathbb{C}$ and $\underline{e}_k \in \mathbf{H}$. The scalar product, section 8.2.1, is

$$\langle \underline{u} | \underline{v} \rangle = \sum_{j,k} \overline{\beta_j} \alpha_k W_{jk}$$

with

$$W_{jk} = \langle \underline{e}_j | \underline{e}_k \rangle$$

and $\underline{u} = \sum_j \beta_j \underline{e}_j$. The linear Hilbert space operators Φ,

$$\Phi \underline{e}_k = \sum_\ell O_{k\ell} \underline{e}_\ell,$$

and Φ^*,

$$\Phi^* \underline{e}_k = \sum_\ell A_{k\ell} \underline{e}_\ell$$

associate with arrays O_{jk} and A_{jk} for the vectors $\underline{e}_j$. Then,

$$\begin{aligned} \langle \underline{u} | \Phi^* \underline{v} \rangle &= \sum_{j,k} \overline{\beta_j} \alpha_k \langle \underline{e}_j | \sum_\ell A_{k\ell} \underline{e}_\ell \rangle \\ &= \sum_{j,k,\ell} \overline{\beta_j} \alpha_k A_{k\ell} M_{j\ell} \end{aligned}$$

if $\underline{v} \in \mathcal{D}_{\Phi^*}$. For the adjoint operator,

$$\begin{aligned} \langle \underline{u} | \Phi^* \underline{v} \rangle &= \langle \Phi \underline{u} | \underline{v} \rangle \\ &= \sum_{j,k} \overline{\beta_j} \alpha_k \langle \sum_\ell O_{j\ell} \underline{e}_\ell | \underline{e}_k \rangle \\ &= \sum_{j,k,\ell} \overline{\beta_j} \alpha_k \overline{O_{j\ell}} M_{\ell k} \end{aligned}$$

if $\underline{v} \in \mathcal{D}_{\Phi^*}$ and $\underline{u} \in \mathcal{D}_\Phi$. From validity of the equality for all α_k, β_k,

$$\begin{aligned} A &= (M^{-1} \overline{O} M)^T \\ &= M^T O^* (M^T)^{-1} \end{aligned}$$

designating the square arrays M_{jk}, O_{jk}, A_{jk} as matrices, O^* as the Hermitian transpose of O, and if M is non-singular. The non-singularity of M follows in a Hilbert space from the linear independence of the $\underline{e}_j$ [27]. For a Hermitian operator, matrices $O = A$ if $\underline{v} \in \mathcal{D}_{\Phi^*} \cap \mathcal{D}_\Phi$ and $\underline{u} \in \mathcal{D}_\Phi$.

8.2.6 Operators in quantum mechanics

Every closed subspace of states corresponds with a projection operator [40]. Fundamental results for Hilbert space operators include the spectral theory for rigged Hilbert space operators [22, 25, 40]; and Stone's theorem [25] for unitarily realized groups of Hilbert space transformations. Spectral theory displays the Hermitian operators that represent observables as limits of linear combinations of projections weighted by representative values of the observable associated with each subspace of states. Projection operators E are bounded, self-adjoint and idempotent, $E = E^* = E^2$, $\|E\psi\| \leq \|\psi\|$. For projections, the range $\mathcal{R}_E = E\mathbf{H} \subset \mathbf{H}$, a proper subset unless $E = \mathbb{I}$. From a projection E, any state $|\psi\rangle \in \mathbf{H}$ may be decomposed as an element $|E\psi\rangle$ within the subspace $E\mathbf{H}$ and an element $|(\mathbb{I} - E)\psi\rangle$ in the orthogonal complement of the subspace, Riesz's theorem.

The states of nature are described by nonnegative ($\langle\psi|\rho\psi\rangle \geq 0$), self-adjoint ($\rho = \rho^*$, $\mathcal{D}_\rho = \mathcal{D}_{\rho^*}$ dense in $\mathbf{H}$), trace-class, normalized operators ρ (Trace$[\rho] = 1$).

$$\text{Trace}[\rho] = \sum_k \langle e_k|\rho e_k\rangle \tag{220}$$

if the $|e_k\rangle$ are an orthonormal basis for the separable Hilbert space $\mathbf{H}$. This basis is not unique and every unitary operator maps a basis to a basis, $|Ue_k\rangle = |e'_k\rangle$. $U^* = U^{-1}$ for a *unitary* operator and unitary operators are bounded. These operators ρ are designated *state density operators* [60] and generalize the vector states generally discussed above. *Vector states* have state density operators $\rho = E$ for E the projection onto a single element $|\psi\rangle$ ($\mathcal{R}_E = \{|a\psi\rangle : a \in \mathbb{C}\}$). Born's rule includes: if a state is described by the state density operator ρ, then the likelihood of observing a state in the subspace $E\mathbf{H}$ corresponding to a projection E given an initial state described by ρ is Trace$[E\rho] \leq 1$.

Operators associated with observables have mean values

$$E[A] = \text{Trace}[A\rho] \in \mathbb{R}$$

for Hermitian operators A and state density operators ρ. Measurements are real numbers and the expectation values of Hermitian operators are real. If A is Hermitian, $A = A^*$ on the domain of A, then for every $\underline{f} \in \mathcal{D}_A$,

$$\begin{aligned}\langle A^*\underline{f}|\underline{f}\rangle &= \langle A\underline{f}|\underline{f}\rangle \\ &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle \underline{f}|A\underline{f}\rangle\end{aligned}$$

from the properties of scalar products and definition of the adjoint operator. Then $A = A^*$ implies that the expectation values $\langle \underline{f}|A\underline{f}\rangle$ are real. And, if the expectation value is real and

$\underline{f} \in \mathcal{D}_A$, then

$$\begin{aligned}\langle \underline{f}|A\underline{f}\rangle &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle A\underline{f}|\underline{f}\rangle\end{aligned}$$

and $A = A^*$ for $\underline{f} \in \mathcal{D}_A$. As a consequence, observables are limited to Hermitian operators.

Significant examples of observables in non-relativistic quantum mechanics are the Hermitian location operators X_ν. In $\mathcal{L}^2$,

$$\langle \psi|\psi\rangle = \int d\mathbf{x}\ |\psi(x)|^2$$

is finite at each time t with $x = ct, \mathbf{x}$. The summation is over three-dimensional space, $\mathbb{R}^3$. Three location operators X_ν, one for each of the three spatial dimensions, are given by

$$X_\nu \psi(x) = x_\nu \psi(x).$$

There is no bound on the values assumed by location in a Euclidean space and consequently the location operator is an unbounded operator. $x_\nu\psi(x)$ is not necessarily square-integrable for square-integrable $\psi(x)$ and as a consequence, $\mathcal{D}_{X_\nu} \subset \mathbf{H}$, a proper subset. Functions $\psi(x) \in \mathcal{L}^2$ such that $x_\nu\psi(x) \in \mathcal{L}^2$ are dense in $\mathcal{L}^2$ but elements of slow decline are excluded from $\mathcal{D}_{X_\nu}$ (e.g., $\psi(x) = (1 + \mathbf{x}^2)^{-1+\delta} \in \mathcal{L}^2$ for $1 \gg \delta > 0$ in three dimensions but $x_\nu\psi(x) \notin \mathcal{L}^2$).

8.2.7 Interpretation of the functions $f_n((x)_n)_{(\kappa)_n}$ in $\underline{f}$

The representation of state (4) is an expansion in powers of fields, and consequently the n-argument functions are not coefficients of n particle states. In the case of free fields, both creation and annihilation contributions are manifest in the expansion (4). An understanding of the functions $f_n((x)_n)_{(\kappa)_n}$ in the state describing function sequences $\underline{f}$ of Borchers' notation (3) for the constructions based on the support limited functions $\underline{\mathcal{P}}$ varies significantly from the descriptions familiar in free field theories and RQFT. Nevertheless, for a limited class of particle-like "macroscopic" state describing functions, interpretation of the state describing functions (3) follows the description of particles from free field theory [24, 25, 52, 61]. When the supports of every argument of a state describing function are "macroscopic," localized and widely space-like separated, then cluster decomposition A.6 provides that the constructed field is approximated by a free field: contributions of the VEV equal free field VEV. In this appendix, these results are illustrated.

The familiar free field example applies when $\underline{\mathcal{S}}$ is the basis space of function sequences. Presented in section 3.2, a free field $\Phi_o(\underline{f})$ is Hermitian (12). For a free field, the scalar product is given by a Wightman functional (13)

$$\langle \underline{h}|\underline{f}\rangle = \mathcal{F}(\underline{h}^* \times \underline{g});$$

the field expands into creation and annihilation operator components

$$\Phi_o(\underline{f}) = \Phi_o^+(\underline{f}) + \Phi_o^-(\underline{f})$$

that have commutation/anti-commutation relations

$$[\Phi_o^-(\underline{f}^*), \Phi_o^+(\underline{g})]_\pm = W_2(\underline{f}^* \times \underline{g})$$

and commute otherwise; and the vacuum state is cyclic with

$$\Phi_o^-(\underline{f})\Omega_o = 0$$

[24, 25, 52, 61]. Notation includes that Φ_o^+ is the creation and Φ_o^- is the annihilation operator, and the only nonzero functions in the arguments $\underline{f}, \underline{g}$ of the field are single argument functions $f_1(x)_\kappa, g_1(x)_\kappa$. $\underline{f}^*$ is from (14). Whether the commutator or anti-commutator is applied depends on the field components labeled by indices κ_j. The statistics of field components are boson or fermion, section 3.2. The algebraic evaluation of the VEV of a free field results in (36),

$$\mathcal{F}_{k,2n-k}((x)_n)_{(\kappa)_n} = \sum_{pairs} \sigma(S,(\kappa)_n) \prod_{j=1}^{n} W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}},$$

a summation over all $(2n)!/(2^n n!)$ ways of pairing indices $i_j, \ell_j \in \{1, 2n\}$, and within each $W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}$ the indices are ordered $i_j < \ell_j$. The $\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ with an odd number of arguments n are zero and $\mathcal{F}_{0,0} = 1$.

Limiting discussion to product functions

$$f_n((x)_n)_{(\kappa)_n} = \prod_{j=1}^{n} f_{1,j}(x_j)_{\kappa_j},$$

elements (5) of the Hilbert space generated from the vacuum by applying products of the field are

$$\begin{aligned} |f_{n,(\kappa)_n}\rangle &= \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{j=1}^{n} \Phi_o(x_j)_{\kappa_j}\Omega\rangle \\ &= | \prod_{k=1}^{n} \Phi_o(f_j)\Omega\rangle \end{aligned}$$

with

$$\Phi_o(f_\alpha) = \int dx \; \Phi_o(x)_{\kappa_\alpha} f_{1,\alpha}(x)_{\kappa_\alpha}$$

and in the abbreviated notation (30) for function sequences $\underline{f}$ with the j-argument functions $f_j = 0$ for $j \neq n$. Expression of the free field in terms of the creation and annihilation operators, annihilation of the vacuum, and the commutation relations result in

$$\begin{aligned} |f_{n,(\kappa)_n}\rangle &= |\prod_{k=1}^{n} \Phi_o(f_j)\Omega\rangle \\ &= |\prod_{k=1}^{n-2} \Phi_o(f_j)(\Phi_o^-(f_{n-1}) + \Phi_o^+(f_{n-1}))\Phi_o^+(f_n)\Omega\rangle \\ &= |\prod_{k=1}^{n-2} \Phi_o(f_j)(\Phi_o^+(f_{n-1})\Phi_o^+(f_n) + W_2(f_{n-1} \times f_n))\Omega\rangle. \end{aligned}$$

In RQFT, products of n creation operators applied to the vacuum describe n particles [24, 25, 52, 61]. Continuation of the evaluation for $|f_n\rangle$ results in an expansion of the element described by state describing functions $f_n((x)_n)_{(\kappa)_n}$ as a summation of descriptions for $n - 2\ell$ particles, $0 \leq n - 2\ell \leq n$ with $\ell \in \mathbb{N}_0$. The n-argument functions $f_n((x)_n)_{(\kappa)_n}$ in (3) for elements of the Hilbert space are the expansion coefficients for products of the quantum field and are not generally interpretable as descriptions of n particles. The functions $f_n((x)_n)_{(\kappa)_n}$ generate a description of a state vector that is a linear combination of descriptions for $k \leq n$ particles and the vacuum if n is even.

However, the interpretation is quite different for the constructions of section 3. Generally there is no unambiguous interpretation of $n \geq 2$, n-argument state describing functions as particles. Applying Born's rule, for the constructions, a state description generally has finite projections onto to all particle descriptions that are kinematically included. If momentum, angular momentum and charges are conserved, then generally two states are not orthogonal due to the connected components of the VEV in the scalar product (8). However, when the supports of the state describing functions satisfy localization (119) and isolation (121), then the states are well approximated by the states of free fields, $\Phi(\underline{f}) \approx \Phi_o(\underline{f})$. The states of free fields have the established interpretation illustrated above. And, if $f_1(x)_\kappa \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$, then the field $\Phi_o(\underline{f})$ is supported only on positive energies due to the zero of $f_1(x)_\kappa$ on the negative energy mass shells, and then $\Phi_o(\underline{f}) = \Phi_o^+(\underline{f})$. $\Phi_o(\underline{f}^*) = \Phi_o^-(\underline{f}^*) = \Phi_o^+(\underline{f})^*$, the operator adjoint. From the lack of support on the negative energy mass shells,

$$W_2(\underline{f}^* \times \underline{g}^*) = W_2(\underline{f} \times \underline{g}) = 0$$

and only $W_2(\underline{f}^* \times \underline{g})$ contribute in an expansion of free field state descriptions if $\underline{f}, \underline{g} \in \mathbf{H}_{\mathcal{P}}$. The interpretation of state describing functions $f_n((x)_n)_{(\kappa)_n}$ as descriptions of n free particles follows from the limited support of the functions in $\underline{\mathcal{P}}$ for "macroscopic" state describing functions that satisfy localization (119) and isolation (121). The cluster decomposition condition A.6 provides that the scalar products of the fields equal the scalar products of free fields if localization and

isolation are satisfied, section 4.3. More generally, the states described by functions $f_n((x)_n)_{(\kappa)_n}$ may have finite projections onto states with distinct numbers and types of particles due to the connected contributions in the constructed VEV.

8.2.8 Observation in Hilbert spaces

The Everett-Wheeler-Graham relative state interpretation of the formalism of quantum mechanics [12] escapes the measurement paradoxes and ad hoc assertions of earlier understandings. The development describes observation from the premise that the "mathematical formalism of quantum mechanics is sufficient as it stands" [12]. This interpretation expedites realization of relativistic quantum mechanics. Non-relativistic concepts such as "collapse of the wave function to an eigenfunction" providing a classical description of the observable are inconsistent with relativity. It would take a period of infinite duration for the support of a typical state describing function such as an energy eigenfunction of the hydrogen atom to collapse to a single point. The nonzero support of the energy eigenfunction is infinite: the speed of light is finite. Causality requires that the time evolution of state describing functions is expressed with operators that satisfy the Poincaré invariance of likelihoods. Time evolution includes measurements. With the observer as well as the observed included in the quantum mechanical description, time evolution is continuous, unitary time translation and there is no need to distinguish which interactions constitute measurement. The relative state description is consistent with our experience as well as consistent with the quantum mechanical description of nature, although it violates longstanding classical intuition.

The process of observation is fundamental in quantum mechanics: observers are not omniscient and the interactions that provide measurements affect the relevant description of the state. This process explains how the perception of nature can differ from the description. In classical physics, the state can be observed without disruption: an observer is external to the systems under consideration and is essentially omniscient. However, in quantum mechanics, observation is an interaction within the quantum description: an observation entangles descriptions of the possible states of an observer with the observed. Observers are also described by quantum mechanics. There is no "macroscopic" or "classical" domain governed by distinct physical principles. A quantum description of state that describes a composite of system under observation and observer evolves into a linear combination that describes distinct alternative possibilities for observation. For example, an initially localized state spreads over space with time and a distinct state of the observer becomes entangled with each possible outcome of a subsequent localizing interaction. It is inconsistent with our experience that there is only one observer state: observers perceive results from among possibilities. To be included in a quantum description and avoid the difficulties described in the Schrödinger cat and EPR measurement paradoxes [51, 17], a distinct state of the observer must be entangled with each of the possibilities for observation. Distinct perceptions are described by distinct states. Within linear expansions of state descriptions, "we" are described by one of the possible states. This relative

state or "many worlds" description [12] derives naturally from the quantum mechanical description of state as elements of Hilbert spaces. Although inherent to quantum mechanics, this natural interpretation is very different from classical concepts and it took several decades after the initial formulations of quantum mechanics for the relative state interpretation to emerge. Hugh Everett III, John Archibald Wheeler, Neill Graham and Bryce DeWitt developed the relative state interpretation of quantum mechanics [12]. Earlier interpretations maintained classical descriptions for observers. Even the designation as "many worlds" betrays a classical predisposition: many classical worlds but a unified, consistent quantum description. That state describing functions expand in linear combinations of other state descriptions is characteristic of Hilbert space descriptions. A result of [12] is that this expansion is equivalent to a mixture, a linear combination of physically independent descriptions. However, none of these descriptions are classically described. That a quantum state description is a distribution over classical state descriptions is a flawed concept inconsistent with the quantum description of nature, section 1. Consideration of the state as a superposition over many classical worlds, each with a state described by classical dynamical variables is an example of a persistent classical predisposition. The relative state interpretation uses the natural quantum mechanical description of states by functions with entanglement leading to a decomposition into relative states labeled by the perceived results of observations.

The quantum mechanical description of observations also explains our comfort with the classical view: often, within the precision of our perceptions, the resulting quantum mechanical description appears to replicate a classical description. However, a richer range of descriptions is included among the descriptions of state in quantum mechanics. A localized interaction results in observation of a localized state; two slits in a screen followed by localizing detectors results in observation of wave-like interference when the initial states are widely supported and propagate toward the screen. In common instances, the perception of a quantum description is indistinguishable from a classical description although in general, in instances such as the uniformly illuminated double slit, a quantum description of state deviates substantially from a classical description. Quantum mechanics supersedes classical descriptions.

The states of an observer are labeled by their perceptions. The relevant final state of the observed is determined by the entangled observer state, the result of an interaction characterized as an observation. The likelihoods of the various possibilities are determined by scalar products, the relevant initial state description, the description of perceived state, and chance. Chance determines our particular perception from among the possibilities. Perceptions such as the track of a planet may be the overwhelming likelihood within our measurement uncertainties, or the likelihoods may deviate substantially from any classical descriptions such as for Young's double slit or the energy levels of an atom. Observation results in a random selection of our perceived state from within a linear expansion of the temporal propagation of the initial state. If $|\underline{\psi}\rangle$ describes an initial state including the description of our history of perceptions, then an interaction of the observer and a system will result in an expansion of the evolved state. The states in the expansion are states labeled by the possibilities for updated histories of the

observer's perceptions,

$$|U(t)\underline{\psi}\rangle = \sum_\theta |E_\theta U(t)\underline{\psi}\rangle = \sum_\theta \sum_k \langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle \, |\underline{g}_{k,\theta}\rangle. \tag{221}$$

The initial state $|\underline{\psi}\rangle$ jointly describes the observer and observed, and the final state expands in entangled descriptions for each of the possible perceived results θ of the observation. Entanglement was introduced in appendix 8.2.1 and entanglement develops in the time evolution of a state that initially has independent descriptions for an observer and an observed. Consistently with the separability of the Hilbert space, the notation in (221) is that this summation is denumerably indexed, our observations have finite resolution. And, the notation applies when the orthogonal subspaces associated with projections E_θ are spanned by a basis of elements $|\underline{g}_{k,\theta}\rangle$, $E_\theta = \sum_k |\underline{g}_{k,\theta}\rangle\langle \underline{g}_{k,\theta}|$. The $|\underline{g}_{k,\theta}\rangle$ jointly describe states of the observer and observed. The orthogonal projections E_θ are onto subspaces of the Hilbert space with descriptions of observers with a history of perceptions θ. Well designed measurements entangle descriptions that approximate eigenfunctions for the measured quantity[p] with the observer's states. Observers often interpret the body's state as an eigenfunction of the quantization of the corresponding classical variable. That is, for example, in a well designed measurement, the observer interprets the perception of a state dominantly supported within a small neighborhood of $\mathbf{x}_o$ as a point-like body located at $\mathbf{x}_o$.

We never observe that recorded observations change, that further observations or communications change the record of past measurements or that the current state is inconsistent with past perceptions. This experience indicates that states labeled by distinct perceptions are orthogonal: the Hilbert space decomposes into orthogonal subspaces labeled by the distinct histories of observations. This decomposition provides no likelihood of transitions from our history of perceptions to a distinct history. The observation may be in error or imperfect, that is, the actual state entangled with an observer may not be the perceived state, but distinct histories of perceptions lie in orthogonal subspaces. Perception do not spontaneously change. Then,

$$E_\theta E_\vartheta = 0 \qquad \text{if} \qquad \theta \neq \vartheta,$$

and $\sum_\theta \; E_\theta = \mathbb{I}$ at least within an orthogonal subspace that includes the initial state $|\underline{\psi}\rangle$ and the states $|\underline{g}_{k,\theta}\rangle$ of interest. Whether this summation includes all possible states in the Hilbert space is not of issue. Our only concern is a particular subspace $|E_\theta U(t)\underline{\psi}\rangle$ that includes our particular history of perceptions, *us*.

The likelihood of adding the observation described by $\underline{g}_{k,\theta}$ to the history of the observer is given by Born's rule,

$$\text{Likelihood=Trace}(E_\theta \rho) = \sum_k |\langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle|^2$$

[p]Discussed in section 2, these eigenfunctions may be distinct from eigenfunctions of the quantizations of the classical dynamical variable.

in the case of a normalized vector state $\rho = |\underline{\psi}\rangle\langle\underline{\psi}|$ and with E_θ the projection onto the subspace with orthonormalized basis $|\underline{g}_{k,\theta}\rangle$.

The Everett-Wheeler-Graham (EWG) interpretation includes a virtual collapse of the full description of nature to a *relative state* entangled with an observer's perception. Demonstrated in [12] and illustrated below, conditional predictions based solely on this relative state agree with predictions using the full description of state. This agrees with our experience; we need only know what we have observed and other possibilities are of no consequence to our ability to predict future outcomes. Alternative "worlds" have no reality for us. The alternatives do not affect our predictions although our experience is one from among many possibilities. This development resolves one of the mysteries of quantum mechanics: why do we never perceive a state in a superposition over contradictory states like a superposition of 'live cat' and 'dead cat' in Schrödinger's cat thought experiment [51]? Our observations entangle us with one or the other state. For the Schrödinger cat thought experiment, the descriptions of final states would be a linear combination of the $\underline{g}_{k,\theta}$ that describes observation of a live cat entangled with a live cat and an undecayed isotope, and an orthogonal state $\underline{g}_{j,\vartheta}$ that describes observation of a dead cat entangled with a dead cat and a decayed isotope. The formalism of quantum mechanics is adequate to describe our experience. There is no need to augment quantum mechanics with state collapse as well as unitary evolution. Early interpretations of quantum mechanics included ad hoc assertions like "collapse of the wave packet" or "hidden variables" in attempts to preserve a classical understanding. Further development has demonstrated that our observations of nature are adequately described by quantum mechanics without such ad hoc augmentations.

Preservation of a "classical domain" are less disruptive to established understanding but incur the famous measurement paradoxes that illustrate the need for a quantum mechanical treatment of measurement. These classical measurement paradoxes include the Einstein-Podolsky-Rosen (EPR) [17], Schrödinger's cat [51], and Wigner's friend [65] paradoxes. The EPR paradox is discussed in appendix 8.6 and illustrates that quantized, conserved quantities can not be classically described. Schrödinger's cat paradox illustrates that the quantum description can not be relegated to a microscopic world and Wigner's friend illustrates the ad hoc nature of assertions necessary to descriptions of measurement as "collapse of the wave packet." The paradoxes emerge from the adoption of contradictory concepts: a classical description for observer with a quantum formalism for dynamics.

The EWG interpretation of quantum mechanics develops the concept of relative (entangled) state. A Hermitian operator A corresponding to an observable has a real expectation values. The conditional expectation is the expected value with states limited to a subspace of the Hilbert space. A Hermitian rigged Hilbert space operator A is represented[q] as

$$A = \sum_k a_k E_k$$

[q] Appropriate limits of these summations are included.

with orthogonal projections E_k from a resolution of unity and $a_k \in \mathbb{R}$ [22, 40]. Eigenvalues a_k are the observed values that follow for states in the subspaces $E_k\mathbf{H}$. The conditional expectation value of the observable A conditioned upon an observer's perception θ is

$$\begin{aligned} E^\theta[A] &= \frac{E[E_\theta A E_\theta]}{E[E_\theta]} \\ &= \frac{\text{Trace}(E_\theta A E_\theta \rho)}{\text{Trace}(E_\theta \rho)} \end{aligned} \tag{222}$$

with ρ the nonnegative, unit trace, state density operator [60] discussed in appendix 8.2.6. The E_θ are the orthogonal projections (221) that project onto subspaces of states that include the observer perceptions labeled by history θ entangled with states of the observed. The E_θ project onto composites of observed plus observer states. Two example constructions of composite states are discussed below.

From the idempotence of projections and transposition invariance of the trace (220) [27], Trace(AB)=Trace(BA),

$$\text{Trace}(E_\theta \rho) = \text{Trace}(E_\theta \rho E_\theta)$$

with $E_\theta \rho E_\theta$ the projection of the state density operator into the subspace labeled by θ. The *relative state density operator* [12], relative to the observer history labeled θ, is

$$\rho^\theta = \frac{E_\theta \rho E_\theta}{E[E_\theta]} \tag{223}$$

normalized to unit trace. The relative state density operators are orthogonal, $\rho^\theta \rho^\vartheta = 0$ if $\theta \neq \vartheta$, and remain orthogonal with time evolution implemented by unitary time translation, $U(t)^* U(t) = 1$. In terms of this relative state density operator, the *conditional expectation* is

$$E^\theta[A] = \text{Trace}(A\rho^\theta).$$

It was argued on physical grounds that states labeled by different histories of perceptions were orthogonal since it is our experience that there is no likelihood of a change to history. As a consequence, there will be a resolution of unity into projection operators E_θ labeled by the perception histories θ.

$$\sum_\theta E_\theta = 1$$

with the θ labeling the possible histories of perceptions, plus one additional orthogonal subspace of all remaining state descriptions. Distinguish this latter projection as E_0. Decomposition of the Hilbert space into orthogonal complements follows from the orthogonality of states with distinct histories, and Riesz's theorem on orthogonal subspaces in a Hilbert space [48].

If the operator A corresponding to an observable quantity is in the commutant of the E_θ from (222), that is, if

$$[A, E_\theta] = 0$$

for each E_θ except possibly with an exception for E_0, then a mixture of the relative state density operators ρ^θ is equivalent to the state density operator ρ. The resolution of unity $\sum E_\theta = 1$, the idempotence of projections, commutation of A and the E_θ, the transposition invariance of the trace (220) and the definition of relative state results in the equivalence.

$$\begin{aligned} E[A] &= \text{Trace}(A\rho) \\ &= \sum_\theta \text{Trace}(E_\theta A\rho) \\ &= \sum_\theta \text{Trace}(E_\theta^2 A\rho) \\ &= \sum_\theta \text{Trace}(E_\theta A E_\theta \rho) \\ &= \sum_\theta \text{Trace}(A E_\theta \rho E_\theta) \\ &= \sum_\theta \text{Trace}(A\rho^\theta) E[E_\theta] \\ &= \sum_\theta E^\theta[A] E[E_\theta]. \end{aligned}$$

Then, using the definition of relative state density operator (223), if an operator A is in the *commutant* of the E_θ, the mixture of relative states,

$$\rho^{eq} = \sum_\theta E[E_\theta]\, \rho^\theta$$

has the same observable properties as the complete state description ρ.

$$\text{Trace}(A\rho) = \text{Trace}(A\rho^{eq}). \tag{224}$$

Each of the relative state density operators ρ^θ provide the expectation values conditioned on a history of observer perceptions θ. The subspace $E_0\mathbf{H}$ is of no interest. This equivalence of a mixture of the mutually orthogonal relative state density operators with the complete description of the state ρ explains why the quantum mechanical description of state is not in contradiction with a classical concept for an observer. Each term in a mixture evolves independently in time: knowledge of the entire state description is not required to propagate the relative states forward in time. Conditional expectations are equivalent to distinguishing one particular history, our history, as a classical observer. All our future observations evolve from our current relative state description ρ^θ. We need not know or account for the "other branches" to predict the future that is relevant to us, the future conditioned on our history. This suggests the "collapse of the wave function" upon observation described in early developments of quantum mechanics [60], but there is no physical distinction between a virtual and actual collapse of the state description ρ to ρ^θ upon observation. An actual collapse would result in

a classical description of the observer but the universality of likelihood conservation, that is, a unitary implementation of time translation, is preserved if the collapse is considered as virtual. The equivalence remains with forward time translation.

$$U(t)\rho\, U(t)^* = \sum_\theta E[E_\theta]\, U(t)\rho^\theta\, U(t)^*.$$

The first example with system observables A and projections E_θ that commute is a tensor product of two Hilbert spaces. The states of the observed are elements of a Hilbert space $\mathbf{H}_1$ and the states of the observer are elements of a Hilbert space $\mathbf{H}_2$. The Hilbert space of interest is the tensor product, $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$. A second example of this commutation is developed in appendix 8.2.9 and is based on the strong cluster decomposition property of the VEV. The second example describes a body that is initially distantly spatially separated from the observer and the commutation is with arbitrarily great likelihood, but inexact since there are no state descriptions of bounded support within $\mathbf{H}_\mathcal{P}$. In both cases, the observables A commute with the projections E_θ.

A *tensor product* is the composition of Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From descriptions of the body $\psi, g \in \mathbf{H}_1$ with scalar product $\langle\psi|g\rangle_1$, and states of the observer described $u, v \in \mathbf{H}_2$ with scalar product $\langle u|v\rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ has states labeled $\psi \otimes u, g \otimes v$ with scalar product

$$\langle \psi \otimes u | g \otimes v\rangle = \langle\psi|g\rangle_1\, \langle u|v\rangle_2.$$

If ${}^1e_k$ and ${}^2e_k$ are orthonormal bases for separable $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively, then an orthonormal basis for the composite Hilbert space has labels

$$e_{kj} = {}^1e_k \otimes {}^2e_j$$

and $\dim(\mathbf{H}) = \dim(\mathbf{H}_1) \times \dim(\mathbf{H}_2)$. Operators defined in $\mathbf{H}_1 \otimes \mathbf{H}_2$ include extensions of the operators from $\mathbf{H}_1$ and $\mathbf{H}_2$. Hilbert space operators in the tensor product include $C = A \otimes B$,

$$\langle \psi \otimes u | C\ g \otimes v\rangle = \langle\psi|Ag\rangle_1\, \langle u|Bv\rangle_2.$$

Then any operators $A \otimes \mathbb{I}$ commute with any $\mathbb{I} \otimes B$. If $\mathbf{H}_1$ includes the descriptions of the observed and $\mathbf{H}_2$ includes the descriptions of the observer, then this example satisfies the assertions above in discussion of the physical equivalence of the mixture of relative states ρ^θ to ρ. For an observation to occur, the time translation $U(t)$ from (221) must couple the constituent Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$. Tensor products demonstrate that the assumed concept of observables for a body in the commutant of the projections onto observer states is realizable.

$\mathcal{B}(\mathbf{H}_1) \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{B}(\mathbf{H}_2)$ demonstrate that there are commuting sets of operators of interest. $\mathcal{B}(\mathbf{H})$ is the set of all bounded operators for the Hilbert space $\mathbf{H}$. That the tensor product composition suffices to define a Hilbert space is discussed further in appendix 8.2.10. A simple example illustrating entanglement and the commutation of observables with projections onto the observer states is in appendix 8.18.

The equivalence of ρ^{eq} and ρ provides that there is no need for a collapse of state description upon measurement. The relative state interpretation is the natural understanding of quantum mechanics, bizarre images summoned by "many worlds" notwithstanding.

8.2.9 Localized observables, separation and independence

The intuitive notion that separation implies independence provides a second example of Hermitian operators A associated with the observed that (nearly) commute with projections E_θ onto states that include a particular history of observer perceptions. Cluster decomposition (74) provides that scalar products factor when the support of the functions that describe the observed and observer are greatly space-like separated. This is suggestive of a tensor product decomposition of Hilbert spaces.

This intuitive, satisfying property that separation impels independence is enabled by consideration of localized observations. Observables essentially limited to bounded, isolated spatial volumes associate observables with bodies. And, if the observed is greatly spatially isolated from the observer, then these observables (arbitrarily nearly) commute with projections onto states that describe the observer. The approximate independence is that

$$0 \approx \|[E_\theta, A]\| \ll \|A\|$$

with A a Hermitian operator representing the localized observable and for projections $\|E_\theta\| = 1$. This commutation is the property used to develop the equivalence of the mixture of relative states with a general state descriptions in the EWG development of observation discussed in appendix 8.2.8. The commutation is only approximate because the supports of state describing functions are not of bounded support: the functions within $\underline{\mathcal{P}}$ are anti-local. In these instances, the equivalence (224) of the mixture of relative states ρ^θ with ρ is to arbitrarily great likelihood.

In the constructions, the cluster decomposition condition (74) provides that the truncated functions defined by cluster expansion (71) are connected functions,

$$^C\mathcal{W}_{k,n-k} = {}^T\mathcal{W}_{k,n-k}.$$

The vanishing of connected functions $^C\mathcal{W}_{k,n-k}$ as the supports of arguments are greatly space-like separated, and satisfaction of cluster decomposition (74) provides that

$$\mathcal{W}_{n,m}((y)_{n+m}) = \mathcal{W}_{j,\ell}((x)_P)\mathcal{W}_{n-j,m-\ell}((x)_{P'})$$

as $\lambda \to \infty$ if $y_{i_k} = x_k$ for $i_k \in P$ and $y_{i_k} = x_k + (0, \lambda \mathbf{a})$ for $i_k \in P' = \{1, n+m\}/P$, the set complement of the $j+\ell$-element set of integers P from $\{1, n+m\}$, with the evident embellishment of the $(x)_n$ notation.

The demonstration is limited to a selected instance. The observed is an elementary particle described by a one-argument function $\psi(x)$. In the final, post measurement phase, the support of $\psi(x)$ is distantly spatially isolated from the support of the description of the observer. $\mathbf{x}_o$ is the time dependent location of the observed. A finite spherical volume centered on $\mathbf{x}_o$ is designated by $V_{\mathbf{x}_o}$. $V_{\mathbf{x}_o}$ includes all but a negligible amount of the support of $\psi(x)$. Assert that there is a choice of eigenfunctions $\{e_\ell(x)\}$ such that: the support of every $e_\ell(x)$ outside of $V_{\mathbf{x}_o}$ is uniformly negligible; and the Hermitian operator $\hat{A}$ associated with the observable is

$$|\hat{A}\underline{f}\rangle = (0, A_1 f(x_1), \ldots, \sum_{j=1}^{n} A_j f_n((x)_n), \ldots) \tag{225}$$

with

$$A_j f_n((x)_n) = \sum_\ell a_\ell e_\ell(x_j) \int dy_1 dy_2 \; W_2(y_1, y_2) \overline{e_\ell(y_1)} f_n(x_1, \ldots x_{j-1}, y_2, x_{j+1}, \ldots x_n).$$

W_2 is the two-point function, and $j \in \{1, n\}$. The a_ℓ are the eigenvalues of the one-argument subspace observable A associated with the eigenfunctions $e_\ell(x)$.

$$A = \sum_\ell a_\ell |e_\ell\rangle\langle e_\ell|$$

and the state describing function expands in a linear combination of eigenvectors

$$\psi(x) \approx \sum_\ell \langle \underline{e}_\ell | \underline{\psi} \rangle \; e_\ell(x). \tag{226}$$

The functions $e_\ell(x)$ are orthonormal,

$$\langle \underline{e}_\ell | \underline{e}_{\ell'} \rangle = \delta_{\ell,\ell'}. \tag{227}$$

The single argument operator A_j is A applied to the jth argument in the n-argument subspace. A is asserted to be bounded and essentially localized within the volume $V_{\mathbf{x}_o}$. A is designated *essentially localized* within $V_{\mathbf{x}_o}$ if cluster decomposition (74) implies that

$$\langle \underline{f} | \hat{A} \underline{f} \rangle \to 0 \tag{228}$$

if the dominant support of a localized state $|\underline{f}\rangle$ is arbitrarily greatly space-like separated from $V_{\mathbf{x}_o}$. These properties restrict the class of observables. For example, the linear harmonic oscillator energy eigenfunctions are localized but not uniformly negligible outside of a bounded

volume: the greater the energy, the broader the support. The volume $V_{\mathbf{x}_o}$ is selected so that the contribution of any $\psi(x)$ or $e_\ell(x)$ of interest beyond $V_{\mathbf{x}_o}$ is arbitrarily negligible. Examples of localized, one argument observables include location with the $e_\ell(x) \approx \delta(\mathbf{x}-\mathbf{x}_\ell)$ and $x_\ell \in V_{\mathbf{x}_o}$.

The second set of operators of interest are projections E_θ onto states that include particular descriptions of the observer. The initial description of the observer is selected to be (nearly) independent of the description of the observed: initially the observer and observed are unentangled with distantly space-like separated supports. E_θ projects a joint, final, post-observation description of observer and observed (221) onto a description with the observer described by a particular new perception added to the history θ. The final state of the evolved observer state is labeled $\underline{h}_0$ and the final observer states with the evolved perceptions are labeled $\underline{h}_\theta$. These descriptions of state do not include description of the observed. The supports of both the $\underline{h}_0$ and $\underline{h}_\theta$ within $V_{\mathbf{x}_o}$ are arbitrarily negligible. Initial and final are long before and long after the interaction that constitutes an observation. The projections onto the final observer states labeled $\underline{h}_\theta$ are

$$E_\theta = \text{orthogonal projection onto the union of the ranges of } E_{\underline{g}_{\ell\theta}} \tag{229}$$

defined from projections

$$|E_{\underline{g}_{\ell\theta}}\, \underline{f}\rangle = \frac{\langle \underline{g}_{\ell\theta}|\underline{f}\rangle}{\langle \underline{g}_{\ell\theta}|\underline{g}_{\ell\theta}\rangle}\, |\underline{g}_{\ell\theta}\rangle$$

onto the product states defined

$$|\underline{g}_{\ell\theta}\rangle = |\underline{e}_\ell \times \underline{h}_\theta\rangle \tag{230}$$

with $\underline{e}_\ell = (0, e_\ell(x_1), 0, \ldots)$ the eigenfunctions of the observable.

Estimates for the commutator of the observed's essentially localized observable A with the projections E_θ onto particular observer states follow from the initial independence, the final distant spatial separation, and cluster decomposition (74). Demonstrated immediately below, the definitions for the observables (225), (227), (226), and (228), projections (229), and joint observer-observed states (230) in the rigged Hilbert spaces of relativistic quantum physics provide that

$$\begin{aligned} |E_\theta\, \underline{\psi} \times \underline{h}_0\rangle &\approx \langle \underline{h}_\theta|\underline{h}_0\rangle\, |\underline{\psi} \times \underline{h}_\theta\rangle \\ |\hat{A}\, \underline{\psi} \times \underline{h}_0\rangle &\approx |(A\underline{\psi}) \times \underline{h}_0\rangle. \end{aligned} \tag{231}$$

The projection onto particular observer states does not affect the description of the observed. The likelihood of a particular final state of the observed is determined by the likelihood that the evolved $\underline{h}_0$ has a nonzero projection onto $\underline{h}_\theta$. The likelihoods of observed values for the localized observable $\hat{A}$ is determined only by the description of the observed.

The demonstration of (231) follows from the scalar product (27).

$$\begin{aligned} \langle \underline{g}_{\ell\theta}|\underline{\psi} \times \underline{h}_0\rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta|\underline{\psi} \times \underline{h}_0\rangle \\ &\approx \langle \underline{e}_\ell|\underline{\psi}\rangle\, \langle \underline{h}_\theta|\underline{h}_0\rangle \end{aligned}$$

as a consequence of the distant space-like final separation of supports and cluster decomposition (74) of the VEV into connected functions. The separation of the volume $V_{\mathbf{x}_o}$ from the final support of the observer also provides that

$$\begin{aligned}\langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta} \rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{e}_\ell \times \underline{h}_\theta \rangle \\ &\approx \langle \underline{e}_\ell | \underline{e}_\ell \rangle \langle \underline{h}_\theta | \underline{h}_\theta \rangle \\ &= 1\end{aligned}$$

with the normalization $\|\underline{h}_\theta\| = 1$. Similarly, $\langle \underline{g}_{\ell\theta} | \underline{g}_{\mu\theta} \rangle \approx 0$ if $\ell \neq \mu$. This approximate orthogonality then provides that

$$E_\theta \approx \sum_\ell E_{\underline{g}_{\ell\theta}} \tag{232}$$

from (229). As a consequence,

$$\begin{aligned}|E_\theta \underline{\psi} \times \underline{h}_0\rangle &\approx \sum_\ell |E_{\underline{g}_{\ell\theta}} \underline{\psi} \times \underline{h}_0\rangle \\ &= \sum_\ell \frac{\langle \underline{g}_{\ell\theta} | \underline{\psi} \times \underline{h}_0 \rangle}{\langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta} \rangle} \, |\underline{g}_{\ell\theta}\rangle \\ &= \sum_\ell \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{\psi} \times \underline{h}_0 \rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\ &\approx \sum_\ell \langle \underline{e}_\ell | \underline{\psi} \rangle \, \langle \underline{h}_\theta | \underline{h}_0 \rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\ &= \langle \underline{h}_\theta | \underline{h}_0 \rangle \, |\underline{\psi} \times \underline{h}_\theta\rangle\end{aligned}$$

from substitution of the expansion (226) of the observed state. E_θ approximates a projection operator with the desired property of projecting onto the observer state of interest independently of the descriptions of the observed.

The form (225) of the local observable $\hat{A}$ provides

$$\begin{aligned}\hat{A}\, \underline{\psi} \times \underline{h}_0 &= (0, A\psi(x_1) f_{o,0}, \ldots \sum_{j=1}^{n} A_j \psi(x_1) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\ &\approx (0, A\psi(x_1) f_{o,0}, \ldots (A\psi(x_1)) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\ &= (A\, \underline{\psi}) \times \underline{h}_0\end{aligned}$$

as a consequence of the distant spatial separations of the support of the $e_\ell(x)$ and $\underline{h}_0$, and cluster decomposition. This completes the demonstration of (231).

From (225), the commutators of the $\hat{A}$ and E_θ follow from

$$\begin{aligned} |\hat{A}E_\theta \ \underline{\psi} \times \underline{h_0}\rangle &\approx \langle \underline{h_\theta}|\underline{h_0}\rangle \ |\hat{A} \ \underline{\psi} \times \underline{h_\theta}\rangle \\ &\approx \langle \underline{h_\theta}|\underline{h_0}\rangle \ |(A\underline{\psi}) \times \underline{h_\theta}\rangle \end{aligned}$$

and

$$\begin{aligned} |E_\theta \hat{A} \ \underline{\psi} \times \underline{h_0}\rangle &\approx |E_\theta \ (A\underline{\psi}) \times \underline{h_0}\rangle \\ &\approx \langle \underline{h_\theta}|\underline{h_0}\rangle \ |(A\underline{\psi}) \times \underline{h_\theta}\rangle. \end{aligned}$$

Then,

$$[E_\theta, \hat{A}] \approx 0$$

for states that describe an observed and observer that are initially unentangled and distantly space-like separated both initially and finally.

This development is not general but illustrates that separation and a lack of entanglement provide independence. This demonstration is an alternative to the tensor product example. This result is enabled within relativistic quantum physics by consideration of observations that are essentially localized, the likely association of locations with the observed, and satisfaction of cluster decomposition (74) in A.6.

8.2.10 Tensor products of linear vector spaces

Discussed in [60] and appendix 8.2.8, a *tensor product* is the composition of linear vector spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From elements described $g_1, g_2 \in \mathbf{H}_1$ with degenerate scalar product $\langle g_1|g_2\rangle_1$, and elements described $f_1, f_2 \in \mathbf{H}_2$ with degenerate scalar product $\langle f_1|f_2\rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ includes states labeled $g_1 \otimes f_1, g_2 \otimes f_2$ with degenerate scalar product

$$\langle g_1 \otimes f_1|g_2 \otimes f_2\rangle = \langle g_1|g_2\rangle_1 \ \langle f_1|f_2\rangle_2. \tag{233}$$

In this appendix, the sufficiency of this assignment to specify a degenerate scalar product on the complete tensor product space is addressed. (233) is evidently a degenerate scalar product of product elements $g_1 \otimes f_1, g_2 \otimes f_2$, but does this degenerate scalar product extend to all elements of $\mathbf{H}$?

First it is established that the Cauchy-Schwarz-Bunyakovsky inequality applies for all product states, and then this result is used to demonstrate that the degenerate scalar product applies to linear combinations of product states. Then, if $\mathbf{H}$ is defined as the completion of the linear span of product functions, then the degenerate scalar product applies to $\mathbf{H}$.

The Cauchy-Schwarz-Bunyakovsky inequality applies for all product states.

$$\begin{aligned}|\langle g_1 \otimes f_1 | g_2 \otimes f_2\rangle|^2 &= |\langle g_1|g_2\rangle_1|^2 |\langle f_1|f_2\rangle_2|^2 \\ &\leq \langle g_1|g_1\rangle_1 \langle g_2|g_2\rangle_1 \langle f_1|f_1\rangle_2 \langle f_2|f_2\rangle_2 \\ &= \langle g_1 \otimes f_1|g_1 \otimes f_1\rangle \langle g_2 \otimes f_2|g_2 \otimes f_2\rangle\end{aligned}$$

from (233) and the Cauchy-Schwarz-Bunyakovsky inequality in the constituent linear vector spaces. As a consequence, (233) extends to linear combinations of product functions.

$$\begin{aligned}|\langle u+v|u+v\rangle|^2 &= \langle u|u\rangle + \langle v|v\rangle + \langle u|v\rangle + \langle v|u\rangle \\ &= \langle u|u\rangle + \langle v|v\rangle + 2\Re e\langle u|v\rangle \\ &\geq \langle u|u\rangle + \langle v|v\rangle - 2\sqrt{\langle u|u\rangle\langle v|v\rangle} \\ &= \left(\sqrt{\langle u|u\rangle} - \sqrt{\langle v|v\rangle}\right)^2 \\ &\geq 0\end{aligned}$$

if the Cauchy-Schwarz-Bunyakovsky inequality applies to the elements u, v. It applies for product elements $g \otimes f$ and then nonnegativity applies to linear combinations of two product elements. Then, if the sum of two product elements has a nonnegative scalar product, then the sum of three also does by a similar argument. Hence, all linear combinations of product elements are nonnegative. The degenerate scalar product extends to an complete tensor product space if the space is defined as follows.

A denumerable basis of elements e_{nm} is defined from products of basis elements for the constituent Hilbert spaces

$$e_{nm} = {}^1e_n \otimes {}^2e_m$$

with ${}^1e_n$ a basis for the separable $\mathbf{H}_1$ and ${}^2e_m$ a basis for the separable $\mathbf{H}_2$. Then (233) provides that

$$\langle e_{nm}|e_{k\ell}\rangle = \delta_{n,k}\delta_{m,\ell}$$

with Kronecker deltas. The e_{nm} are a basis for $\mathbf{H}_1 \otimes \mathbf{H}_2$ defined as the completion $\mathbf{H}$ of all linear combinations of product states $g \otimes f$ in the norm from (233). Nonnegativity extends to all convergent limits.

$$\langle u|u\rangle = \langle \sum_{n,m} a_{nm}e_{nm}| \sum_{k,\ell} a_{k\ell}e_{k\ell}\rangle = \sum_{n,m}\sum_{k,\ell} \overline{a_{nm}}a_{k\ell}\delta_{n.k}\delta_{m,\ell} = \sum_{n,m} |a_{nm}|^2$$

that is manifestly nonnegative.

8.3 Translations, location operators and relativistically invariant localized states

Hermitian Hilbert space operators that correspond to location are discussed in this appendix.

Discussed below, spatial translations of the Fourier transforms $\tilde{f}_n((p)_n)$ of functions are generated by p_ν and momentum translations of $f_n((x)_n)$ by x_ν are generated by x_ν. Translation invariance of the scalar product and Stone's theorem leads to the conclusion that momenta are densely defined Hermitian operators. However, in relativistic physics, the scalar product is not invariant to translations in momenta. In a non-relativistic development, the $\mathcal{L}^2$ scalar product is invariant to translations in momenta but in a relativistic development, the universality of the speed of light dictates a Lorentz invariant scalar product. As a consequence, x_ν does not generate a unitarily implemented symmetry in a relativistic development. If multiplication by x_ν were a densely defined Hermitian operator, then translations in momentum would be unitarily implemented and the scalar product would be momentum translation invariant. Multiplication of functions by x_ν can not correspond to the location operator in relativistic physics since it cannot be Hermitian, and as developed in appendix 8.2.6, only Hermitian operators are associated with observables. It is demonstrated in section 7 that the operator X_ν that quantizes x_ν is not Hermitian. Despite its commutation with the momentum operators, the "position" operators X_ν are not Hermitian for the relativistic scalar product (27).

From the properties (24) of the Fourier transform, translation of a function corresponds to multiplication of the Fourier transform by e^{-ipa}. Recall that $px = p_0ct - \mathbf{k}\cdot\mathbf{x}$, a Lorentz invariant. For both free fields and the constructions, the Hamiltonian is $H = \hbar c p_0$ [33]. From the scalar product (27), translation of the function to $x - a$ corresponds to translation of a field by $x + a$. Fourier transforms correspond

$$e^{-ipa}\tilde{\psi}(p) \leftrightarrow \psi(x-a)$$

and then for sufficiently small $\|a\|$, Taylor theorem polynomial approximation of the exponential function and $\psi(x)$ provide that

$$(1 - ipa)\tilde{\psi}(p) \leftrightarrow \psi(x) - a\cdot\frac{d\psi(x)}{dx}.$$

Then the operator P_ν that corresponds to multiplication of the Fourier transform by p_ν is

$$P_\nu = -i\hbar g_{\nu\nu}\frac{d}{dx_\nu}.$$

The Minkowski signature g is from (25). With

$$U(a)\psi(x) = \psi(x-a),$$

the translation invariance of the scalar product provides that translations $U(a)$ are unitary and the generators,

$$U(a) = \exp(-iap),$$

are the energy-momentum operators $P_\nu = \hbar p_\nu$. From Stone's theorem, the energy-momentum operators are densely defined and Hermitian. P_ν is evidently unbounded. Similarly, multiplication of functions by x_ν generates translations in energy-momenta. Fourier transforms correspond

$$e^{iqx}\psi(x) \leftrightarrow \tilde{\psi}(p-q)$$

and then for sufficiently small $\|q\|$,

$$(1+iqx)\psi(x) \leftrightarrow \tilde{\psi}(p) - q \cdot \frac{d\tilde{\psi}(p)}{dp}.$$

Then the operator X_ν that corresponds to multiplication of a function by x_ν is

$$X_\nu = ig_{\nu\nu}\frac{d}{dp_\nu}.$$

With

$$T(q)\tilde{\psi}(p) = \tilde{\psi}(p-q),$$

if the scalar product were energy-momentum translation invariant then the

$$T(q) = \exp(iqX)$$

would be unitary and its Hermitian generators would be the location operators X_ν. The correspondence of X_ν and P_ν with location and momentum respectively is established in the interpretation of the support of the arguments of functions as spacetime coordinates and of the Fourier transforms as energy-momenta in units of wavenumber. Considering only spatial coordinates, $\nu = 1, 2, 3$,

$$X_\nu = x_\nu, -i\frac{d}{dp_\nu} \qquad \text{and} \qquad P_\nu = i\hbar\frac{d}{dx_\nu}, \hbar p_\nu \tag{234}$$

that apply to functions or the Fourier transforms of functions, respectively. Signs are determined by the convention for the Fourier transform (22). The X_ν and P_ν given by (234) canonically commute. But the properties of X_ν and P_ν as Hilbert space operators depend on the Hilbert space realizations, that is, depend on the scalar product. In the Hilbert space $\mathcal{L}^2$ appropriate for non-relativistic physics [52], the scalar product is both spacetime and energy-momentum translation invariant and X_ν and P_ν are both densely defined Hermitian operators. For a relativistic scalar product, only the P_ν are Hermitian since in relativistic physics, there is only translation invariance of the scalar product. Demonstrated in section 7, multiplication by x_ν is not realized as a Hermitian Hilbert space operator for the relativistically invariant scalar products of elementary particle states.

The identification of X_ν as the location operator and P_ν as the momentum operator in non-relativistic physics results in the Born-Heisenberg-Jordan relation for their commutator.

$$[X_\nu, P_\nu] = -i\hbar.$$

This commutation implies the Heisenberg uncertainty relation as discussed in appendix 8.7 and a Baker–Campbell–Hausdorff identity implies that similarity transforms of the location operators translate the eigenvalues.

$$\begin{aligned} e^{-ia_\nu P_\nu/\hbar} X_\nu e^{ia_\nu P_\nu/\hbar} &= X_\nu + \frac{ia_\nu}{\hbar}\,[X_\nu, P_\nu] \\ &= X_\nu + a_\nu. \end{aligned} \tag{235}$$

As a consequence, there is a simultaneous eigenfunction of the X_ν for every $\mathbf{a} \in \mathbb{R}^3$ given a function that is in the union of the null spaces of the X_ν, $\nu = 1, 2, 3$. From $X_\nu \psi_0(\mathbf{x}) = 0$,

$$\psi_{\mathbf{a}}(\mathbf{x}) = e^{ia_1 P_1/\hbar} e^{ia_2 P_2/\hbar} e^{ia_3 P_3/\hbar} \psi_0(\mathbf{x}) \qquad \text{satisfies} \qquad X_\nu \psi_{\mathbf{a}}(\mathbf{x}) = a_\nu \psi_{\mathbf{a}}(\mathbf{x}).$$

This result followed solely from the Born-Heisenberg-Jordan relation and there are many other differential operators $\widehat{X}_\nu$ that also canonically commute with the momentum operators P_ν. These differential operators $\widehat{X}_\nu$ also have eigenfunctions associated with every location in $\mathbb{R}^3$.

From Clairaut's theorem, the operations

$$\widehat{X}_\nu = -iu(p)\frac{d}{dp_\nu}u^{-1}(p) \tag{236}$$

mutually commute as long as $u(p)$ is twice continuously differentiable. The $\widehat{X}_\nu$ canonically commute with the energy-momentum operators,

$$[\widehat{X}_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$$

with $\nu, \mu = 1, 2, 3$. Then, there is a class of differential operators $\widehat{X}_\nu$ parameterized by twice continuously differentiable functions $u(p)$ that satisfy the Born-Heisenberg-Jordan commutation relations $[\widehat{X}_\nu, P_\nu] = -i\hbar$ (241) and that mutually commute, $[P_\nu, P_\mu] = [\widehat{X}_\nu, \widehat{X}_\mu] = 0$. The operations potentially correspond with observables only if they are Hermitian in the scalar product of the Hilbert space realization of interest.

The location operators $\widehat{X}_\nu$ are determined by the Hilbert space realization of quantum mechanics. Three distinct sets of location operators are developed in this appendix: the elevation of x_ν that applies in a canonical quantization of non-relativistic physics (234); an $\widehat{X}_\nu$ for the relativistic free field realized in Fock space; and an $\widehat{X}_\nu$ for the relativistic constructions based upon the basis function spaces $\underline{\mathcal{P}}$ from section 3.4. The eigenfunctions of these operators are labeled by locations $\mathbf{a} \in \mathbb{R}^3$, are orthogonal for distinct locations as a consequence of Hermiticity

[3], and are dominantly supported near each location **a** although they are not of point support in the relativistic cases. In relativistic physics, the lack of negative energy support and Lorentz covariance of the states implies that Dirac delta functions are not generalized eigenfunctions of a Hermitian operator within the Hilbert space.

The evident correspondence of particles with state descriptions provides that location operators apply for descriptions of free fields [31, 61]. A free field decomposes into canonically commuting particle creation and annihilation operators; every state can be represented as a linear combination of particle creation operators applied to the vacuum state. When there is interaction, as discussed in section 3.1, these location operators apply to states in the one particle subspace and more generally provide a location for indeterminant bodies. If high order connected functions contribute significantly, then locations do not associate with specific species or numbers of particles. The extension of a single particle operator to multiple particle states (21) is a second quantization that follows (225) for the constructions when interaction is lacking. The discussion of section 3.1 illustrates that general state descriptions are not associated with determined numbers nor types of particles. A particle interpretation is a classically inspired perception of the quantum description of state. Generally, the quantum descriptions decompose as linear combinations of states interpretable as particles.

In (236), $u(p) = 1$ is the Hermitian location operator X_ν in the Hilbert space $\mathcal{L}^2$ applicable in non-relativistic physics. The eigenfunctions of the X_ν are Dirac delta functions and ideally associated with classical locations. More generally, in non-relativistic limits, $\|\hbar\mathbf{p}\| \ll mc$, $\widehat{X}_\nu \approx X_\nu$ if $u(p) \approx u(mc/\hbar, 0, 0, 0)$. However, as discussed in section 2, eigenfunctions for a Hermitian operator are orthogonal in the Hilbert space scalar product.

The Hermitian operator associated with location is in the form (240) derived from the association of classical bodies with states that are dominantly supported within small isolated volumes. From the discussion of Heisenberg's uncertainty principle in appendix 8.7, product states with factors

$$\tilde{f}(p) = \frac{(p_0 + \omega)\tilde{\varphi}(\mathbf{p})}{\sqrt{2\omega}}$$

have scalar products that coincide with the $\mathcal{L}^2$ scalar product of the $\tilde{\varphi}(\mathbf{p})$ in the one-particle subspace or more generally for states that are well-described by classical particles. Using the translates (235), and a selection of Dirac delta sequences

$$\tilde{\psi}_0(\mathbf{p}) = \omega^{1/2} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \omega^{1/2}$$

with $L \to 0$ for $\varphi(\mathbf{x})$ result in the relativistically invariant localized functions of [43]. For the relativistic free field construction based upon the basis function spaces $\underline{\mathcal{P}}$ [31], the Hermitian operator that corresponds to location has the additional constraint that the eigenfunctions must not be supported on negative energy mass shells.

$$\tilde{\psi}_0(\mathbf{p}) = \frac{(p_0 + \omega)}{\sqrt{2\omega}} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \frac{(p_0 + \omega)}{\sqrt{2\omega}}.$$

That is, $\tilde{\psi}_0(\mathbf{p}) \in \underline{\mathcal{P}}$ is required. The translations (235) produce state describing functions labeled by distinct locations $\mathbf{a} \neq \mathbf{a}'$ and these functions become orthogonal as $L \to 0$. There is a (generalized) eigenfunction associated with every location $\mathbf{a} \in \mathbb{R}^3$. In both cases, the functions $\tilde{\psi}_{\mathbf{a}}(\mathbf{p})$ are functions in the constructed Hilbert space $\mathbf{H}_{\mathcal{P}}$ when L is finite and nonnegative.

If $u(p) = \omega^{1/2}$ or $u(p) = (p_0 + \omega)/\omega^{1/2}$ with ω from (10), then the resulting $\widehat{X}_\nu$ is Hermitian in the one particle subspace for the relativistic scalar product. From (35),

$$\begin{aligned}
\langle f_1 | \widehat{X}_\nu g_1 \rangle &= \int \frac{dp}{2\omega}\, \delta(p_0 - \omega) \overline{\tilde{f}_1}(p) \left(-i\omega^{1/2} \frac{d}{dp_\nu}\, \omega^{-1/2} \tilde{g}_1(p) \right) \\
&= -i \int \frac{d\mathbf{p}}{2}\, \omega^{-1/2} \overline{\tilde{f}_1}(\omega, \mathbf{p}) \frac{d}{dp_\nu}\, \omega^{-1/2} \tilde{g}_1(\omega, \mathbf{p}) \\
&= i \int \frac{d\mathbf{p}}{2}\, \left(\frac{d}{dp_\nu}\, \omega^{-1/2} \overline{\tilde{f}_1}(\omega, \mathbf{p}) \right) \omega^{-1/2} \tilde{g}_1(\omega, \mathbf{p}) \\
&= \langle \widehat{X}_\nu f_1 | g_1 \rangle
\end{aligned} \tag{237}$$

from integration by parts.

An alternative method to associate $\widehat{X}_\nu$ with location is to set $\widehat{X}_\nu$ equal to a Hermitian operator derived from X_ν.

$$\widehat{X}_\nu = \frac{1}{2}(X_\nu + X_\nu^*)$$

with the adjoint operators X_ν^* defined for the free field scalar product (237). Then

$$\begin{aligned}
\widehat{X}_\nu \tilde{f}_1(p) &= -\frac{i}{2} \left(\frac{d}{dp_\nu} + \omega \frac{d}{dp_\nu} \omega^{-1} \right)\, \tilde{f}_1(p) \\
&= -ir \left(\tilde{f}_1'(p) - \frac{\omega'}{2\omega}\, \tilde{f}_1(p) \right) \\
&= -i\omega^{1/2} \frac{d}{dp_\nu}\, \omega^{-1/2} \tilde{f}_1(p)
\end{aligned}$$

from $\langle X_\nu^* f_1 | g_1 \rangle = \langle f_1 | X_\nu g_1 \rangle$, (237), integration by parts, and with the prime designating differentiation with respect to p_ν. However, this method is not general. In this case of location, the adjoint shares a dense domain with X_ν. The adjoint of the field $\Phi(\underline{f})^*$ from (31) is generally undefined for the constructions and $\widehat{\Phi}(\underline{f}) = \frac{1}{2}(\Phi(\underline{f}) + \Phi(\underline{f})^*)$ is then not Hermitian.

The relativistic location eigenfunctions, the relativistically invariant localized functions, are orthogonal. The two-point function for a free field (35) is expressed using the Fourier transforms,

$$W_2(\psi^* g) = \int \frac{d\mathbf{p}}{2\omega}\, \overline{\tilde{\psi}(\omega, \mathbf{p})}\, \tilde{g}(\omega, \mathbf{p})$$

and then the relativistically invariant localized functions centered on distinct locations $\mathbf{a}$ and $\mathbf{a}'$ are orthogonal.

$$\begin{aligned} W_2(\psi^*_{\mathbf{a}}\psi_{\mathbf{a}'}) &= \tfrac{1}{2}\int d\mathbf{p}\; e^{-i\mathbf{p}\cdot(\mathbf{a}'-\mathbf{a})} \\ &= \tfrac{1}{2}(2\pi)^3\;\delta(\mathbf{a}-\mathbf{a}'). \end{aligned}$$

To characterize the spacetime support of $\tilde{\psi}_{\mathbf{a}}(\mathbf{p})$, [43] uses a transform with a Lorentz covariant measure supported solely on the mass shell,

$$\begin{aligned} f_{\mathbf{a}}(x) &= 2\lambda_c^2\int \frac{dp}{(2\pi)^{3/2}}\;\theta(E)\delta\left(p^2-\lambda_c^{-2}\right)\;e^{ipx}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) \\ &= \lambda_c^2\int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\;\frac{1}{\omega}\;e^{i\omega ct-i\mathbf{p}\cdot\mathbf{x}}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}). \end{aligned}$$

In the limit $L \to 0$ and for $t = 0$, $\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) = \sqrt{\lambda_c\omega}\, e^{i\mathbf{p}\cdot\mathbf{a}}$ and

$$f_{\mathbf{a}}(x) = F(r/\lambda_c)$$

with

$$F(r/\lambda_c) = c_F\,(\lambda_c/r)^{5/4}H^{(1)}_{5/4}(ir/\lambda_c), \tag{238}$$

a Hankel function $H^{(1)}_{5/4}(z)$, dimensionless real constant c_F, and $r^2 = (\mathbf{x}-\mathbf{a})^2$ [1, 43]. The dominant support of $F(r/\lambda_c)$ is near $r = 0$ due to divergence as $r^{-5/2}$ at the origin and a rapid decline bounded by $\exp(-r/\lambda_c)$ at large r. λ_c is the reduced Compton wavelength. The relativistically invariant localized functions $f_{\mathbf{a}}(x)$ do not equal zero within any spatial neighborhood.

By the characterizations developed in section 2, states such as relativistically invariant localized states would typically be perceived as localized states despite their lack of bounded support. The relativistically invariant localized states are essentially localized within a volume of radius proportional to the Compton wavelength of the body, 3.0×10^{-13} m for an electron.

8.4 Location, a prototype correspondence in relativistic quantum physics

Location contradicts a canonical quantization in relativistic physics. Location is a classical dynamical variable represented by a spatial argument of a state describing function in a canonical quantization. Elevation of location to quantum mechanical operator is not Hermitian in relativistic physics [43, 66]. Location demonstrates that a canonical quantization is not generally available. One significant distinction between relativistic and non-relativistic physics is that the scalar product is invariant to velocity shifts in a non-relativistic development but a universal

and finite speed of light precludes velocity shift invariance in relativistic developments. The elevation of $\mathbf{x}$ is Hermitian in non-relativistic but not relativistic physics as a consequence.[r] The elevation of location is the Hermitian location operator in non-relativistic physics, but a Hermitian elevation of location is precluded by relativity even for free fields, [43, 66] and section 7. Yet, clearly, location remains an observable.

The quantization of location, also designated the elevation of location, is the Hilbert space operator with eigenfunctions that are Dirac delta functions over space. Every point in $\mathbb{R}^3$ is an eigenvalue of the quantization of location and the eigenfunctions are generalized eigenfunctions. The number of orthogonal generalized eigenfunctions of location is uncountable and these eigenfunctions can therefore not be elements of a separable rigged Hilbert space, section 7. The point support of a Dirac delta function corresponds with the classical concept of location as a point in $\mathbb{R}^3$. The Hermitian Hilbert space operator that corresponds to location is distinct from the canonical quantization of location. Canonical quantizations are not necessarily Hermitian. From [14, 60] and appendix 8.2.6, to correspond to a classical dynamical variable, that is, to have a real expected value for all state descriptions, a Hilbert space operator must be Hermitian. However, the relativistic Hilbert space scalar product is not compatible with eigenfunctions that are delta functions. Eigenfunctions of a Hermitian Hilbert space operator with distinct eigenvalues are necessarily orthogonal in the Hilbert space scalar product [3]. A relativistic scalar product has a Källén-Lehmann form two-point function to achieve the physically necessary properties of Poincaré invariance and positive energies. Dirac delta functions are not orthogonal in this scalar product.

$$\langle f(x_0)\delta(\mathbf{x}-\mathbf{y}_1)|f(x_0)\delta(\mathbf{x}-\mathbf{y}_2)\rangle \neq 0 \qquad \text{when} \quad \mathbf{y}_1 \neq \mathbf{y}_2.$$

Hence, Dirac delta functions over space can not be the eigenfunctions of a Hermitian operator in relativistic physics [3, 43, 66].

With the relaxed, approximate and conditional quantum-classical correspondence, there are Hermitian operators $\widehat{X}_\nu$ that correspond to location in relativistic physics. The eigenfunctions of these operators are Theodore Newton and Eugene Wigner's relativistically invariant localized functions [43]. In the momentum domain, the relativistically invariant localized functions $\psi_x(x_1)$ [43] are

$$\tilde{\psi}_x(p) = (2\omega)^{\frac{1}{2}}\ e^{-ipx} \tag{239}$$

labeled by spacetime points $x \in \mathbb{R}^4$. These functions are generalized eigenfunctions of a Hermitian operator defined in the Hilbert space for a relativistic free field. There is an eigenfunction with eigenvalue $\mathbf{x}$ for every location $\mathbf{x} \in \mathbb{R}^3$. For non-relativistic momenta, $\tilde{\psi}_x(p) \approx (2m)^{\frac{1}{2}}\ e^{-ipx}$ and in this sense the relativistically invariant localized functions approximate Dirac delta functions. For the constructions, one-argument functions always describe a single elementary parti-

[r] As a consequence of $\langle e^{i\mathbf{q}\cdot\mathbf{x}}f(x)|e^{i\mathbf{q}\cdot\mathbf{x}}g(x)\rangle = \langle f(x)|g(x)\rangle$ for the $\mathcal{L}^2$ scalar product applicable in non-relativistic physics, the generator of velocity shifts, the quantization of $\mathbf{x}$, is Hermitian. The scalar product in relativistic physics uses a Källén-Lehmann form two-point function [10] and then $e^{i\mathbf{q}\cdot\mathbf{x}}$ is not unitary.

cle and the $\widehat{X}_\nu$ are location operators for that elementary particle in the one-particle subspace. In subspaces with a greater number of arguments and with relativistic momenta in a multiple species construction, the $\widehat{X}_\nu$ are not associated with a determined number nor species of particles. These functions are also discussed in appendix 8.3.

Hermitian location operators $\widehat{X}_\nu$ are described by spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [22], lemma 5.6.7 [40], and chapters 7-10 [25]). Restricted to the one-argument subspace within $\mathbf{H}_\mathcal{P}$, the three Hermitian location operators are

$$\widehat{X}_\nu = \int_{\mathbb{R}^3} d\mathbf{x}\; x_\nu E_x. \tag{240}$$

$E_x \sim |\psi_x\rangle\langle\psi_x|$ projects one-argument functions to the relativistically invariant localized functions located near $\mathbf{x}$ at time $x_0 = ct$ (239). $\nu = 1, 2, 3$, $x = x_0, \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^3$ and x_ν are the three Cartesian components of $\mathbf{x}$. The relativistically invariant localized functions are mutually orthogonal, $\langle\psi_{y_1}|\psi_{y_2}\rangle = 0$ if $\mathbf{y}_1 \neq \mathbf{y}_2$ and $y_{10} = y_{20}$. The relativistically invariant localized functions are generalized eigenfunctions of the $\widehat{X}_\nu$.

$$\widehat{X}_\nu \psi_x(x_1) = x_\nu \psi_x(x_1),$$

and

$$Q_\chi = \int_\chi d\mathbf{x}\; E_x$$

are projection operators that provide a resolution of unity. $d\mathbf{x}$ is Lebesgue measure on $\mathbb{R}^3$, and the χ refer to measurable volumes within $\mathbb{R}^3$. From the orthogonality of the $\psi_x(x_1)$, $E_x E_{x'} = 0$ if $x_0 = x'_0$ and $\mathbf{x} \neq \mathbf{x}'$. The Q_χ are projections (idempotent, self-adjoint operators)[s] in the one particle subspace of $\mathbf{H}_\mathcal{P}$. The functions $\psi_x(x_1)$ are essentially localized but not strictly localized. The ranges of the projections Q_χ are subspaces of positive energy, Poincaré covariant, essentially localized states and orthogonal projection operators are Birkhoff and von Neumann's experimental propositions [7, 60].

There are many first order linear differential operators that canonically commute with the Hermitian momentum operators P_ν. Examples are constructed in appendix 8.3 and include: the Hermitian quantization X_ν of $\mathbf{x}$ applicable in $\mathcal{L}^2$ Hilbert spaces of non-relativistic physics; the Hermitian operator $\widehat{X}_\nu$ that has Newton and Wigner's relativistically invariant localized functions [43] as eigenfunctions and applies in Fock space developments of the relativistic free field; and the Hermitian operator that applies for the section 3 construction of a Hilbert space based on $\underline{\mathcal{P}}$.

[s] $E_x E_{x'} = 0$ if $\mathbf{x} \neq \mathbf{x}'$ and $t = t'$, and then $Q_\chi Q_{\chi'} = 0$ if $\chi \cap \chi' = \emptyset$ for spatial volumes χ, χ' within a constant time plane. The functions $\psi_x(x_1)$ and $\psi_{x'}(x_1)$ are orthogonal at coincident times. However, evaluated at distinct times $t \neq t'$, $E_x E_{x'} \neq 0$. If there were $\mathcal{Q}_\Delta$ such that $\mathcal{Q}_\Delta \mathcal{Q}_{\Delta'} = 0$ for projections onto spacetime volumes $\Delta, \Delta' \subset \mathbb{R}^4$ with $\Delta \cap \Delta' = \emptyset$ and space-like separated, then those $\mathcal{Q}_\Delta = 0$ [66]. There are no such projections. The volumes with $Q_\chi Q_{\chi'} = 0$ are insufficient to conclude from Bogolubov's edge of the wedge theorem that $Q_\chi = 0$.

For free fields, the Hermitian location operator $\widehat{X}_\nu$ in (240) that applies in the one-particle subspace extends to multiple argument states by second quantization. If the interaction of states is significant then multiple-argument components of states are not necessarily interpretable as determined numbers or species of particles, section 3.1. The number and species of particles and even whether the state is perceived as particles is not evident unless interaction is negligible for the state of interest. That is, a free field location operator necessarily applies approximately only for states that are well-represented by classical particles. Free field and non-relativistic states are readily interpreted as consisting of particular numbers and species of particle. Otherwise, the dominant supports of multiple argument functions correspond to likely locations but the numbers and species of particles at those locations is indeterminant.

Relativistic location illustrates a key difference between the less constrained, alternative development and a canonical quantization: the distinction between an elevation of a classical dynamical variable and a Hermitian operator that corresponds to the classical dynamical variable. For location, three quantities are distinguished: $\mathbf{x}$, X_ν, and $\widehat{X}_\nu$. The vector $\mathbf{x} \in \mathbb{R}^3$ specifies a classical location and a point in the domain of the state describing functions. The X_ν for $\nu = 1, 2, 3$ are elevations of the three components of $\mathbf{x}$, operators with Dirac delta functions as generalized eigenfunctions. And, the $\widehat{X}_\nu$ are Hermitian Hilbert space operators with generalized eigenfunctions that are the relativistically invariant localized functions of Newton and Wigner [43]. These generalized eigenfunctions describe the natural generalization of localized states in a relativistic development. The "elevation of c-number to q-number" conjecture is that the X_ν should equal $\widehat{X}_\nu$. The contradiction to the Hermiticity of X_ν in relativistic physics, included in section 7, is a "localization problem" of RQFT. The relativistically invariant localized functions conditionally approximate Dirac delta functions. That a small neighborhood of a location $\mathbf{x}$ includes all perceptions of location to arbitrarily great likelihood suffices physically as the description of localized. Location in relativistic physics is also discussed in [43, 66], section 3.1, and appendix 8.3. The "localization problem" as well as the lack of nontrivial realizations in relativistic quantum physics are overcome by adopting more appropriate relativistic quantum-classical correspondences.

8.5 Inconsistency of the classical description with nature

Quantum mechanics is a striking change from classical descriptions. Several competing schools of thought persist on whether quantum mechanical descriptions actually depict nature. The difficulty for many is that quantum mechanics forces us to abandon well-developed classical intuition.

The considerations listed below require a general abandonment of classical concepts. The conflict of classical concepts with nature is illustrated by many considerations including:

1. Planck's calculation of the spectrum of black body radiation

2. the heat capacity of solids

3. the photoelectric effect
4. Stokes fluorescence and Compton scattering
5. Gibb's paradox
6. the creation and annihilation of particles
7. interference patterns in a Michelson interferometer at very low "single quantum at a time" light intensity
8. discrete energy bands in the radiation spectra from atoms
9. the Einstein-Podolsky-Rosen paradox for conserved quantities such as angular momentum, discussed in appendix 8.6.

Taken together, these considerations do not rectify with a classical world view. Each is a motivation for quantum mechanics, and together with principles of simplicity and universality, a quantum mechanical description for nature is indicated.

For consistency with observations, Max Planck's calculation for the spectrum of black body radiation included that the energy in electromagnetic radiation is quantized in discrete particle-like amounts with an energy E proportional to photon frequency ν, $E = h\nu$. h became known as Planck's constant. Planck's revelation and Albert Einstein's insight that the photoelectric effect is also explained by a quantized photon energy produced agreement with observation. The observed photoelectric effect is strong evidence for the interpretation of electromagnetic radiation as photons. In the photoelectric effect, electrons are not emitted from a surface until the light frequency is sufficient, that is, until the light quanta include sufficient energy that the dominant reaction, interaction of an electron with a single photon, results in an electron with sufficient energy to escape the surface. This observation is (nearly) independent of the amplitude of the incident radiation. The exception is nonlinear optics, if the electron absorbs the energy of multiple photons. These results contradict the classical description of electromagnetism as waves and energy as a freely specified real parameter. If the illumination is a wave, we should anticipate that once the amplitude of the wave was sufficient that electrons would escape the surface. In his identification of the quantized energy $E = h\nu$ of an electromagnetic field, Einstein was also motivated by the observed heat capacity of solids, in particular, contradictions to the equipartition of energy in classical statistical mechanics, and Stokes fluorescence [55], that the frequency of a photon emitted by a body was less than the illuminating frequency. (Stokes fluorescence has since been supplemented with observations of anti-Stokes radiation, when vibrational modes of an emitter contribute to the energy of the re-emitted photon.) Together with Joseph John Thomson's observations of the electron, Robert Milliken and Harvey Fletcher's measurement of a discrete electric charge, and Jean Perrin's observations and Albert Einstein's description of Brownian motion, these findings provided motivation for the description of matter as a composition of atoms.

The indistinguishability of bodies of the same mass, spin, polarization and charges naturally resolves Gibb's paradox. Gibb's paradox arises if entropy is not an extensive quantity. In statistical physics, an extensive quantity is proportional to the amount of substance, that is, for an extensive entropy, twice the volume of gas should have twice the entropy. Were entropy not extensive, then Gibbs paradox is a violation of the second law of thermodynamics, that entropy of isolated systems does not decrease. Other solutions have been suggested to resolve Gibb's paradox, but the indistinguishability of bodies is a natural resolution consistent with the Bose-Einstein or Fermi-Dirac statistics of similarly described bodies in quantum mechanics. Note that this indistinguishability is in contradiction to the classical concept that the trajectories of individual bodies can be distinguished and followed.

The creation and annihilation of particles is another contradiction to the classical concept of continuous trajectories. In these cases, trajectories disappear and trajectories with distinct descriptions appear. Such transformations of identifiable bodies are not described by classical physics. With creation and annihilation, one cannot follow the trajectory of a single identifiable entity, nor generally even determine the number of bodies in a relativistic state description.

A Michelson interferometer consists of two light paths split and later recombined using a half-silvered mirror, two reflecting mirrors, and an optical path matching slab of clear glass. An interference pattern of light and dark intensity rings is visible on detectors at an end of the optical path, for example, on photographic film. This pattern persists even as the intensity of the light is lowered. From the model of light quanta established in the photoelectric effect and Planck's calculation of black body radiation spectra, it follows that the illumination can be reduced to less than a single photon at a time within the interferometer on average. This establishes that each photon interferes with itself, and since the arms of the interferometer are separated and can be of unequal length, that the photon takes both paths. This contradicts a classical description of the photon as a classical particle with a definite trajectory. The interference pattern is consistent with the classical concept of electromagnetism as a wave, but that description is contradicted by the photoelectric effect. The interferometer is a strong argument for the reality of the quantum description of state. It is difficult to understand how at low intensities the interference pattern can be responsive to changes in arm path length without a physical presence in each of the two separate arms for each photon.

Discrete line spectra observed in emissions of light, for example, from a gas of hydrogen or sodium atoms, contradict that energy is a real number parameter, an initial condition for classical equations of motion. Line spectra contradict that any in a range of real numbers can provide the initial conditions that result in a hydrogen atom. There is no mechanism within classical descriptions that results in the observed atomic spectra (nor the observed heat capacities of solids at low temperatures). One of the predictions of non-relativistic quantum mechanics is line spectra, and observations of the Lamb shift of hydrogen atom energy levels is one of the most precise tests in physics. The Lamb shift is correctly estimated by the Feynman series rules for quantum electrodynamics.

Quantum mechanics resolves many observed flaws of classical descriptions, from extensive

entropy to conservation laws for quantized quantities to discrete atomic spectra to the consistency of observed interference patterns with particle-like descriptions of waves. Despite all this, classical descriptions of physics continue to be used as the underpinnings of quantum physics. The concept of a continuous evolution of distinguishable bodies traveling trajectories, the descriptions of Newtonian physics and Einstein's geometrodynamics, are evidently useful approximations to quantum physics but in limited instances. The concept that quantum dynamics is the "quantization" of these classical dynamic descriptions rests on expedience, experience with non-relativistic quantum mechanics, and successful but limited phenomenology. Any correspondence of quantum with classical need only occur in appropriate instances, and canonical quantization is an unjustified extrapolation otherwise.

8.6 The Einstein-Podolsky-Rosen paradox

The Einstein-Podolsky-Rosen (EPR) [17] paradox and confirmation of Bell's inequalities [6] illustrate that a classical description is inconsistent with nature. The quantum mechanical description of nature has been described as "bizarre" but, this description reflects that quantum mechanics demands rejection of long established classical concepts. The EPR paradox [17] illustrates that the quantum mechanical description of nature contradicts classical concepts. Einstein, Podolsky and Rosen develop the argument that quantum mechanics must be incomplete because the quantum description conflicts with a classical concept of state. Of course, the alternative is that the classical description is inconsistent with nature. To develop the EPR paradox, it is observed that spin angular momentum is quantized with the same discrete values on any axis of observation. The paradox arises if a spin zero particle decays into two spin one-half particles that subsequently fly apart. Sufficiently separated, we can determine the spin state of an arbitrarily distant particle by observing the paired particle: the distant spin is the one that paired with the near spin conserves angular momentum. The paradox is that we determine the spin of the non-causally related distant particle differently as determined by our selection of measurement axis. If the distant particle is classically described, then it has a determined spin unaffected by our observation of the nearby particle. The resolution to the conflict is entanglement, a concept in quantum mechanics that is not supported in a classical description. That is, the resolution of the EPR paradox is not a contradiction to quantum mechanics but a contradiction to classical concepts, to the classical description of state. Entanglement is natural in the Hilbert space description of multiple particle states [50]. The paradox is resolved by the realization that a classical description is not consistent with nature. Although Einstein motivated quantum mechanics with studies of the photoelectric effect and the heat capacity of solids [55], he is perhaps most celebrated for his understanding of time and motion described in special relativity and geometrodynamical gravity. Einstein's derivation of gravity relies on classical description of nature. Expressed in the EPR paradox, Einstein did not embrace implications of quantum mechanics. I presume that Einstein was persuaded by his immensely successful insights into time and motion using classical concepts that are in con-

flict with quantum mechanics. Also, more consistent perspectives on quantum mechanics [12] developed later. The principle of equivalence, that acceleration is equivalent to a gravitational force, relies on the classical description of a body characterized by a trajectory. In a quantum mechanical description, the characterization of the velocity of a body deteriorates with enhancements in location accuracy: both location and velocity are never known with a precision that exceeds the Heisenberg uncertainty bound. We cannot arbitrarily precisely associate the acceleration of a body with its locations.[t] It is a great irony that just as problematic aspects of Newtonian mechanics were resolved with dynamical, relativistic time and the equivalence of acceleration with gravitation, atomic physics necessitated the new mechanics that supersedes classical developments.

Flaws in classical physics such as a non-causal radiation reaction force in classical electrodynamics [29] can be tolerated since classical descriptions only approximate the more fundamental, and causal, quantum descriptions. Indicated in Feynman's quote in section 5, it could be anticipated that the transition from the well-established and successful practice of classical physics to quantum mechanics would be slow. Even though it was observation that necessitated the development of the new mechanics, adoption of quantum mechanics is inhibited by the established reliance on classical concepts. Although the classical perspective is contradicted by nature and quantum mechanics is the more comprehensive and unified model, the classical perspective is an accurate approximation of the quantum description in our common experience.

8.7 Heisenberg uncertainty

One of the great insights in the development of quantum mechanics is that location and momentum are not independently specified descriptions of bodies. Both location and momenta derive from a state describing function $\psi(x)$. The supports of functions $\psi(x)$ are associated with the likelihoods of locations and the supports of the Fourier transforms $\tilde{\psi}(p)$ are associated with the likelihoods of momenta. Likelihoods are provided by Born's rule from the Hilbert space scalar product. Mean values of location and momentum are independently specified but variances are constrained as described by the Heisenberg uncertainty principle. The Heisenberg uncertainty principle follows from the quantum mechanical description of states. If the scalar product is $\mathcal{L}^2$, appropriate in non-relativistic physics, then the location operator corresponds to the Hermitian operator realized by multiplication of the function $\psi(x)$ by the value of an argument $\mathbf{x}_\nu$ and the momentum operator corresponds to the Hermitian operator that is the similarly quantized multiplication of $\tilde{\psi}(p)$ by $\mathbf{p}_\nu$ in the momentum domain. Using the Fourier

[t] Although for typical non-relativistic Hamiltonians acceleration and location commute, $[X_\nu, [H, P_\nu]] = 0$, an estimate for acceleration from a sequence of localizing measurements fails due to the lack of a trajectory precisely associated with a quantum description. Each observation conveying location knowledge contributes velocity uncertainty. There are no classical state descriptions included in quantum mechanics. For many states, particularly state descriptions with high energies and overlapping supports, the principle of equivalence does not even approximate the evolution of the state describing functions.

transform relations (24) and linearity, in $\mathcal{L}^2$ location corresponds to $-i\hbar d/dp_\nu$ and momentum to $i\hbar d/dx_\nu$. This is discussed further in appendix 8.3. If at a time t the support of $\psi(x)$ is dominantly supported in the neighborhood of a point $\mathbf{y}(t)$, then

$$\langle\psi|X_\nu\psi\rangle = \int d\mathbf{x}\ \mathbf{x}_\nu|\psi(x)|^2 \approx \mathbf{y}_\nu(t)\int d\mathbf{x}\ |\psi(x)|^2 = \mathbf{y}_\nu(t)$$

for a normalized $\psi(x)$. $\mathbf{y}(t)$ is any suitable representative of the neighborhood of support. From Parseval's equality (23), the definition of Fourier transform (22) and implied identities, if at a time t the support of the Fourier transform $\tilde{\psi}(\mathbf{p},t)$ is dominantly supported near a momentum $\mathbf{q}(t) = \hbar\mathbf{p}(t)$, then with $\nu = 1,2,3$,

$$\langle\psi|P_\nu\psi\rangle = \hbar\int d\mathbf{p}\ \mathbf{p}_\nu|\tilde{\psi}(\mathbf{p},t)|^2 = i\hbar\int d\mathbf{x}\ \overline{\psi(\mathbf{x},t)}\ \frac{d\psi(\mathbf{x},t)}{dx_\nu} \approx \mathbf{q}_\nu(t).$$

There is no knowledge of momentum for a location eigenfunction, and no knowledge of location for a momentum eigenfunction. Are there classical particle-like selections for functions that describe quantum mechanical states? Here, a nearly classical state is one with both location and velocity known precisely. For finite masses, the time derivative of location is velocity $\mathbf{v}$ and in the non-relativistic approximation $\mathbf{p} \approx m\mathbf{v}$. There is no bound on the precision of knowledge of the mass. Are there states that approximate both location and momentum arbitrarily well? The result is that there is an optimal accuracy that can be simultaneously obtained for knowledge of location and momentum, and a choice of state describing functions that provides the optimal simultaneous knowledge. This result is the *Heisenberg uncertainty principle*. In quantum mechanics, not only is dynamics no longer described by a smooth trajectory specified by an initial location and velocity, but we cannot know both the location and the time derivative of the location of a body sufficiently well to specify a trajectory. That is, our description of state does not support the concept of Newtonian mechanics. The Heisenberg uncertainty principle is a lower bound on the breadth of the supports in location and momentum in a quantum mechanical description. We can approximate the location and time derivative of location for a body, and this approximation becomes better the heavier the body, but the concept of Newtonian mechanics is not supported by the quantum description of state. Also, in quantum mechanics, we cannot necessarily identify a particular body to propagate forward in time. Similarly described bodies are indistinguishable and cannot be labeled to identify a trajectory except while a single body is sufficiently isolated from other bodies to reliably identify the body. While isolated, a dominant volume in the support of a state describing function can be identified and followed with a high likelihood until it approaches other similar bodies. At this point, its identity becomes lost in ambiguity.

The Heisenberg uncertainty principle is derived in ordinary, non-relativistic quantum mechanics. It is a result that follows from the $\mathcal{L}^2$ scalar product applicable in non-relativistic physics. In $\mathcal{L}^2$, three location operators $X_\nu = x_\nu$ and three momentum operators $P_\nu = i\hbar d/dx_\nu$,

one for each spatial dimension, are self-adjoint in $\mathcal{L}^2$ and satisfy the Born-Heisenberg-Jordan relation

$$[X_\nu, P_\nu] = -i\hbar, \tag{241}$$

a canonical commutation relation (CCR). For an arbitrary state $|\psi\rangle$ labeled by a function $\psi(x) \in \mathcal{L}^2$ in the intersection of the domains of X_ν and P_ν (with convergent $|\langle\psi|X_\nu\psi\rangle|$ and $|\langle\psi|P_\nu\psi\rangle|$), define operators

$$A = X_\nu - \langle\psi|X_\nu\psi\rangle, \quad \text{and} \qquad B = P_\nu - \langle\psi|P_\nu\psi\rangle.$$

$\nu = 1, 2$ or 3 and the intersection of the domains includes states labeled by the Schwartz tempered functions that are dense in $\mathcal{L}^2$. From the commutation of X_ν and P_ν it follows that $[A, B] = -i\hbar$ and that A, B are mean zero for the state $|\psi\rangle$. From the interpretation of $\langle\psi|A\psi\rangle$ as the mean value of the quantity associated with the operator A for normalized states $\langle\psi|\psi\rangle = 1$, identify variances of location and momenta as

$$\sigma_x^2 = \langle\psi|A^2\psi\rangle, \quad \text{and} \qquad \sigma_p^2 = \langle\psi|B^2\psi\rangle.$$

Self-adjointness of A and B follows from the self-adjointness of X_ν and P_ν on $\mathcal{L}^2$. Self-adjointness and the Cauchy-Schwarz-Bunyakovsky inequality then provide that

$$|\langle\psi|AB\psi\rangle|^2 = |\langle A\psi|B\psi\rangle|^2 \leq \langle A\psi|A\psi\rangle \langle B\psi|B\psi\rangle = \langle\psi|A^2\psi\rangle \langle\psi|B^2\psi\rangle = \sigma_x^2\sigma_p^2.$$

The lower bound, equality, is achieved if $|A\psi\rangle = c|B\psi\rangle$ for a complex constant c. From the definition of commutator, sesquilinearity of scalar products, and the property of scalar products that $\langle u|v\rangle = \overline{\langle v|u\rangle}$, identify that

$$\langle\psi|[A, B]\psi\rangle = \langle\psi|AB\psi\rangle - \langle\psi|BA\psi\rangle = \langle\psi|AB\psi\rangle - \langle AB\psi|\psi\rangle = 2i\Im m(\langle\psi|AB\psi\rangle),$$

twice the imaginary part. The imaginary part of z has a magnitude bounded by the magnitude of z. This bound and the commutation relation for A and B result in that

$$|\langle\psi|[A, B]\psi\rangle|^2 = |i\hbar\langle\psi|\psi\rangle|^2 = \hbar^2 = |2i\Im m(\langle\psi|AB\psi\rangle)|^2 \leq 4|\langle\psi|AB\psi\rangle|^2 \leq 4\sigma_x^2\sigma_p^2.$$

This is Heisenberg's uncertainty principle. For states in $\mathcal{L}^2$,

$$\sigma_x\sigma_p \geq \hbar/2,$$

a lower limit on the geometric mean of the variances of location and momentum. This limits the simultaneous accuracy of location and momentum descriptions in each dimension ν. The lower bound is achieved if $\langle\psi|AB\psi\rangle$ is imaginary and $|A\psi\rangle = c|B\psi\rangle$ for $c \in \mathbb{C}$. The labels $\psi(x)$ with $x \in \mathbb{R}^3$ for the states that meet both lower bounds are Gaussian functions (105),

$$\psi(x) = \frac{e^{-(x-x_o)^2/(4L^2)+ip_o x/\hbar}}{(2\pi L^2)^{\frac{1}{4}}}.$$

These Gaussian functions satisfy

$$\langle\psi|AB\psi\rangle = i\int_{\mathbb{R}^3} dx\ \frac{(x-x_o)^2}{2L^2}|\psi(x)|^2$$

that is imaginary and

$$A\psi(x) = (x-x_o)\psi(x) = cB\psi(x) = c(-i\hbar\frac{d\psi(x)}{dx} - p_o\psi(x)) = c\ \frac{i\hbar(x-x_o)}{2L^2}\psi(x)$$

for real $x_o = \langle\psi|X_\nu\psi\rangle$ and $p_o = \langle\psi|P_\nu\psi\rangle$, $c = 2L^2/(i\hbar)$ and $\langle\psi|\psi\rangle = 1$. The parameter $L = \sigma_x$ and $\sigma_p = \hbar/(2L)$. This is the function shape in each dimension. The spatial function that labels a minimum uncertainty state is the product of three factors, one for each spatial dimension. These Gaussian functions are denoted minimum packets and are the most classical particle-like state describing functions in the sense that the geometric mean of the uncertainties in simultaneous knowledge of the location and momentum of the state is minimized.

The Heisenberg uncertainty principle applies in relativistic physics with some revision. In a relativistic development, the operators X_ν ($|X_\nu\psi\rangle = |x_\nu\psi\rangle$) are not self-adjoint and therefore are not the quantization of location. As a consequence, the development above of the Heisenberg uncertainty relation does not apply for operator pairs X_ν, P_ν. Nevertheless, the Heisenberg uncertainty principle applies when classical approximations to the relativistic physics apply, and with X_ν replaced by the Hermitian $\widehat{X}_\nu = -i\hbar\omega^{1/2}d/dp_\nu\omega^{-1/2}$, the relativistic, single body location operator [43]. Classical particle approximations apply when states are described by functions with isolated (121) concentrations of support well-represented by a single location (119) and momentum (120). The eigenfunctions of $\widehat{X}_\nu$ are the relativistically invariant localized functions. If the $\widehat{X}_\nu$ is used in the development of the Heisenberg uncertainty relation, then the minimum uncertainty packets are not Gaussian functions. The relativistic minimum uncertainty packets are normalized inverse Fourier transforms of

$$\tilde{\psi}(\mathbf{p}) = \omega^{1/2}e^{-\sigma_o^2(\mathbf{p}-\mathbf{w})^2}e^{i\mathbf{p}\cdot\mathbf{x}_o/\hbar}.$$

If the state description consists of sufficiently isolated concentrations in the support of each argument and these supports are well-represented by single locations and momenta, then the state can be represented by classical particles and the Heisenberg uncertainty principle applies for functions $\psi(\mathbf{x})$ used to define functions in $\mathbf{H}_\mathcal{P}$.

$$\tilde{\varphi}(p) = \frac{(p_0+\omega)\tilde{\psi}(\mathbf{p})}{\sqrt{2\omega}} \in \mathbf{H}_\mathcal{P}. \tag{242}$$

Due to cluster decomposition, the contributions to scalar products of the high order connected functions are negligible when a state consists solely of widely spatially isolated bodies. In these

cases, the $\mathcal{W}_{k,n-k}$ are well-approximated by the free field contribution (42) to the VEV. From (42), the significant contributions to scalar products (27) for product states f_n,

$$f_n((x)_n) = \prod_{j=1}^{n} \varphi_j(x_j) \in \mathbf{H}_{\mathcal{P}}$$

and functions φ_j with Fourier transforms of the form (242) consists of sums of products of factors $W_2(\varphi_j^* \varphi_\ell)$. Substitution results in

$$\begin{aligned} W_2(\varphi_j^* \varphi_\ell) &= \int dp_1 dp_2 \; \tilde{W}_2(p_1, p_2) \overline{\tilde{\varphi}_j(-p_1)} \tilde{\varphi}_\ell(p_2) \\ &= \int dp_1 dp_2 \; \delta(\mathbf{p}_1 + \mathbf{p}_2) \; \frac{\delta(p_{10} + \omega_1)}{\sqrt{2\omega_1}} \; \frac{\delta(p_{20} - \omega_2)}{\sqrt{2\omega_2}} \; \overline{\tilde{\varphi}_j(-p_1)} \tilde{\varphi}_\ell(p_2) \\ &= \int d\mathbf{p}_2 \; \overline{\tilde{\psi}_j(\mathbf{p}_2)} \tilde{\psi}_\ell(\mathbf{p}_2) \\ &= \int d\mathbf{x} \; \overline{\psi_j(\mathbf{x})} \psi_\ell(\mathbf{x}). \end{aligned} \tag{243}$$

This is the $\mathcal{L}^2$ scalar product of the $\psi_j(\mathbf{x})$. The Heisenberg uncertainty bound applies to the breadths of location and momentum support of $W_2(\varphi_j^* \varphi_j)$ with $\psi_j(\mathbf{x})$ substituted for $\varphi_j(x)$.

Gaussian functions provide an explicit example of an expansion of a state description as a linear combination of more localized Gaussian elements in the Hilbert space. The identity

$$e^{-\frac{x^2}{4L_0^2}} = \sqrt{\frac{1}{\epsilon^2(1-\epsilon^2)4L_0^2\pi}} \int ds \; e^{-\frac{s^2}{4(1-\epsilon^2)L_0^2}} e^{-\frac{(x-s)^2}{4\epsilon^2 L_0^2}}$$

with $0 < \epsilon \ll 1$ is an expansion of the more broadly supported Gaussian function as a linear combination of arbitrarily narrower Gaussian functions centered on s. This expansion is in one dimension. The likelihood of a transition from the normalized state characterized by

$$\psi(x) = N_\psi e^{-\frac{x^2}{4L_0^2}}$$

to the more localized, normalized state characterized by

$$g_s(x) = N_s e^{-\frac{(x-s)^2}{4\epsilon^2 L_0^2}}$$

is

$$|\langle g_s | \psi \rangle|^2 = \frac{2\epsilon}{1+\epsilon^2} \; e^{-\frac{s^2}{4(1+\epsilon^2)L_0^2}}$$

with $\langle \psi | \psi \rangle = \langle g_s | g_s \rangle = 1$. As anticipated, the likelihood of transition to a function centered on s declines rapidly the farther s is from the center of support at $x = 0$. And, the likelihood of transition to a function of width $\epsilon\sigma$ declines slowly with reductions in relative function spread.

8.8 Realizations of $Q_{2,2}$

This appendix illustrates the arrays $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ in (54) in section 3.3.3. The arrays determine $n \geq 4$, n-argument connected VEV functions that describe interaction in the constructed realizations of RQP. The illustrated example is $n = 4$, $k = 2$, and any number of component fields N_c. The matrices $\mathfrak{h}(p)$ in (55) are therefore $N_c \times N_c$ and two factors of $\mathfrak{h}(p)$ appear in each term of the expansion for $Q_{2,2}((p)_4)_{(\kappa)_4}$. The full set of illustrations assumes that the array $\mathfrak{h}(p)$ includes both boson and fermion components.

Keeping only the detail necessary to identify terms, terms are distinguished by transpositions of the arguments p_j, κ_j of the four fields in a 4-point VEV, $j = 1, 2, 3, 4$. Transposition of arguments corresponds with commutation of the field operators within VEV. Selection of $\kappa_j \in \{1, N_c\}$ determines a type, boson or fermion, and an associated mass m_{κ_j} for each j. Argument type of the jth argument follows from the values of κ_j using (39) in section 3.2. There are four energy-momenta $(p)_4$ and N_c^4 possible selections for $(\kappa)_4$.

The constructions begin with expansions in the constituents $\mathfrak{B}(p)$ and $\Upsilon(p)$ of $\mathfrak{h}(p)$. The summation $a, b \in \mathbb{J}_{2,2}$ in (57) with $a \in \{1, 2\}$ and $b \in \{3, 4\}$results in

$$\begin{aligned}\exp(\sum_{a,b\in\mathbb{J}_{2,2}} \rho_a\rho_b\mathfrak{h}_{\kappa_a\kappa_b}) = \exp(\rho_1\rho_2\mathfrak{B}(p_1+\alpha_o p_2)_{\kappa_1\kappa_2})\ \exp(\rho_3\rho_4\mathfrak{B}(\alpha_o p_3+p_4)_{\kappa_3\kappa_4}) \\ \times \exp(\rho_2\rho_3\Upsilon(-p_2+p_3)_{\kappa_2\kappa_3} + \rho_1\rho_4\Upsilon(-p_1+p_4)_{\kappa_1\kappa_4}).\end{aligned} \tag{244}$$

Notation is from section 3.3.3.

In the abbreviated notation (47), $i_1i_2i_3i_4$ designates one of the permutations π_j of the four sets of field arguments $(p, \kappa)_4$ and $\sigma(i_1i_2i_3i_4) = \sigma(\pi_j, (\kappa)_4)$ designates the associated sign. The associated sign is identified below (48) in section 3.3.2. The reference order A_o is 2134. Similarly to the abbreviated notation (47), designate

$$\begin{aligned}\mathfrak{B}_i(ab) &= \mathfrak{B}(p_a+\alpha_o p_b)_{\kappa_a\kappa_b} \\ \mathfrak{B}_o(ab) &= \mathfrak{B}(\alpha_o p_a+p_b)_{\kappa_a\kappa_b} \\ \Upsilon(ab) &= \Upsilon(-p_a+p_b)_{\kappa_a\kappa_b}.\end{aligned} \tag{245}$$

In this abbreviated notation and neglecting the factors of $\delta(p_j^2 - \lambda_{cj}^2)$, the construction of $Q_{2,2}((p)_4)_{(\kappa)_4}$ follows from the split signed symmetrization (49) of $\mathfrak{q}_{2,2}(1234)$. From (54) and (244),

$$\begin{aligned}\mathfrak{q}_{2,2}(1234) &= \mathfrak{q}_{2,2}((p)_4)_{(\kappa)_4} \\ &= \frac{d}{d\rho_1}\frac{d}{d\rho_2}\frac{d}{d\rho_3}\frac{d}{d\rho_4}\ \exp(\sum_{a,b\in\mathbb{J}_{2,2}} \rho_a\rho_b\mathfrak{h}_{\kappa_a\kappa_b}) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34) + \Upsilon(23)\Upsilon(14)\end{aligned} \tag{246}$$

after evaluation with all $\rho_j = 0$. The split signed symmetrization (49) produces

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \sum_{\pi_2} \sigma(\pi_2,(\kappa)_n) \left(\sum_{\pi_1} \sigma(\pi_1,(\kappa)_n)\ \mathfrak{q}_{2,2}(\pi_2(\pi_1(\{1,n\})))\right) \\ &= \sum_{\pi_2} \sigma(\pi_2,(\kappa)_n) \Big(\sigma(1234)\mathfrak{q}_{2,2}(\pi_2(1234)) + \sigma(2134)\mathfrak{q}_{2,2}(\pi_2(2134))\Big) \\ &= \sigma(1234) \Big(\sigma(1234)\mathfrak{q}_{2,2}(1234) + \sigma(2134)\mathfrak{q}_{2,2}(2134)\Big) \\ &\quad +\sigma(1243) \Big(\sigma(1234)\mathfrak{q}_{2,2}(1243) + \sigma(2134)\mathfrak{q}_{2,2}(2143)\Big). \end{aligned} \tag{247}$$

Five cases of interest follow from the index types of selections for the $(\kappa)_4$. The five cases have species types: 1,2,3,4=boson; 1,2,3,4=fermion; 1,2=boson and 3,4=fermion; 1,2=fermion and 3,4=boson; and 1,3=boson and 2,4=fermion. Other cases are zero or related by split signed symmetry. These cases associate with boson scattering, fermion scattering, fermion pair production from bosons, fermion pair annihilation to bosons, and fermion-boson scattering, respectively.

Boson scattering: 1,2,3,4=boson. All signs $\sigma(i_1 i_2 i_3 i_4)$ are positive.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} =&\ \mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) + \mathfrak{q}_{2,2}(1243) + \mathfrak{q}_{2,2}(2143) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34) + \Upsilon(23)\Upsilon(14) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) + \Upsilon(13)\Upsilon(24) \\ &\quad +\mathfrak{B}_i(12)\mathfrak{B}_o(43) + \Upsilon(24)\Upsilon(13) + \mathfrak{B}_i(21)\mathfrak{B}_o(43) + \Upsilon(14)\Upsilon(23) \\ &= (\mathfrak{B}_i(12) + \mathfrak{B}_i(21))(\mathfrak{B}_o(34) + \mathfrak{B}_o(43)) + 2\Upsilon(13)\Upsilon(24) + 2\Upsilon(23)\Upsilon(14). \end{aligned} \tag{248}$$

Fermion scattering: 1,2,3,4=fermion. Each transposition from the reference order results in a negative sign. $\sigma(1234) = \sigma(2143) = -1$.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} =&\ -\mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) + \mathfrak{q}_{2,2}(1243) - \mathfrak{q}_{2,2}(2143) \\ &= -\mathfrak{B}_i(12)\mathfrak{B}_o(34) - \Upsilon(23)\Upsilon(14) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) + \Upsilon(13)\Upsilon(24) \\ &\quad +\mathfrak{B}_i(12)\mathfrak{B}_o(43) + \Upsilon(24)\Upsilon(13) - \mathfrak{B}_i(21)\mathfrak{B}_o(43) - \Upsilon(14)\Upsilon(23) \\ &= (\mathfrak{B}_i(21) - \mathfrak{B}_i(12))(\mathfrak{B}_o(34) - \mathfrak{B}_o(43)) - 2\Upsilon(23)\Upsilon(14) + 2\Upsilon(13)\Upsilon(24). \end{aligned} \tag{249}$$

Fermion pair production from bosons: 1,2=boson and 3,4=fermion. Only $\sigma(2143) = -1$ and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) - \mathfrak{q}_{2,2}(1243) - \mathfrak{q}_{2,2}(2143) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) - \mathfrak{B}_i(12)\mathfrak{B}_o(43) - \mathfrak{B}_i(21)\mathfrak{B}_o(43) \\ &= (\mathfrak{B}_i(21) + \mathfrak{B}_i(12))(\mathfrak{B}_o(34) - \mathfrak{B}_o(43)). \end{aligned} \tag{250}$$

Fermion pair annihilation to bosons: 1,2=fermion and 3,4=boson. Only $\sigma(1234) = -1$ and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= -\mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) - \mathfrak{q}_{2,2}(1243) + \mathfrak{q}_{2,2}(2143) \\ &= -\mathfrak{B}_i(12)\mathfrak{B}_o(34) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) - \mathfrak{B}_i(12)\mathfrak{B}_o(43) + \mathfrak{B}_i(21)\mathfrak{B}_o(43) \\ &= (\mathfrak{B}_i(21) - \mathfrak{B}_i(12))(\mathfrak{B}_o(34) + \mathfrak{B}_o(43)). \end{aligned} \tag{251}$$

Boson-fermion scattering: 1,4=boson and 2,3=fermion. All signs are positive and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) + \mathfrak{q}_{2,2}(1243) + \mathfrak{q}_{2,2}(2143) \\ &= \Upsilon(23)\Upsilon(14) + \Upsilon(14)\Upsilon(23) \\ &= 2\Upsilon(23)\Upsilon(14). \end{aligned} \tag{252}$$

The expressions for $Q_{2,2}((p)_4)_{(\kappa)_4}$ each include a factor

$$\prod_{j=1}^{4} \delta(p_j^2 - \lambda_{cj}^{-2})$$

that constrains energy-momentum support to mass shells. These factors are neglected in the abbreviated notation.

With a transposition of indices, $\mathfrak{B}_i(ab) = \mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a \kappa_b}$ becomes

$$\begin{aligned} \mathfrak{B}(\alpha_o p_a + p_b)^T &= \int d\mu_{\mathfrak{B}}(s)\; e^{is(\alpha_o p_a + p_b)} M(s)^T \\ &\neq \mathfrak{B}(p_a + \alpha_o p_b). \end{aligned}$$

if $\alpha_o \neq 1$ despite the symmetry, $M(s) = M(s)^T$ in the example realization of $M(s)$ in section 3.2.2. If the fermion component of $\mathfrak{B}$ were symmetric with transposition of fermion type arguments, $\mathfrak{B}_\nu(ab) = \mathfrak{B}_\nu(ba)$ for $\nu = i, o$, then fermion pair production and fermion pair annihilation amplitudes would vanish. The fermion component of $\mathfrak{B}$ is symmetric with transposition of fermion type arguments if $\alpha_o = 1$ for the example realization of a fermion in section 3.2.2.

8.9 Feynman-Dyson series and the constructions

A renormalized perturbative series for scattering amplitudes [24, 52, 61] is discussed in this appendix to contrast Feynman (also referred to as Feynman-Dyson, or Dyson) series scattering amplitudes with the explicit scattering amplitudes from the constructions in section 3. In

this appendix, the comparison of quantum mechanical constructions with Feynman series is illustrated for the example of one neutral scalar field $\Phi(x)$, $N_c = 1$. Feynman series apply only to scattering [9], to infinite interval transition amplitudes.

From section 4.5, the constructed scattering amplitudes are infinite interval limits of more general state transition amplitudes. Appropriate states are interpretable as classical bodies described by momenta using plane wave limits of localized states. For states (4) described by product functions of point support at times λ_j and with momentum support centered on $\mathbf{q}_j$,

$$f_n((x)_n) = \prod_{j=1}^{n} \ell(x_j; \lambda_j, \mathbf{q}_j)$$

[33, 35, 37], a transition from m to n freely propagating particles is described by the LSZ (Lehmann-Symanzik-Zimmermann) expressions for scattering amplitudes [10],

$$\begin{aligned} S_{n.m} &= \lim_{\lambda\to\infty} \langle U(\lambda)\ell(\lambda, \mathbf{q}_{m+1}) \dots \ell(\lambda, \mathbf{q}_{m+n}) | U(-\lambda)\ell(-\lambda, \mathbf{q}_1) \dots \ell(-\lambda, \mathbf{q}_m)\rangle \\ &= \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+1})) \dots \Phi(\ell(\lambda, \mathbf{q}_{m+n}))\Omega | U(-\lambda)\Phi(\ell(-\lambda, \mathbf{q}_1)) \dots \Phi(\ell(-\lambda, \mathbf{q}_m))\Omega\rangle \end{aligned} \tag{253}$$

and

$$S_{n.m} = \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+n}))U(\lambda)^{-1} \dots U(\lambda)\Omega | U(-\lambda)\Phi(\ell(-\lambda, \mathbf{q}_1))U(-\lambda)^{-1} \dots U(-\lambda)\Omega\rangle$$

with $U(\lambda)$ the unitary time translation operator, and applying the definition of field (4) with the notation of section 3.1.3. The state describing functions $\ell(x_j; \lambda_j, \mathbf{q}_j)$ are described in (104) in section 4.5.

$$\tilde{\ell}(p_j; \lambda_j, \mathbf{q}_j) = e^{ip_{j0}\lambda_j}(\omega_j + p_{j0})\tilde{f}(\mathbf{p}_j - \mathbf{q}_j)$$

with $\lambda_j \in \mathbb{R}$, $\mathbf{q}_j \in \mathbb{R}^3$ and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. $\ell(x_j; \lambda, \mathbf{q}_j)$ is within the completion $\mathbf{H}_{\mathcal{P}}$ of the $\mathcal{P}(\mathbb{R}^4)$ described in section 3.4, Ω designates the vacuum and the VEV are constructed in section 3. $\mathcal{S}(\mathbb{R}^3)$ includes functions with Fourier transforms that are delta sequences with supports concentrated near the momenta $\mathbf{q}_j$. A convenient choice for function $\tilde{f}(\mathbf{p})$ is the Gaussian function (105), a point-wise nonnegative delta sequence. The cluster decomposition axiom A.6 provides that the Gaussian functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are described at large times as free particles if the functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are translated to center the spatial support on free particle trajectories with distinct momenta $\mathbf{q}_j$.

The LSZ scattering amplitudes (253) are VEV of products of temporal translations of fields $\Phi(\ell(t, \mathbf{q}))$. From section 4.5 and with $\lambda_j = \lambda$, temporal translations of these fields are independent of time.

$$U(t)\Phi(\ell(t, \mathbf{q}))U(t)^{-1} = \Phi(\ell(0, \mathbf{q}))$$

due to the limitation of the spectral support of VEV from section 3 to mass shells and the selection of compensating phases in $\ell(x_j; \lambda_j, \mathbf{q}_j)$.

The RQFT [8, 24, 52, 61] rule for calculation of plane wave scattering amplitudes is to evaluate the generalized functions

$$S_{n,m}((p)_{n+m}) = \langle \widetilde{\Phi_o^+}(p_n) \dots \widetilde{\Phi_o^+}(p_1)\Omega_o | U_D(t,-t) \widetilde{\Phi_o^+}(p_{n+1}) \dots \widetilde{\Phi_o^+}(p_{n+m})\Omega_o \rangle \tag{254}$$

for $t = \infty$. These scattering amplitudes are expanded in free field VEV. Free field operators are distinguished in this appendix by the notation $\Phi_o(x)$. The Dyson operator $U_D(t_1, t_2)$ is provided below in (259). Creation components $\Phi_o^+(x)$ of free fields are introduced in section 3.2. The scattering amplitudes $S_{n,m}((p)_{n+m})$ apply the Fourier transform of generalized functions [20],

$$\Phi_o^+(\underline{f}) = \widetilde{\Phi_o^+}(\underline{\tilde{f}}). \tag{255}$$

Introduced in section 3.2, the free field

$$\Phi_o = \Phi_o^+ + \Phi_o^-$$

has a cyclic vacuum state Ω_o and an annihilation component $\Phi_o^- \, \Omega_o = 0$. From section 3.1.4, the operator adjoint of Φ_o^+ is

$$\Phi_o^+(\underline{f})^* = \Phi_o^-(\underline{f}^*) \tag{256}$$

using the dual (14) for functions on the right-hand side and the Hilbert space operator adjoint on the left-hand side. The adjoint is

$$\widetilde{\Phi}_o^+(p)^* = \widetilde{\Phi}_o^-(-p)$$

from the Fourier transform (255) and $*$-dual (14). For $f \in \mathcal{P}(\mathbb{R}^4)$, $\Phi_o(\underline{f}) = \Phi_o^+(\underline{f})$ as a consequence of the support constraint on functions in $\underline{\mathcal{P}}$. From section 3.2, the commutators of free field creation Φ_o^+ and annihilation Φ_o^- components provide that

$$[\Phi_o^-(x), \widetilde{\Phi_o^+}(p)] = \frac{\delta^+(p) e^{ipx}}{(2\pi)^2} \tag{257}$$

using the definition of Fourier transforms for generalized functions (255), the Fourier transform (22), and the notation

$$\delta^{\pm}(p) = \theta(\pm p_0)\delta(p^2 - \lambda_c^{-2})$$

from (35). The adjoint results in

$$[\widetilde{\Phi_o^-}(-p), \Phi_o^+(x)] = \frac{\delta^+(p) e^{-ipx}}{(2\pi)^2}.$$

The commutation relations and annihilation of the vacuum by Φ_o^- provide the convenient result

$$\langle \Omega_o | (\Phi_o^-(x))^k \prod_{j=1}^{n} \widetilde{\Phi_o^+}(p_j) \, \Omega_o \rangle = \frac{n! \, \delta_{k,n}}{(2\pi)^{2n}} \prod_{j=1}^{n} \delta^+(p_j) e^{ip_j x} \tag{258}$$

demonstrated by induction and using the Kronecker delta, $\delta_{k,n}$.

The Feynman series for scattering amplitudes results from the Neumann series solution with the Dyson operator substituted into the scattering amplitudes (254). The Dyson operator is

$$\begin{aligned} U_D(\lambda_1, \lambda_2) &= e^{iH_0\lambda_1} e^{-iH(\lambda_1-\lambda_2)} e^{-iH_0\lambda_2} \\ &= \mathbb{I} - i\int_{\lambda_2}^{\lambda_1} ds\; H_{int}(s)\, U_D(s, \lambda_2) \end{aligned} \tag{259}$$

evaluated with $\lambda_1 = -\lambda_2 \to \infty$. The resulting Volterra equation of the second kind for $U_D(\lambda_1, \lambda_2)$ is satisfied formally by the Dyson operator. In (254), $\Phi_o(x)$ is a neutral scalar free quantum field and H_0 generates time translations of these free fields. The Hamiltonian H is expressed in free fields.

$$H = H_0 + H_{int}$$

with a conjectured interaction Hamiltonian for a self-interacting, neutral scalar field of

$$H_{int}(x_0) = \sum_{\ell \geq 4} a_\ell \int d\mathbf{x}\; :(\Phi_o(x_0, \mathbf{x}))^\ell: . \tag{260}$$

The summation is over all $\mathbf{x} \in \mathbb{R}^3$. The notation $:(\Phi_o)^\ell:$ designates *normal ordering* of the Hamiltonian (260). Normal ordering designates that the factors of Φ_o^+ and Φ_o^- in the binomial expansion of $(\Phi_o)^\ell$ are ordered with every Φ_o^- to the right of any Φ_o^+ [8, 24, 52, 61]. Normal ordering sets $\langle \Omega_o | H_{int}\, \Omega_o \rangle = 0$ but does not place the vacuum $|\Omega_o\rangle$ in the null space of the Hamiltonian, $|H_{int}\, \Omega_o\rangle \neq 0$. From the Campbell-Baker-Hausdorff expression for this example of a boson field,

$$\begin{aligned} :\exp(\alpha\Phi_o): &= :\exp(\alpha\Phi_o^- + \alpha\Phi_o^+): \\ &= \exp(\alpha\Phi_o^-)\exp(\alpha\Phi_o^+) \\ &= \exp(\alpha\Phi_o^- + \alpha\Phi_o^+)\exp(-\tfrac{\alpha^2}{2}[\Phi_o^-, \Phi_o^+]). \end{aligned}$$

From section 3.2, the commutator is central, commutes with both Φ_o^+ and Φ_o^-, in the algebra generated by Φ_o^+ and Φ_o^-. Note that $[\Phi_o^-(x), \Phi_o^+(y)]$ diverges for $x = y$ and this is one of many divergences encountered in the elevation of classical to quantum Hamiltonian in the canonical formalism. Normal ordering is not a linear operation on the algebra generated from Φ_o^+ and Φ_o^-. A contradiction to a general specification for normal order as a linear operation in the algebra of fields is illustrated by the free field commutation relation

$$[\Phi_o^-(f_1), \Phi_o(f_2)] = W_2(f_1\, f_2).$$

Normal ordering produces

$$:[\Phi_o^-(f_1), \Phi_o^+(f_2)]: = 0 \neq :W_2(f_1\, f_2): = W_2(f_1\, f_2).$$

The choice of Hamiltonian (260) associates RQFT with a classical field model. Normal ordering determines an order for the non-commuting operators that elevate commuting classical dynamical variables.

The first contributing order of the Neumann series for the Dyson operator $U_D(\lambda_1, \lambda_2)$ approximates

$$U_D(\lambda_1, \lambda_2) \approx \mathbb{I} - i \int_{\lambda_2}^{\lambda_1} ds\ H_{int}(s).$$

Then, the first contributing order to the Feynman rules scattering amplitude (254) is the generalized function

$$\begin{aligned}
S_{k,n-k}((p)_n) &= \langle \widetilde{\Phi_o^+}(p_k) \dots \widetilde{\Phi_o^+}(p_1)\Omega_o | U_D(\infty, -\infty) \widetilde{\Phi_o^+}(p_{k+1}) \dots \widetilde{\Phi_o^+}(p_n)\Omega_o \rangle \\
&\approx \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\ (\mathbb{I} - i \int_{-\infty}^{\infty} ds\ H_{int}(s))\ \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j)\Omega_o \rangle \\
&= \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\ (\mathbb{I} - i \sum_{\ell} a_\ell \int dx\ \sum_{\nu} \tbinom{\ell}{\nu} (\Phi_o^+(x))^\nu (\Phi_o^-(x))^{\ell-\nu})\ \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j)\Omega_o \rangle.
\end{aligned}$$

The normal ordered binomial expansion and commutation relations (257) provide that the only term that does not include forward scattering contributions has $\ell = n$ and $\nu = k$. Other terms include forward contributions and are discarded in this characterization of the connected function ${}^C\mathcal{W}_{k,n-k}$ associated with a non-forward scattering likelihood. An associated connected VEV derives from the non-forward contribution to

$$\begin{aligned}
&-ia_n \tbinom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\ (\Phi_o^+(x))^k (\Phi_o^-(x))^{n-k}\ \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j)\Omega_o \rangle \\
&\quad = -ia_n \tbinom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\ (\Phi_o^+(x))^k \Omega_o \rangle \langle \Omega_o (\Phi_o^-(x))^{n-k}\ \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j)\Omega_o \rangle \\
&\quad = -i \frac{n!\, a_n}{(2\pi)^{2n-4}}\ \delta(-p_1 \dots - p_k + p_{k+1} \dots + p_n) \prod_{\ell=1}^{n} \delta^+(p_\ell) \\
&\quad = -ic_n\ \delta(-p_1 \dots - p_k + p_{k+1} \dots + p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_c^{-2}) \\
&\quad = -i\ {}^C\widetilde{\mathcal{W}}_{k,n-k}((-p)_k, (p)_{k+1,n}).
\end{aligned}$$

Commutations of $\widetilde{\Phi_o^-}(-p_j)$ with $\widetilde{\Phi_o^+}(p_\ell)$ produce forward contributions and are discarded in the second line. Substitution of (258) and the Fourier expansion of the delta function results in the third line. Equality of the third lines and fourth lines applies for the energy support constrained

functions from the completion of basis spaces $\underline{\mathcal{P}}$, and not for all tempered functions $\underline{\mathcal{S}}$. Factors $\theta(p_{j0})$ in the $\delta^+(p_j)$ are redundant with the support constraints for either the appropriate function in $\underline{\mathcal{P}}$ or $*$-dual (14) of $\underline{\mathcal{P}}$. The coefficients a_ℓ in the Hamiltonian (260) and the c_n in the description of the VEV (52) are related

$$c_n = \frac{n!\, a_n}{(2\pi)^{2n-4}}.$$

In the final line, a neutral scalar field connected VEV ${}^C\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.3 is identified and substituted. For Hamiltonians (260), the first order contribution to the Feynman series scattering amplitude (254) coincides with the Fourier transform of a connected VEV ${}^C\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.3. While the Hamiltonian (102) of the construction is distinct from the conjectured canonical formalism Hamiltonian (260), the weak coupling scattering amplitudes of the two developments nearly coincide: the Feynman series continues with renormalized, higher powers of the interaction Hamiltonian. The phase 'i' is irrelevant except for phase differences between forward and non-forward contributions, and the effects of these phases vanish in the scattering limit. The phase difference implements nonnegativity of the scalar product A.2.

The first contributing order of the scattering amplitudes (254) coincide up to a phase with the constructed scattering amplitudes (106) for a neutral scalar field. Then, the scattering likelihoods of low order expansion in RQFT and the constructions are the same: the Feynman rules scattering likelihoods asymptotically coincide at weak coupling with the scattering likelihoods from the fully quantum mechanical constructions. This equality persists in multiple component field examples if relativistic corrections to the Feynman series are made in appropriate cases [35].

The RQFT rule (254) follows from a conjectured asymptotic equality of free and interacting fields. If the free field were unitary similar to the interacting field at asymptotic times, $\lambda \to \pm\infty$,

$$e^{-iH\lambda}\Phi(f)e^{iH\lambda} = e^{-iH_0\lambda}\Phi_o(f)e^{iH_0\lambda}$$

with

$$\Omega_o = e^{iH\lambda}e^{-iH_0\lambda}\Omega,$$

then the scattering amplitude (254) would follow. The unitary similarity of interacting and free fields, and translation invariance of vacuums would provide that

$$\begin{aligned}&\langle \Phi(f_k)\dots\Phi(f_1)\Omega|\Phi(f_{k+1})\dots\Phi(f_n)\Omega\rangle \\ &\qquad = \langle \Phi_o(f_k)\dots\Phi_o(f_1)\Omega_o|U_D(\lambda_{out},\lambda_{in})\Phi_o(f_{k+1})\dots\Phi_o(f_n)\Omega_o\rangle.\end{aligned} \tag{261}$$

However, neither precedent is true.

1. If a unitary similarity implied (261), that similarity would contradict the Haag (Haag-Hall-Wightman-Greenberg) theorem [10, 56]. Haag's theorem demonstrates that unitary

similarity of free and interacting fields is not possible. Unitary similarity applies to fields Φ and Φ_o that are densely defined operators in a common Hilbert space [25].

2. In the Fock space of the free field, translation invariance of the free field VEV (42) provides that the free field vacuum Ω_o is translation invariant. But, this vacuum is not invariant to $e^{iH\lambda}$, $H_{int}\Omega_o \neq 0$. The all creation operator term contributes. For example, from the commutation relation (257) and VEV (258), the Fourier transform of delta functions, and the annihilation of the vacuum by Φ_o^-, it follows that

$$\begin{aligned}\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|H_{int}\Omega_o\rangle &= a_k\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|\int d\mathbf{x}\ (\Phi_o^+(x))^k\Omega_o\rangle \\ &= \frac{k!\,a_k}{(2\pi)^{2k-3}}\ \delta(\mathbf{p}_1+\dots+\mathbf{p}_k)\prod_{j=1}^{k}e^{i\omega_j x_0}\delta^-(p_j) \\ &\neq 0\end{aligned}$$

for the Hamiltonians (260).

If (259) were used to extrapolate transition amplitudes (254) to finite time intervals $\lambda_1 - \lambda_2$, these transition amplitudes would not be Lorentz covariant.

Higher order contributions to the Feynman series for scattering amplitudes (254) diverge but terms are renormalized (regularized) to achieve convergent contributions in each order of the iteration (254). If a finite number of regularizations suffice to achieve convergent terms to all orders, then the interaction Hamiltonian is denoted *renormalizable*. The series are considered to be asymptotic to scattering amplitudes.

8.10 Temporal evolution of function supports

For every $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$,

$$\tilde{\varphi}(p) = (p_0+\omega)e^{-iE\lambda}\tilde{f}(\mathbf{p})$$

is a function within $\mathbf{H}_{\mathcal{P}}$. λ is real parameter that translates the temporal support. The inverse Fourier transform of the state describing function $\tilde{\varphi}$ is

$$\begin{aligned}\varphi(x) &= \int\frac{dp}{(2\pi)^2}\ e^{ipx}(p_0+\omega)\tilde{f}(\mathbf{p}) \\ &= \left(-i\frac{\partial}{\partial x_0}+\sqrt{\lambda_c^{-2}-\Delta}\right)\int\frac{dp_0}{(2\pi)^{\frac{1}{2}}}\ e^{ip_0(x_0-\lambda)}\int\frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\ e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\omega\lambda}\tilde{f}(\mathbf{p}) \\ &= (2\pi)^{\frac{1}{2}}\left(-i\frac{\partial}{\partial x_0}+\sqrt{\lambda_c^{-2}-\Delta}\right)\delta(x_0-\lambda)f(\lambda,\mathbf{x})\end{aligned} \tag{262}$$

with Δ the Laplacian for $\mathbf{R}^3$, the derivative of $\delta(t)$ is a generalized function [20], and the function $f(\lambda, \mathbf{x}) \in \mathcal{S}(\mathbb{R}^3)$ for any real λ.

$$f(\lambda, \mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda} \tilde{f}(\mathbf{p}) \tag{263}$$

with $\omega = \omega(\mathbf{p})$ from (10) and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. With the understandings of section 5.1, $\varphi(x)$ defines functions supported on time $x_0 = \lambda$ with spatial support described by $f(\lambda, \mathbf{x})$.

For a rotationally invariant function $\tilde{f}$, the z-axis of the parametrization of $\mathbf{p}$ can be aligned with the z-axis for any selected spatial vector $\mathbf{x}$. In this instance, changing summation variables to polar coordinates results in

$$\mathbf{p} \cdot \mathbf{x} = \rho r \cos\phi$$

with

$$\begin{aligned} \rho^2 &= \mathbf{p}^2 \\ r^2 &= \mathbf{x}^2 \end{aligned}$$

and

$$f(\lambda, \mathbf{x}) = \int_0^\infty \rho^2 d\rho \int_0^\pi \sin\phi d\phi \int_0^{2\pi} \frac{d\theta}{(2\pi)^{\frac{3}{2}}}\; e^{-i\rho r\cos\phi} e^{i\omega\lambda} \tilde{f}(\mathbf{p}).$$

The Cartesian components of these polar coordinates are

$$\mathbf{p} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}. \tag{264}$$

The angle summations are elementary leaving a single summation to evaluate (263) for a rotationally symmetric, even $\tilde{f}(\mathbf{p})$.

$$\begin{aligned} f(\lambda, \mathbf{x}) &= \frac{-i}{\sqrt{2\pi}\, r} \int_0^\infty \rho d\rho\; \left(e^{i\rho r} - e^{-i\rho r}\right) e^{i\omega\lambda} \tilde{f}(\rho) \\ &= \frac{-i}{\sqrt{2\pi}\, r} \int_{-\infty}^\infty \rho d\rho\; e^{i\rho r} e^{i\omega\lambda} \tilde{f}(\rho) \\ &= -\frac{1}{\sqrt{2\pi}\, r} \frac{\partial}{\partial r} \int_{-\infty}^\infty d\rho\; e^{i\rho r} e^{i\omega\lambda} \tilde{f}(\rho) \end{aligned} \tag{265}$$

from a change of summation variable $\rho \to -\rho$ in the second term and the dominated convergence theorem.

Non-relativistically supported Gaussian functions provide an elementary example. Gaussian, minimum uncertainty functions centered of zero momentum,

$$\tilde{f}(\mathbf{p}) = e^{-\sigma^2 \mathbf{p}^2}, \tag{266}$$

are even and rotationally symmetric with a length parameter σ that characterizes the breadth of the spatial support of (263). Gaussian functions achieve the Heisenberg uncertainty lower bound on location and momentum support spread. If the support of $\tilde{f}(\mathbf{p})$ is over non-relativistic ρ, $\mathbf{p}^2 = \rho^2 \ll \lambda_c^{-2}$ within the dominant support, then

$$\omega \approx \lambda_c^{-1} + \frac{1}{2}\lambda_c \rho^2 - \frac{1}{8}\lambda_c^3 \rho^4 \ldots$$

from Taylor series expansion of (10). If λ is sufficiently small, then

$$\exp(i\omega\lambda) \approx \exp(i\lambda_c^{-1}\lambda + \frac{i}{2}\lambda_c \rho^2 \lambda).$$

λ is sufficiently small if $\lambda_c^3 \rho^4 \lambda \ll 16\pi$ within the dominant support of $\tilde{f}(\mathbf{p})$, and this is implied if

$$\lambda \ll 16\pi\lambda_c$$

since $\sigma^2 \gg \lambda_c^2$ for non-relativistically supported $\tilde{f}(\mathbf{p})$ and $\sigma\rho < \kappa$ for $\kappa \sim 10$ dependent on the confidence in likelihood. For non-relativistically supported $\tilde{f}(\mathbf{p})$ and sufficiently small λ, (265) is approximated with $\omega \approx \lambda_c^{-1} + \frac{1}{2}\lambda_c\rho^2$,

$$\begin{aligned} f(\lambda, \mathbf{x}) &\approx -\frac{e^{i\frac{\lambda}{\lambda_c}}}{\sqrt{2\pi}\, r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\rho\, e^{i\rho r} e^{-(\sigma^2 - i\frac{1}{2}\lambda_c\lambda)\rho^2} \\ &= -\frac{e^{i\frac{\lambda}{\lambda_c}}}{\sqrt{2}\, r\, (\sigma^2 - i\frac{1}{2}\lambda_c\lambda)^{\frac{1}{2}}}\frac{\partial}{\partial r}\, e^{-r^2/(4(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))} \\ &= \frac{e^{i\frac{\lambda}{\lambda_c}}}{(2(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))^{\frac{3}{2}}}\, e^{-r^2/(4(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))} \end{aligned}$$

from the Gaussian summation (281) in appendix 8.15.

Even, rotationally symmetric $\tilde{f}(\mathbf{p})$ with $\tilde{\varphi}_2 \in \mathbf{H}_{\mathcal{P}}$ and relativistic time translations that preserve the form are

$$\tilde{f}(\mathbf{p}) = \tilde{h}(\rho^2) e^{-2\sigma^2 \lambda_c^{-2}(\lambda_c\omega - 1)}$$

for continuously differentiable, polynomially bounded growth functions $\tilde{h}(\rho^2)$. The time translates have complex support spread parameters σ^2 and a phase shift independent of $\mathbf{p}$.

$$e^{-i\omega\lambda}\tilde{f}(\mathbf{p}) = \tilde{h}(\rho^2) e^{-(2\sigma^2\lambda_c^{-2} + i\lambda_c^{-1}\lambda)(\lambda_c\omega - 1)} e^{-i\lambda_c^{-1}\lambda}.$$

If $\tilde{h}(\rho^2) \approx 1$ for $\lambda_c \rho \ll 1$, these $\tilde{f}(\mathbf{p})$ approximate the Gaussian functions (266)

$$\tilde{f}(\mathbf{p}) \approx e^{-\sigma^2 \rho^2}$$

if momentum support is non-relativistic, $\lambda_c \rho \ll 1$ within the dominant support of $\tilde{f}(\mathbf{p})$. The approximation derives from (10),

$$\lambda_c \omega \approx 1 + \frac{(\lambda_c \rho)^2}{2}.$$

In very relativistic instances $\omega \approx \rho$ and $f(\lambda, \mathbf{x})$ becomes a more Lorentzian than Gaussian function over $r = \|\mathbf{x}\|$. Without approximation of the Hamiltonian, estimates for $f(\lambda, \mathbf{x})$ (265) apply for all λ.

8.11 Center-of-momentum reference frames

For any energy-momentum vector q with $q^2 > 0$ and momentum $\mathbf{q}$, Poincaré invariance of the scalar product can be exploited to transform to a primed inertial reference frame with $\mathbf{q}$ transformed to $\mathbf{q}' = 0$. In particular, Poincaré invariance of the scalar product can be exploited to transform to the center-of-momentum frame for n particles, a reference frame with $\mathbf{p}_1 + \mathbf{p}_2 + \ldots \mathbf{p}_n = 0$. Two particles of energy-momenta q_1, q_2 with $q_1^2 = q_2^2 = m^2$ and momenta $\mathbf{q}_1, \mathbf{q}_2$ illustrate the result. There is a Lorentz boost to a reference frame with $\mathbf{q}_2' = -\mathbf{q}_1'$. The Poincaré transformation to a center-of-momentum frame is developed in this appendix. A zero momentum is unaffected by a coordinate frame translation. Following a boost with translation, the center-of-mass of n particles may be collocated with the origin of coordinates in a center-of-momentum frame.

The transformation from the original energy-momentum coordinates to coordinates in the primed reference frame is

$$p_j' = \Lambda p_j$$

with Λ a Lorentz transform. The center of momentum frame is defined by this linear transformation Λ that sets

$$\Lambda q_1 = (\omega(\mathbf{q}_1'), \mathbf{q}_1') \quad \text{and} \qquad \Lambda q_2 = (\omega(\mathbf{q}_1'), -\mathbf{q}_1').$$

Both q_1 and q_2 are on mass shells defined by finite rest mass m. The transformation Λ is a proper (det(Λ)=1), orthochronous ($\Lambda_{00} > 0$) Lorentz transformation

$$\Lambda = \mathcal{B}(\beta)\mathcal{R}$$

consisting of a rotation $\mathcal{R}$ and a boost $\mathcal{B}(\beta)$. To evaluate Λ, designate the center-of-momentum for q_1, q_2 by

$$q = q_1 + q_2 = \begin{pmatrix} \omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) \\ \rho \cos\theta \sin\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\phi \end{pmatrix} \tag{267}$$

in polar coordinates with the momentum $\mathbf{q} = (q_x, q_y, q_z)$,

$$\rho = \|\mathbf{q}\| = \|\mathbf{q}_1 + \mathbf{q}_2\|,$$

and

$$\begin{aligned} \cos\phi &= \frac{q_z}{\rho}, & \sin\phi &= \frac{\sqrt{q_x^2 + q_y^2}}{\rho} \\ \cos\theta &= \frac{q_x}{\sqrt{q_x^2 + q_y^2}}, & \sin\theta &= \frac{q_y}{\sqrt{q_x^2 + q_y^2}} \end{aligned} \tag{268}$$

with quadrants selected for θ, ϕ to correspond with the signs of q_x, q_y, q_z. $\theta \in \{0, 2\pi\}$ is the anticlockwise angle of $\mathbf{q}$ from the x-axis in the x-y plane, and $\phi \in \{0, \pi\}$ is the angle of $\mathbf{q}$ from the z-axis in the plane containing $\mathbf{q}$ and the z-axis. The rotation $\mathcal{R}$ aligns the momentum $\mathbf{q}$ with the primed z-axis,

$$\begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ 0 \\ 0 \\ \rho \end{pmatrix} = \mathcal{R}q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ 0 & \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}$$

and the boost $\mathcal{B}(\beta)$ zeros the momentum.

$$\begin{pmatrix} \omega(\mathbf{q}_1') + \omega(\mathbf{q}_2') \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(\beta) & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & \gamma(\beta) \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ 0 \\ 0 \\ \rho \end{pmatrix}$$

with

$$\beta = \frac{\rho}{\sqrt{(\omega(\mathbf{q}_1) + \omega(\mathbf{q}_2))^2 - \rho^2}}$$

and $\gamma(\beta) = \sqrt{1 + \beta^2}$.

The transformation to the center-of-momentum frame determined by $\mathbf{q}_1$ and $\mathbf{q}_2$ is the Lorentz transform $\Lambda = \mathcal{B}(\beta)\mathcal{R}$.

$$\begin{pmatrix} \omega(\mathbf{q}_1') + \omega(\mathbf{q}_2') \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(\beta) & -\beta\frac{q_x}{\rho} & -\beta\frac{q_y}{\rho} & -\beta\frac{q_z}{\rho} \\ 0 & \frac{q_y}{\rho_{xy}} & -\frac{q_x}{\rho_{xy}} & 0 \\ 0 & \frac{q_x q_z}{\rho_{xy}\rho} & \frac{q_y q_z}{\rho_{xy}\rho} & -\frac{\rho_{xy}}{\rho} \\ -\beta & \gamma(\beta)\frac{q_x}{\rho} & \gamma(\beta)\frac{q_y}{\rho} & \gamma(\beta)\frac{q_z}{\rho} \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) \\ q_x \\ q_y \\ q_z \end{pmatrix} \tag{269}$$

with $\rho_{xy} = \sqrt{q_x^2 + q_y^2}$.

The inverse transformation Λ^{-1} is

$$\Lambda^{-1} = (\mathcal{B}(\beta)\mathcal{R})^{-1} = \mathcal{R}^T\mathcal{B}(-\beta) = \mathcal{R}^T\mathcal{B}(-\beta)^T = (\mathcal{B}(-\beta)\mathcal{R})^T$$

or

$$\Lambda^{-1} = \begin{pmatrix} \gamma(\beta) & 0 & 0 & \beta \\ \beta\frac{q_x}{\rho} & \frac{q_y}{\rho_{xy}} & \frac{q_x q_z}{\rho_{xy}\rho} & \gamma(\beta)\frac{q_x}{\rho} \\ \beta\frac{q_y}{\rho} & -\frac{q_x}{\rho_{xy}} & \frac{q_y q_z}{\rho_{xy}\rho} & \gamma(\beta)\frac{q_y}{\rho} \\ \beta\frac{q_z}{\rho} & 0 & -\frac{\rho_{xy}}{\rho} & \gamma(\beta)\frac{q_z}{\rho} \end{pmatrix}.$$

8.12 Similarity transforms and compositions of $M(p), D, S(A)$ triples

A common similarity transform of a triple of arrays $M(p), D, S(A)$ that satisfy (43) also satisfy (43) possibly with exception of the direct sum decomposition of $M(p)$ into boson $B(p)$ and fermion $F(p)$ components. The similarity transform applies a real ($\overline{V} = V$) or pure imaginary ($\overline{V} = -V$) orthogonal matrix ($V^T = V^{-1}$). To preserve the decomposition $M(p) = B(p) \oplus F(p)$ with nonzero $B(p)$ and $F(p)$, the similarity transform is limited to apply individually to $B(p)$ and $F(p)$, $V = V_B \oplus V_F$ with V_B of the same dimension as $B(p)$ followed by a second similarity transformation by a permutation matrix [27]. Then the triple

$$VM(p)V^T, VDV^T, VS(A)V^T \tag{270}$$

satisfies (43) if $M(p), D, S(A)$ satisfy (43). Demonstration reveals the sufficiency of a real or pure imaginary orthogonal transform and $V = V_B \oplus V_F$,

$$\begin{aligned}
\overline{VDV^T}VDV^T &= V\overline{D}V^TVDV^T = V\overline{D}DV^T = \mathbb{I}_{N_c} \\
VDV^TVM(p)V^T &= VDM(p)V^T = (VC(p)V^T)^*VC(p)V^T \\
VM(\Lambda^{-1}p)V^T &= VS(A)M(p)S(A)^TV^T = (VS(A)V^T)VM(p)V^T(VS(A)V^T)^T \\
VDV^TVS(A)V^T &= \overline{VS(A)V^T}VDV^T \\
VM(p)V^T &= (V_B \oplus V_F)B(p) \oplus F(p)(V_B^T \oplus V_F^T) = (V_BB(p)V_B^T) \oplus (V_FF(p)V_F^T) \\
(V_BB(p)V_B^T)^T &= V_BB(-p)V_B^T \\
(V_FF(p)V_F^T)^T &= -V_FF(-p)V_F^T.
\end{aligned}$$

A common similarity transform also preserves satisfaction of (79).

$$VS(A)V^T = VS(A_1)S(A_2)V^T = VS(A_1)V^TVS(A_2)V^T$$

from the invertability of V. A final, common permutation matrix similarity transform O permutes the indices κ_j but applies individually to the boson and fermion components, and effectively preserves a decomposition of $M(p)$ into boson and fermion constituents.

Given two solutions $M_1(p), D_1, S_1(A)$ and $M_2(p), D_2, S_2(A)$ of dimensions N_1 and N_2 respectively that satisfy the conditions to construct a physically nontrivial realization of relativistic quantum physics (43) and (79), their direct sums

$$M(p) = M_1(p) \oplus M_2(p), D = D_1 \oplus D_2, S(A) = S_1(A) \oplus S_2(A) \tag{271}$$

and Kronecker products

$$M(p) = M_1(p) \otimes M_2(p), D = D_1 \otimes D_2, S(A) = S_1(A) \otimes S_2(A) \tag{272}$$

also satisfy (43). The direct sum results in arrays

$$U \oplus V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

of dimension $N_c = N_1 + N_2$. $U, V = M_j(p), D_j, S_j(A)$ and $j = 1, 2$. The components $B_j(p)$ of dimension N_{B_j} and $F_j(p)$ of dimension N_{F_j} are the boson and fermion components of $M_j(p)$. The Kronecker product is

$$U \otimes V = \begin{pmatrix} U_{11}V & U_{12}V & \cdots & U_{1N_1}V \\ U_{21}V & U_{22}V & \cdots & U_{2N_1}V \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_1 1}V & U_{N_1 2}V & \cdots & U_{N_1 N_1}V \end{pmatrix}$$

with U_{ij} the elements of the $N_1 \times N_1$ matrix array U and V is the $N_2 \times N_2$ matrix. The result is an $(N_1 N_2) \times (N_1 N_2)$ array. That both the direct sum and Kronecker product of solutions also satisfy the required conditions (43) and (79) follows from the mixed product properties of direct sums of matrices

$$(U \oplus V)(C \oplus D) = UC \oplus VD$$

and Kronecker products of matrices,

$$(U \otimes V)(C \otimes D) = UC \otimes VD,$$

the linearity of the direct sum

$$(U + C) \oplus (V + D) = U \oplus V + U \oplus D + C \oplus V + C \oplus D$$

and Kronecker product

$$(U + C) \otimes (V + D) = U \otimes V + U \otimes D + C \otimes V + C \otimes D,$$

and the transpose properties

$$U^T \oplus V^T = (U \oplus V)^T$$

and

$$U^T \otimes V^T = (U \otimes V)^T.$$

To satisfy these properties, U and C have the same dimension N_1, and V and D have the same dimension N_2.

The demonstrations are for the Kronecker product but the common properties provide that the demonstrations also apply for the direct sum with the evident modifications.

If $\overline{D}_j D_j = \mathbb{I}_{N_j}$ for $j = 1, 2$, then

$$\begin{aligned} \overline{D}D &= (\overline{D}_1 \otimes \overline{D}_2)(D_1 \otimes D_2) \\ &= (\overline{D}_1 D_1) \otimes (\overline{D}_2 D_2) \\ &= \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \\ &= \mathbb{I}_{N_c} \end{aligned}$$

from the mixed product property and the Kronecker product of identity matrices.

If $\overline{D}_j M_j = C_j^* C_j$ for $j = 1, 2$, then

$$\begin{aligned} \overline{D}M &= (\overline{D}_1 \otimes \overline{D}_2)(M_1 \otimes M_2) \\ &= (D_1 M_1) \otimes (D_2 M_2) \\ &= (C_1^* C_1) \otimes (C_2^* C_2) \\ &= (C_1^* \otimes C_2^*)(C_1 \otimes C_2) \\ &= C^* C \end{aligned}$$

with $C = C_1 \otimes C_2$ from the mixed product and transpose properties.

If $S_j M_j S_j^T = M_j'$ with $M_j = M_j(p)$, $M_j' = M_j(\Lambda^{-1} p)$ and $j = 1, 2$, then

$$\begin{aligned} SMS^T &= (S_1 \otimes S_2)(M_1 \otimes M_2)(S_1 \otimes S_2)^T \\ &= ((S_1 M_1) \otimes (S_2 M_2))(S_1^T \otimes S_2^T) \\ &= (S_1 M_1 S_1^T) \otimes (S_2 M_2 S_2^T) \\ &= M_1' \otimes M_2' \\ &= M' \end{aligned}$$

from the mixed product and transpose properties.

If $D_j S_j = \overline{S}_j D_j$ for $j = 1, 2$, then

$$\begin{aligned} DS &= (D_1 \otimes D_2)(S_1 \otimes S_2) \\ &= (D_1 S_1) \otimes (D_2 S_2) \\ &= (\overline{S}_1 D_1) \otimes (\overline{S}_2 D_2) \\ &= (\overline{S}_1 \otimes \overline{S}_2)(D_1 \otimes D_2) \\ &= \overline{(S_1 \otimes S_2)}(D_1 \otimes D_2) \\ &= \overline{S} D \end{aligned}$$

from the mixed product property and complex conjugation.

If both $M_1(p)$ and $M_2(p)$ satisfy (39) and (40), then a permutation matrix O similarity transform produces

$$M(p) = M_1(p) \oplus M_2(p), \quad \text{or} \quad M(p) = M_1(p) \otimes M_2(p)$$

of the form

$$O M(p) O^T = B(p) \oplus F(p).$$

For the block diagonal direct sum it follows that $B(p) = B_1(p) \oplus B_2(p)$ and $F(p) = F_1(p) \oplus F_2(p)$. For the Kronecker product, define $N_j \times N_j$ matrices that fill out the arrays B_j and F_j with zeros such that

$$M_j = \begin{pmatrix} B_j & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_j \end{pmatrix}.$$

Then

$$M = M_1 \otimes M_2$$

equals

$$M_1 \otimes M_2 = \begin{pmatrix} B_1 \otimes B_2 & 0 & 0 & 0 \\ 0 & B_1 \otimes F_2 & 0 & 0 \\ 0 & 0 & F_1 \otimes B_2 & 0 \\ 0 & 0 & 0 & F_1 \otimes F_2 \end{pmatrix}$$

from linearity. $B_1 \otimes B_2$ is a square matrix of dimension $N_{B_1} N_{B_2}$ and $F_1 \otimes F_2$ is a square matrix of dimension $N_{F_1} N_{F_2}$. The transpose property provides that both transform with a positive sign. $B_1 \otimes F_2$ is a square matrix of dimension $N_{B_1} N_{F_2}$ and $F_1 \otimes B_2$ is a square matrix of dimension $N_{B_1} N_{F_2}$. The transpose property provides that both transform with a negative sign. A permutation matrix O places M into the direct sum form

$$O(M_1 \otimes M_2) O^T = \begin{pmatrix} B(p) & 0 \\ 0 & F(p) \end{pmatrix}$$

with

$$B(p) = \begin{pmatrix} B_1 \otimes B_2 & 0 \\ 0 & F_1 \otimes F_2 \end{pmatrix}$$

and

$$F(p) = \begin{pmatrix} B_1 \otimes F_2 & 0 \\ 0 & F_1 \otimes B_2 \end{pmatrix}.$$

The dimension of the direct sum of the four block diagonal matrix components is

$$N_c = N_{B_1} N_{B_2} + N_{B_1} N_{F_2} + N_{F_1} N_{B_2} + N_{F_1} N_{F_2}$$

from $N_c = N_1 N_2$ and $N_j = N_{B_j} + N_{F_j}$.

If $S_j(A) = S_j(A_1) S_j(A_2)$ for $j = 1, 2$, then

$$\begin{aligned} S(A) &= S_1(A) \otimes S_2(A) \\ &= (S_1(A_1) S_1(A_2)) \otimes (S_2(A_1) S_2(A_2)) \\ &= (S_1(A_1) \otimes S_2(A_1))(S_1(A_2) \otimes S_2(A_2)) \\ &= S(A_1) S(A_2) \end{aligned}$$

from the mixed product property.

A triple $M(p), D, S(A)$ is designated as *decomposable* if there is a real ($V = \overline{V}$) or imaginary ($V = -\overline{V}$) orthogonal matrix V ($V^{-1} = V^T$) such that

$$\begin{aligned} V M(p) V^T &= M_1(p) \oplus M_2(p) \\ V D V^T &= D_1 \oplus D_2 \\ V S(A) V^T &= S_1(A) \oplus S_2(A) \end{aligned} \tag{273}$$

with $N_1 \times N_1$ dimensional triple $M_1(p), D_1, S_1(A)$ and $(N_c - N_1) \times (N_c - N_1)$ dimensional triple $M_2(p), D_2, S_2(A)$ that individually satisfy (43) and (79). To preserve satisfaction of axioms A.1-7, and physically, to preserve superselection, V can not connect boson and fermion indices κ. Then, $OV = V_B \oplus V_F$ with V_B of the same dimension as $B(p)$, and O is the permutation matrix [27] that sets

$$\begin{aligned} V M(p) V^T &= M_1(p) \oplus M_2(p) \\ &= (B_1(p) \oplus F_1(p)) \oplus (B_2(p) \oplus F_2(p)) \\ &= O^T (B_1(p) \oplus B_2(p)) \oplus (F_1(p) \oplus F_2(p)) O. \end{aligned}$$

Then the additional real orthogonal similarity transform O follows V and transforms the decomposable triple $M(p), D, S(A)$ into the required form (39). Permutation matrices are real and orthogonal [27].

A low dimension example of a decomposable (273) Kronecker product of triples is the product of the triples $M(p), D, S(A)$ for two charged scalar boson representations. The Kronecker products are

$$M(p) \otimes M(p) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, $D \otimes D = M(p) \otimes M(p)$ and $S(A) \otimes S(A) = \mathbb{I}_4$. $N_c = 2^2 = 4$. The permutation matrix

$$O = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

produces

$$O(M(p) \otimes M(p))O^T = O \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} O^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the Kronecker product composed triple decomposes as the direct sums

$$\begin{aligned} O(M(p) \otimes M(p))O^T &= M(p) \oplus M(p) \\ O(D \otimes D)O^T &= D \oplus D \\ O(S(A) \otimes S(A))O^T &= S(A) \oplus S(A). \end{aligned}$$

The triple from the Kronecker products of two charged scalar field representations is permutation matrix similar to the direct sum of two, two component scalar field representations. A charge operator is the generator of the symmetry $S_\phi \otimes S_\phi$,

$$Q = -i \, \frac{d}{d\phi}(S_\phi \otimes S_\phi)\Big|_{\phi=0}$$

with S_ϕ the charged scalar boson charge symmetry operator (44).

$$O(S_\phi \otimes S_\phi)O^T = O \begin{pmatrix} e^{2i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i\phi} \end{pmatrix} O^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2i\phi} & 0 \\ 0 & 0 & 0 & e^{-2i\phi} \end{pmatrix}$$

and the charges are $0, 0, 2, -2$ for the four component functions $(O\cdot)_1 f_1(x)_\kappa$, $1 \leq \kappa \leq 4$ in the one-argument subspace, with the shorthand notation (15). The Kronecker product of two charged scalar field representations includes an uncharged particle-antiparticle pair and a charge two particle-antiparticle pair.

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

8.13 Two body classical trajectories

In this appendix, notation is established and results for two body trajectories from classical mechanics are collected.

non-relativistic two body problems provide explicit examples of classical trajectories $\mathbf{u}_j(\lambda)$. The motion of two classical bodies interacting by a scalar pair potential is executed within a plane. A pair potential depends solely on the body separation and constant properties of the bodies. The location of bodies in a plane at each time λ is specified by Cartesian spatial coordinates,

$$\mathbf{u}_j = \begin{pmatrix} r_j \cos\theta_j \\ r_j \sin\theta_j \\ 0 \end{pmatrix} \tag{274}$$

with $r_j = \|\mathbf{u}_j\|$ the Euclidean length. Notation is abbreviated, $\mathbf{u}_j = \mathbf{u}_j(\lambda)$ and similarly for r_j, θ_j.

The Lagrangian for a two body problem in the notation of this note is

$$\mathcal{L} = \frac{1}{2} m_1 c^2 \dot{\mathbf{u}}_1^2 + \frac{1}{2} m_2 c^2 \dot{\mathbf{u}}_2^2 - V(\|\mathbf{u}_1 - \mathbf{u}_2\|)$$

with V the pair potential. The separation of the two bodies is

$$\mathbf{u}_s = \mathbf{u}_1 - \mathbf{u}_2.$$

In a reference frame with the center-of-mass coincident with the origin of coordinates,

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = 0.$$

The solutions for $\mathbf{u}_i$ given the separation $\mathbf{u}_s$ in the center-of-mass coordinate frame are

$$\mathbf{u}_1 = \frac{m_2 \mathbf{u}_s}{m_1 + m_2}, \qquad \mathbf{u}_2 = -\frac{m_1 \mathbf{u}_s}{m_1 + m_2}.$$

Substitution results in a single body Lagrangian for $\mathbf{u}_s$,

$$\mathcal{L} = \frac{1}{2} \mu c^2 \dot{\mathbf{u}}_s^2 - V(\|\mathbf{u}_s\|),$$

with μ denoted the reduced mass.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This results in Newton's equation of motion for the separation,

$$\mu c^2 \ddot{\mathbf{u}}_s = -\frac{\partial V}{\partial \mathbf{u}_s} \tag{275}$$

with

$$\frac{\partial V}{\partial \mathbf{u}_s} = \nabla_{\mathbf{u}_s} V$$

a gradient vector. For equal masses in the center-of-mass reference frame,

$$\mu = \frac{m}{2}, \qquad \mathbf{u}_2 = -\mathbf{u}_1$$

and with a $-g/r$ pair potential, $V = -gmc^2/\|\mathbf{u}_s\|$, substitution of $2\|\mathbf{u}_1\| = \|\mathbf{u}_s\|$ for the separation results in equation of motion

$$\ddot{\mathbf{u}}_1 = -\frac{g}{4}\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|^3}. \tag{276}$$

The interaction is characterized by the length g. For gravity, $g = Gm^2/mc^2$ and for electrostatics $g = Ke^2/mc^2$. For a mass of one a.m.u. and an elementary charge of one, $g = 1.23 \times 10^{-53}$ m for gravity and $g = 1.54 \times 10^{-18}$ m for electrostatics.

With

$$L = \mathbf{u}_1^2 \dot{\theta}_1,$$

the relationships of Cartesian and radial coordinates for two equal mass bodies $m = m_1 = m_2$ in the center-of-mass reference frame ($\mathbf{u}_2 = -\mathbf{u}_1$) include

$$\begin{aligned} \mathbf{u}_j^2 &= r_j^2 \\ \dot{\mathbf{u}}_j \cdot \mathbf{u}_j &= \dot{r}_j r_j \\ \dot{\mathbf{u}}_j^2 &= \dot{r}_j^2 + \frac{L^2}{r_j^2}. \end{aligned} \tag{277}$$

The trajectory $\mathbf{u}_1$ in the center-of-mass reference frame for the pair of equal mass bodies interacting via a $-g/r$ pair potential are conveniently parameterized by $\theta = \theta_1$ from (274). A solution to Kepler's problem is

$$r_1 = \frac{L^2/4g}{1 - \epsilon_r \cos\theta}$$

and

$$\lambda(\theta) = \frac{1}{L}\int_{\theta_0}^{\theta} d\phi \; r_1(\phi)^2$$

with $\lambda(\theta_0) = 0$ and

$$\epsilon_r = \sqrt{1 + 32e_1L^2/g^2}.$$

For $e_1 > 0$, the solution is unbound and diverges when $\epsilon_r \cos\theta = 1$. For these unbound trajectories, θ is constrained to the interval $(\beta, 2\pi - \beta)$ with $\beta = \cos^{-1}(\epsilon_r^{-1})$. Bound states have $e_1 \leq 0$ and

$$e_1 \geq -\frac{g^2}{32L^2}.$$

$e_1 = -g^2/32L^2$ are the circular orbits. For the circular orbits of $-g/r$ pair potentials,

$$4L^2 = gr_1.$$

8.14 Non-relativistic energy approximations

In this appendix, non-relativistic approximations to the Hamiltonian (102) are developed.

A non-relativistic approximation of the energies ω_k from (10) derives from the Taylor theorem polynomial approximation for momenta $\mathbf{p}_k$ near $\mathbf{q}_k$. To second order in $\mathbf{p}_k - \mathbf{q}_k$,

$$\omega_k \approx \omega(\mathbf{q}_k) + \frac{\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)}{\omega(\mathbf{q}_k)} + \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)} - \frac{(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2}{2\omega(\mathbf{q}_k)^3}$$

and the Cauchy-Schwarz-Bunyakovski inequality provides that

$$(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2 \leq \mathbf{q}_k^2 \, (\mathbf{p}_k - \mathbf{q}_k)^2.$$

For non-relativistic velocities

$$\mathbf{q}_k^2 \ll \omega(\mathbf{q}_k)^2$$

and then

$$\frac{(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2}{2\omega(\mathbf{q}_k)^3} \ll \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)}.$$

This justifies neglect of the last term from the Taylor series in (278) if the support of state describing functions excludes relativistic momenta, if, for example, (156) of section 5.4 applies. The resulting non-relativistic approximation for ω_k is

$$\omega_k \approx \omega(\mathbf{q}_k) + \frac{\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)}{\omega(\mathbf{q}_k)} + \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)}. \tag{278}$$

The approximation applies for non-relativistically supported state describing functions. If $\mathbf{q}_k$ is a representative for the support of a state describing function and $\mathbf{p}$ is any point from the non-relativistic dominant support, then $\|\mathbf{p}_k - \mathbf{q}_k\| \ll \lambda_c^{-1} \leq \omega(\mathbf{q}_k)$. (278) provides a convenient approximation for the Hamiltonian (102) applied in each n-argument subspace,

$$H = \sum_k \omega_k.$$

If $\mathbf{q}_k$ and $\mathbf{p}_k$ are both non-relativistic, $\|\mathbf{q}_k\|, \|\mathbf{p}_k\| \ll \lambda_c^{-1}$, then both correction terms to $\omega(\mathbf{q}_k)$ in (278) are second order in small quantities. Neglect of the term proportional to $(\mathbf{p}_2 - \mathbf{q}_k)^2$ in (154) results in a convenient linear in $\mathbf{p}_2$ approximation for the energy. However, with non-relativistic momenta, this approximation is also second order in small quantities and neglect of the quadratic term is not justified. Neglect of the second correction requires that $(\mathbf{p}_k - \mathbf{q}_k)^2 \ll |\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)|$ in addition to non-relativistic momenta.

A non-relativistic approximation (125) of the energies $\omega(\mathbf{q}_k)$ also derives from a Taylor theorem polynomial approximation if $\|\mathbf{q}_k\| \ll \omega(\mathbf{q}_k)$.

$$\begin{aligned} \omega(\mathbf{q}_k) &\approx \frac{1}{\lambda_c} + \frac{\lambda_c \mathbf{q}_k^2}{2} \\ &= \frac{1}{\lambda_c}\left(1 + \frac{\dot{\mathbf{u}}_k(\lambda)^2}{2}\right) \end{aligned}$$

from the non-relativistic relation between momentum and velocity (125) and with the reduced Compton wavelength (11). This approximation applies for non-relativistic velocities of the corresponding classical trajectories $\mathbf{u}_k(\lambda)$ in appropriate reference frames.

The center-of-mass and the relative motion of bodies decouple in non-relativistic instances. For two argument functions (145) in section 5.4, the assumption that decouples the motions is that the momentum of the center-of-momentum $\mathbf{p}'_3 = \mathbf{p}_3 + \mathbf{p}_4$ from (145) in section 5.4 is non-relativistic. Then, from (125),

$$\mathbf{p}_3'^2 \ll \frac{1}{\lambda_c^2} < \omega(\mathbf{p}'_3)^2.$$

In these non-relativistic instances, Taylor expansion results in

$$\begin{aligned} \omega_3 + \omega_4 &= \omega(\tfrac{\mathbf{p}'_3 + \mathbf{p}'_4}{2}) + \omega(\tfrac{\mathbf{p}'_3 - \mathbf{p}'_4}{2}) \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}'_4) + \frac{\mathbf{p}_3'^2}{4\omega(\frac{1}{2}\mathbf{p}'_4)} \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}'_4) + \tfrac{1}{4}\lambda_c \mathbf{p}_3'^2 \end{aligned}$$

using (10), (125) and (278). In this non-relativistic approximation, the argument of the conservation of energy delta function becomes independent of $\mathbf{p}'_1$ and $\mathbf{p}'_3$ as a consequence of momentum conservation, $\mathbf{p}'_1 = \mathbf{p}'_3$.

$$\begin{aligned} \omega_1 + \omega_2 - \omega_3 - \omega_4 &\approx 2\omega(\tfrac{1}{2}\mathbf{p}'_2) + \tfrac{1}{4}\lambda_c \mathbf{p}_1'^2 - 2\omega(\tfrac{1}{2}\mathbf{p}'_4) - \tfrac{1}{4}\lambda_c \mathbf{p}_3'^2 \\ &= 2\omega(\tfrac{1}{2}\mathbf{p}'_2) - 2\omega(\tfrac{1}{2}\mathbf{p}'_4). \end{aligned} \tag{279}$$

This approximation appears in the energy conservation delta function independently of the propagation interval λ, unlike the generation of time translation with λ multiplying any error in the Hamiltonian.

8.15 Gaussian quadratures

In this section, the non-relativistic, brief interval, limited acceleration approximation of the functional $Q(1)$ is evaluated as elementary functions for a range of trajectories determined by initial conditions $\mathbf{u}(0), \dot{\mathbf{u}}(0)$. From (177), the $Q(F)$ of interest follow as derivatives of $Q(1)$. For this evaluation, the free field VEV contribution is distinguished from the connected VEV contribution,

$$Q(F) = Q_F(1) + Q_C(1), \tag{280}$$

with

$$Q_F(1) = \frac{a_g}{\lambda_c^2} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2' - \mathbf{p}_4') \; e^{\mathbf{p}_2' \cdot \mathbf{b}_2 + \mathbf{p}_4' \cdot \mathbf{b}_4} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}$$

and

$$Q_C(1) = \frac{a_g c_4}{8\lambda_c} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2) \; e^{\mathbf{p}_2' \cdot \mathbf{b}_2 + \mathbf{p}_4' \cdot \mathbf{b}_4} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}$$

from the definitions (152) for T_4 and (175) for $Q(F)$, and in the abbreviated notation (183).

Evaluation of $Q_F(1)$ and $Q_C(1)$ follows from the Gaussian quadrature

$$\sqrt{\sigma^2} \int_{-\infty}^{\infty} ds \; e^{-\sigma^2 s^2 + \beta s} = \sqrt{\pi} \; e^{\beta^2/(4\sigma^2)} \tag{281}$$

for $\sigma^2, \beta \in \mathbb{C}$ with $\mathbb{R}e(\sigma^2) > 0$.

It is convenient to introduce compact notation for the complex length parameter,

$$\sigma^2 = \sigma_R^2 + i\sigma_Q^2 \tag{282}$$

with $\sigma_R^2, \sigma_Q^2 \in \mathbb{R}$ and $\sigma_R^2 > 0$.

From (280), evaluation of the momentum conservation delta function, $e^{a+b} = e^a e^b$, translation of the summation variables, and substitution of the Gaussian summation (281) result in

$$\begin{aligned} Q_F(1) &= \frac{a_g}{\lambda_c^2} \int d\mathbf{p}_2' \; e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \; e^{-2\sigma_R^2 (\mathbf{p}_2' - 2\mathbf{w})^2} \\ &= \frac{a_g}{\lambda_c^2} e^{2\mathbf{w} \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \int d\mathbf{p}_2' \; e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \, e^{-2\sigma_R^2 \, \mathbf{p}_2'^2} \\ &= \frac{a_g}{\lambda_c^2} \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{3}{2}} e^{2\mathbf{w} \cdot (\mathbf{b}_2 + \mathbf{b}_4)} e^{\frac{(\mathbf{b}_2 + \mathbf{b}_4)^2}{8\sigma_R^2}} \end{aligned} \tag{283}$$

for the free field VEV contribution to $Q(1)$.

Expressing the summations in spherical coordinates is convenient to evaluate the energy conservation delta function in the connected VEV contribution $Q_C(1)$ to $Q(1)$. Factoring the Gaussian functions in (280) provides that

$$\begin{aligned} Q_C(1) &= \frac{c_4 a_g}{8\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2) \\ &\quad \times e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})} e^{\mathbf{p}_4' \cdot (\mathbf{b}_4 + 4\sigma^2 \mathbf{w})} e^{-\overline{\sigma^2}\mathbf{p}_2'^2} e^{-\sigma^2 \mathbf{p}_4'^2}. \end{aligned} \tag{284}$$

A selection of $\mathbf{b}_2 = \overline{\sigma^2}\mathbf{c}_2$, $\mathbf{b}_4 = \sigma^2 \mathbf{c}_4$ with $\mathbf{c}_2, \mathbf{c}_4 \in \mathbb{R}^3$ results in $\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w}$ and $\mathbf{b}_4 + 4\sigma^2\mathbf{w}$ that are complex constants times real spatial vectors. Then, spherical coordinates simplify the summations in (284). Subsequently, the result from evaluation of the energy conserving delta function and summations is analytically extended to evaluate $Q_C(1)$ at the physical values of interest, $\mathbf{b}_4 = -\mathbf{b}_2 = i\mathbf{u}$.

A change of summation variables to spherical coordinates

$$\mathbf{p}_j' = (\rho_j' \cos\theta_j \cos\phi_j, \rho_j' \sin\theta_j \cos\phi_j, \rho_j' \sin\phi_j)$$

with the z_j-axes aligned with the real vectors $\mathbf{c}_j + 4\mathbf{w}$ sets

$$\mathbf{p}_j' \cdot (\mathbf{b}_j + 4\sigma^2 \mathbf{w}) = \rho_j' r_j \sin\phi_j.$$

$j = 2, 4$ and

$$\begin{aligned} r_2 &= \overline{\sigma^2} \left((\mathbf{c}_2 + 4\mathbf{w})^2 \right)^{\frac{1}{2}} \\ &= \left((\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})^2 \right)^{\frac{1}{2}} \\ r_4 &= \sigma^2 \left((\mathbf{c}_4 + 4\mathbf{w})^2 \right)^{\frac{1}{2}} \\ &= \left((\mathbf{b}_4 + 4\sigma^2 \mathbf{w})^2 \right)^{\frac{1}{2}} \end{aligned} \tag{285}$$

with $\mathbb{R}e(r_j) > 0$ from $\mathbb{R}e(\sigma^2) > 0$. Rotational invariance of the state describing function (155) justifies the selection of axes. With the change to spherical coordinates,

$$\begin{aligned} Q_C(1) &= \frac{c_4 a_g}{8\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int_0^\infty \rho_2'^2 d\rho_2' \int_0^\infty \rho_4'^2 d\rho_4' \; \delta(\rho_2'^2 - \rho_4'^2) \; e^{-\overline{\sigma^2}\rho_2'^2} e^{-\sigma^2 \rho_4'^2} \\ &\quad \times \int_0^{2\pi} d\theta_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_2 d\phi_2 \; e^{\rho_2' r_2 \sin\phi_2} \int_0^{2\pi} d\theta_4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_4 d\phi_4 \; e^{\rho_4' r_4 \sin\phi_4}. \end{aligned}$$

The θ_j and ϕ_j summations are elementary.

$$\int_0^{2\pi} d\theta_j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_j \, d\phi_j \; e^{\rho_j' r_j \sin\phi_j} = 2\pi \; \frac{e^{\rho_j' r_j} - e^{-\rho_j' r_j}}{\rho_j' r_j}.$$

The ρ'_j summations are

$$\begin{aligned}\int_0^\infty \rho_j'^2 d\rho'_j \ \frac{e^{\rho'_j r_j} - e^{-\rho'_j r_j}}{\rho'_j r_j} \ h(\rho_j'^2) &= \int_0^\infty \rho'_j d\rho'_j \ \frac{e^{\rho'_j r_j}}{r_j} h(\rho_j'^2) - \int_0^\infty \rho'_j d\rho'_j \ \frac{e^{-\rho'_j r_j}}{r_j} h(\rho_j'^2) \\ &= \int_0^\infty \rho'_j d\rho'_j \ \frac{e^{\rho'_j r_j}}{r_j} h(\rho_j'^2) - \int_0^{-\infty} \rho'_j d\rho'_j \ \frac{e^{\rho'_j r_j}}{r_j} h((-\rho'_j)^2) \\ &= \int_{-\infty}^\infty \rho'_j d\rho'_j \ \frac{e^{\rho'_j r_j}}{r_j} \ h(\rho_j'^2)\end{aligned}$$

from reflection of the summation variable in the second term. Denoted by $h(\rho_j'^2)$, both the Gaussian functions and energy conservation delta function are even functions of the ρ'_j. Substitution of these summations into $Q_C(1)$ then provides that

$$Q_C(1) = (2\pi)^2 \frac{c_4 a_g}{8\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int_{-\infty}^\infty \rho'_2 d\rho'_2 \int_{-\infty}^\infty \rho'_4 d\rho'_4 \delta(\rho_2'^2 - \rho_4'^2) e^{-2\sigma_R^2 \rho_2'^2} \ \frac{e^{\rho'_2 r_2 + \rho'_4 r_4}}{r_2 r_4}.$$

The delta function [20] is

$$\delta(\rho_2'^2 - \rho_4'^2) = \frac{\delta(\rho'_2 - \rho'_4)}{2|\rho'_2|} + \frac{\delta(\rho'_2 + \rho'_4)}{2|\rho'_2|}$$

and then

$$Q_C(1) = \frac{\pi^2 c_4 a_g}{4\lambda_c r_2 r_4} e^{-8\sigma_R^2 \mathbf{w}^2} \int_{-\infty}^\infty |\rho'_2| d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho'_2 (r_2 + r_4)} - e^{\rho'_2 (r_2 - r_4)} \right). \tag{286}$$

Reflecting the summation variable in the domain $(-\infty, 0)$, the remaining summation reorganizes to

$$\begin{aligned}&\int_{-\infty}^\infty |\rho'_2| d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho'_2 (r_2 + r_4)} - e^{\rho'_2 (r_2 - r_4)} \right) \\ &\quad = \int_0^\infty |\rho'_2| d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho'_2 (r_2 + r_4)} - e^{\rho'_2 (r_2 - r_4)} \right) \\ &\qquad\quad - \int_\infty^0 |\rho'_2| d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{-\rho'_2 (r_2 + r_4)} - e^{-\rho'_2 (r_2 - r_4)} \right) \\ &\quad = \int_0^\infty \rho'_2 d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho'_2 (r_2 + r_4)} - e^{\rho'_2 (r_2 - r_4)} + e^{-\rho'_2 (r_2 + r_4)} - e^{-\rho'_2 (r_2 - r_4)} \right) \\ &\quad = \int_0^\infty \rho'_2 d\rho'_2 \ e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho'_2 r_2} - e^{-\rho'_2 r_2} \right) \left(e^{\rho'_2 r_4} - e^{-\rho'_2 r_4} \right).\end{aligned}$$

For large real components of r_2, r_4, a convenient approximation applies. If

$$\mathbb{R}e(r_2), \mathbb{R}e(r_4) \gg 0,$$

then

$$e^{\rho_2' r_j} \gg e^{-\rho_2' r_j}$$

and the dominant support of the integrand is for large ρ_2'. Neglect of the smaller terms and subsequent inclusion of the weakly weighted summation over $\rho_2' \in (-\infty, 0)$ approximates (286) when $\mathbb{R}e(r_2), \mathbb{R}e(r_4)$ from (285) are sufficiently large. With this approximation,

$$\begin{aligned}
\int_0^\infty \rho_2' d\rho_2' \; e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2' r_2} - e^{-\rho_2' r_2} \right) \left(e^{\rho_2' r_4} - e^{-\rho_2' r_4} \right) &\approx \int_{-\infty}^\infty \rho_2' d\rho_2' \; e^{-2\sigma_R^2 \rho_2'^2} e^{\rho_2'(r_2+r_4)} \\
&= \frac{\partial}{\partial \mu} \int_{-\infty}^\infty d\rho_2' \; e^{-2\sigma_R^2 \rho_2'^2} e^{\rho_2'(r_2+r_4+\mu)} \\
&= \frac{\partial}{\partial \mu} \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{1}{2}} e^{\frac{(r_2+r_4+\mu)^2}{8\sigma_R^2}} \\
&= \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{1}{2}} \left(\frac{r_2+r_4}{4\sigma_R^2} \right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}}
\end{aligned}$$

from the Gaussian summation (281) and evaluated at $\mu = 0$. Sufficiently large r_2, r_4 is determined to set the peak of the dominant support of $e^{-2\sigma_R^2 \rho_2'^2 + \rho_2'(r_2+r_4)}$ much greater than the width of the support. The envelope of the support is determined by the real component σ_R^2 of σ^2. The width of support is determined from

$$e^{-2\sigma_R^2 \rho_2'^2 + \rho_2' \mathbb{R}e(r_2+r_4)} = e^{-2\sigma_R^2 (\rho_2' - \rho_o)^2} e^{4\sigma_R^2 \rho_o^2}$$

for $\rho_o = \mathbb{R}e(r_2 + r_4)/(4\sigma_R^2)$, the value of ρ_2' at the peak of the envelope of support. Then large r_j is

$$\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\sigma_R^2} \tag{287}$$

for r_2, r_4 that are the analytic extensions of (285). This approximation cannot be arbitrarily accurate with $\sigma_R^2 \to 0$ due to the bound that ensures the support of the state describing function (155) is non-relativistic.

$$\sigma_R^2 \gg \lambda_c^2$$

from (156). Finally, substitution provides

$$Q_C(1) \approx \frac{\pi^2 c_4 a_g}{4\lambda_c} \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{1}{2}} \frac{e^{-8\sigma_R^2 \mathbf{w}^2}}{4\sigma_R^2} \left(\frac{r_2 + r_4}{r_2 r_4} \right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}} \tag{288}$$

for $\mathbf{b}_2 = \overline{\sigma^2} \mathbf{c}_2$ and $\mathbf{b}_4 = \sigma^2 \mathbf{c}_4$ with real $\mathbf{c}_j$.

The value of $Q_C(1)$ of physical interest has $\mathbf{b}_j$ from (176). In (288), the $\mathbf{b}_j$ are extended in $\mathbb{C}^3$. After analytic extension,

$$\begin{aligned}
r_2 &= \left((-i\mathbf{u} + 4\overline{\sigma^2}\mathbf{w})^2 \right)^{\frac{1}{2}} \\
r_4 &= \left((i\mathbf{u} + 4\sigma^2 \mathbf{w})^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{289}$$

and then

$$\overline{r_2} = r_4.$$

Singularities in the analytic extension of the approximation (288) for $Q_C(1)$ include simple divergences at $r_j = 0$ and cut lines due to the multiple sheets of the square root (285) in r_j. Cut lines are oriented toward negative real values of the r_j to avoid the large positive real values of interest. With

$$r_j^2 = a_j + ib_j,$$

$a_j, b_j \in \mathbb{R}$, the root of r_j^2 is selected to set $\mathbb{R}e(r_j) \geq 0$. Then

$$\begin{aligned} \mathbb{R}e(r_j) &= \sqrt{\tfrac{1}{2}\left(\sqrt{a_j^2 + b_j^2} + a_j\right)} \\ \mathbb{I}m(r_j) &= \operatorname{sgn}(b_j)\sqrt{\tfrac{1}{2}\left(\sqrt{a_j^2 + b_j^2} - a_j\right)} \end{aligned} \tag{290}$$

from half-angle formulas for cosine and sine. With the notation (282) for complex $\sigma^2 = L(0)^2$, the physical values of interest follow from (289),

$$\begin{aligned} a_4 &= a_2 \quad = 16(\sigma_R^4 - \sigma_Q^4)\mathbf{w}^2 - \mathbf{u}^2 - 8\sigma_Q^2 \mathbf{u}\cdot\mathbf{w} \\ b_4 &= -b_2 \quad = 32\sigma_R^2\sigma_Q^2\mathbf{w}^2 + 8\sigma_R^2\mathbf{u}\cdot\mathbf{w}. \end{aligned} \tag{291}$$

The analytic extension of the quadrature of interest (284) with $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ equals the summation (286) when

$$\begin{aligned} \mathbf{b}_2 &= \overline{\sigma^2}\mathbf{c}_2 \\ \mathbf{b}_4 &= \sigma^2\mathbf{c}_4 \end{aligned}$$

and $\mathbf{c}_j \in \mathbb{R}^3$. The summation (286) analytically extends for $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ and within the regions of holomorphy, this extension equals the extension of (284) by the identity theorem. Both (286) and the approximation (288) are functions of r_2, r_4 that, with exclusion of the isolated singularities and cut lines, are functions over $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$. The approximation (288) applies for $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ with $\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\mathbb{R}e(L(0)^2)}$. If $L(0)^2$, $\mathbf{u}^2$ and $\dot{\mathbf{u}}^2$ satisfy the large r_j condition (287), then the analytic extension of the approximation (288) approximates $Q_C(1)$.

Substitution of the values (176) into (283) and (288) provides the value of interest for $Q(1)$ in the notation (280),

$$\begin{aligned} Q_F(1) &= \frac{a_g}{\lambda_c^2}\left(\frac{\pi}{2L(0)_R^2}\right)^{\frac{3}{2}} \\ Q_C(1) &= \frac{\pi^2 c_4 a_g}{8L(0)_R^2\lambda_c}\left(\frac{\pi}{2L(0)_R^2}\right)^{\frac{1}{2}} e^{-8l(0)_R^2\mathbf{w}^2}\left(\frac{1}{r_2} + \frac{1}{r_4}\right) e^{\frac{(r_2+r_4)^2}{8L(0)_R^2}}. \end{aligned} \tag{292}$$

The r_j are from (290) with (291). The conditions that produce the approximation of (292) are:

1. non-relativistic momenta (125), $\dot{\mathbf{u}}^2 \ll 1$, $L(0)_R^2 \gg \lambda_c^2$ and $\lambda_c \mathbf{w} = \dot{\mathbf{u}}$ from (124). Non-relativistic approximation of the Hamiltonian applies for a limited interval λ

2. significant body separation, $\mathbf{u}^2 \gg L(0)_R^2$, enables the isolation (121) of support to identify a classical body with a region of space and neglect the cross term in the VEV

3. sufficiently brief intervals to neglect $\mathcal{O}(\lambda^2)$ corrections to evolution of the classical dynamical variables (166)

4. limited acceleration (171), $\|\ddot{\mathbf{u}}(0)\| \leq \epsilon \lambda_c / \mathbf{u}^2$, justifies neglect of the envelope evolution correction. Limited error implies a sufficiently limited interval

5. large r_j (287) enables approximation of the scalar products with elementary forms.

8.16 Expected value of the energy of $\varphi_2(0)$

In this appendix, approximations to expected value of the energy

$$E_q = \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\|\varphi_2(0)\|^2}$$

are developed.

H is the Hamiltonian (102) in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$ and the state describing functions $\varphi_2(0)$ are of the form (144) with (148) and (155). The evaluation is a non-relativistic and large r_j approximation. With appropriate selection of parameters $\mathbf{u}(0)$, $\dot{\mathbf{u}}(0)$ and $L(0)^2$, these state describing functions exhibit quantum-classical correspondences.

The Hamiltonian (102) in the two-argument subspace is $\omega_1 + \omega_2$. Similarly to the development in section 5.4.1, for non-relativistic momenta $\lambda_c^2 \mathbf{p}_1'^2 \ll 1$,

$$\omega_1 + \omega_2 \approx 2\omega(\frac{1}{2}\mathbf{p}_2') + \frac{1}{4}\lambda_c \mathbf{p}_1'^2$$

in the Jacobi coordinates (145). The contribution $\frac{\lambda_c}{4}\mathbf{p}_1'^2$ of the center-of-mass to the energy is negligible if either the representative momentum condition (120) applies or the center-of-momentum description $\tilde{f}_M$ has a zero expectation. For $\lambda_c^2 \mathbf{p}_2'^2 \ll 1$, the approximation (154),

$$2\omega(\frac{\mathbf{p}_2'}{2}) \approx 2\lambda_c^{-1} + \frac{\lambda_c}{4}\mathbf{p}_2'^2,$$

the definition of $Q(F)$ (175), and the relation of powers of momenta with derivatives of $Q(1)$ (177) results in

$$\langle H\varphi_2(0)|\varphi_2(0)\rangle \approx \left(2\lambda_c^{-1} + \frac{\lambda_c}{4}\nabla_{\mathbf{b}_2}^2\right) Q(1)$$

with a Laplacian

$$\nabla^2_{\mathbf{b}_2} = \sum_\nu \frac{\partial^2}{\partial \mathbf{b}^2_{2\nu}}.$$

and $\mathbf{b}_2$ is the parameter of $Q(1)$ in (175) from section 5.4.3. The $\mathbf{b}_{2\nu}$ are the three components of $\mathbf{b}_2$, $\nu = x, y, z$.

Evaluation of $\nabla^2_{\mathbf{b}_2} Q(1)$ follows from the indicated differentiations of $Q(1)$. $Q(1)$ is the sum of free field and connected VEV contributions (280) evaluated in (283) and (288) in appendix 8.15.

$$\begin{aligned} Q_F(1) &= \frac{a_g}{\lambda_c^2}\left(\frac{\pi}{2\sigma_R^2}\right)^{\frac{3}{2}} e^{2\mathbf{w}\cdot(\mathbf{b}_2+\mathbf{b}_4)} e^{\frac{(\mathbf{b}_2+\mathbf{b}_4)^2}{8\sigma_R^2}} \\ Q_C(1) &\approx \frac{\pi^2 c_4 a_g}{4\lambda_c}\left(\frac{\pi}{2\sigma_R^2}\right)^{\frac{1}{2}} \frac{e^{-8\sigma_R^2\mathbf{w}^2}}{4\sigma_R^2}\left(\frac{1}{r_2}+\frac{1}{r_4}\right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}} \end{aligned}$$

if $\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\sigma_R^2}$. From (285) in appendix 8.15,

$$\begin{aligned} r_2 &= \left((\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})^2\right)^{\frac{1}{2}} \\ r_4 &= \left((\mathbf{b}_4 + 4\sigma^2\mathbf{w})^2\right)^{\frac{1}{2}} \end{aligned}$$

and the physical values of interest for the parameters $\mathbf{b}_2$, $\mathbf{b}_4$ and σ^2 are (176). $\mathbf{w}$ and $\mathbf{u}$ are the abbreviated notation (183) for initial momenta and positions.

To evaluate the gradient of $Q(1)$, it is convenient to designate $A \in \{F, C\}$, $\nu \in \{x, y, z\}$, $j \in \{2, 4\}$, and derivatives as

$$\partial_{j\nu} g(\mathbf{b}_2, \mathbf{b}_4) = \frac{\partial g(\mathbf{b}_2, \mathbf{b}_4)}{\partial \mathbf{b}_{j\nu}}.$$

Functions $f_{nA}((j\nu)_n)$ are defined

$$\prod_{k=1}^{n} \partial_{j_k\nu_k} Q_A(1) = f_{nA}((j\nu)_n) Q_A(1)$$

with a recursive definition for the functions f_{nA} from the product rule for derivatives.

$$\begin{aligned} f_{1A}(j\nu) &= \frac{\partial_{j\nu} Q_A(1)}{Q_A(1)} \\ f_{k+1,A}((j\nu)_{k+1}) &= (\partial_{j_{k+1}\nu_{k+1}} f_{kA}((j\nu)_k)) + f_{1A}(j_{k+1}\nu_{k+1}) f_{kA}((j\nu)_k). \end{aligned} \tag{293}$$

For $A = F$, differentiation of $Q_F(1)$ (283) results in

$$\begin{aligned} f_{1F}(j\nu) &= 2\mathbf{w}_\nu + \frac{\mathbf{b}_{2\nu} + \mathbf{b}_{4\nu}}{4\sigma_R^2} \\ \partial_{j\nu} f_{1F}(j\nu) &= \frac{1}{4\sigma_R^2}. \end{aligned}$$

Then the recursive definition (293) provides that

$$\begin{aligned}\frac{\lambda_c^2}{4}\nabla^2_{\mathbf{b}_2}Q_F(1) &= \frac{\lambda_c^2}{4}\sum_\nu \left(\partial_{j\nu}f_{1F}(j\nu)+f_{1F}(j\nu)^2\right)Q_F(1)\\ &= \left(\frac{3\lambda_c^2}{16\sigma_R^2}+\dot{\mathbf{u}}^2\right)Q_F(1)\end{aligned}$$

with $\mathbf{b}_2=-\mathbf{b}_4$ from (176). $\lambda_c\mathbf{w}=\dot{\mathbf{u}}$ in the non-relativistic approximation.

For $A=C$, differentiation of $Q_C(1)$ (288) results in

$$\begin{aligned}f_{1C}(j\nu) &= \frac{r_2+r_4}{4\sigma_R^2}\partial_{j\nu}r_j-\frac{(\partial_{j\nu}r_j)}{r_j}\frac{r_{j'}}{r_2+r_4}\\ \partial_{j\nu}f_{1C}(j\nu) &= \frac{r_2+r_4}{4\sigma_R^2}\partial^2_{j\nu}r_j+\frac{(\partial_{j\nu}r_j)^2}{4\sigma_R^2}\\ &\quad -\frac{(\partial^2_{j\nu}r_j)}{r_j}\frac{r_{j'}}{r_2+r_4}+\frac{(\partial_{j\nu}r_j)^2}{r_j^2}\frac{r_{j'}}{r_2+r_4}+\frac{(\partial_{j\nu}r_j)^2}{r_j(r_2+r_4)^2}\,r_{j'}.\end{aligned}$$

with introduction of the notation

$$j'=\begin{cases}2 & \text{if } j=4\\ 4 & \text{if } j=2.\end{cases}$$

Derivatives of r_j with respect to the components of $\mathbf{b}_\ell$ follow from (285).

$$r_2=\left((\mathbf{b}_2+4\overline{\sigma^2}\mathbf{w})^2\right)^{\frac{1}{2}}$$

and then

$$\begin{aligned}\partial_{2\nu}r_2 &= \frac{\mathbf{b}_{2\nu}+4\overline{\sigma^2}\mathbf{w}_\nu}{r_2}\\ \partial^2_{2\nu}r_2 &= \frac{1}{r_2}-\frac{(\mathbf{b}_{2\nu}+2\overline{\sigma^2}\mathbf{w}_\nu)^2}{r_2^3}.\end{aligned}$$

Then, the recursive definition (293) provides

$$\begin{aligned}\nabla^2_{\mathbf{b}_2}Q_C(1) &= \left(\sum_\nu \partial_{2\nu}f_{1C}(2\nu)+f_{1C}(2\nu)^2\right)Q_C(1)\\ &\approx \left(\sum_\nu \frac{r_2+r_4}{4\sigma_R^2}\partial^2_{2\nu}r_2+\frac{(\partial_{2\nu}r_2)^2}{4\sigma_R^2}+\left(\frac{r_2+r_4}{4\sigma_R^2}\partial_{2\nu}r_2\right)^2\right)Q_C(1)\\ &\approx \left(\frac{r_2+r_4}{4\sigma_R^2}\right)^2\sum_\nu(\partial_{2\nu}r_2)^2\,Q_C(1)\end{aligned}$$

applying the large r_j approximation to retain only the most significant among the proliferation of terms. From (285),

$$\sum_\nu (\partial_{2\nu} r_2)^2 = \frac{(\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})^2}{r_2^2} = 1.$$

Collecting results,

$$\frac{\lambda_c}{4}\nabla^2_{\mathbf{b}_2} Q(1) = \left(\frac{3\lambda_c^2}{16\sigma_R^2} + \dot{\mathbf{u}}^2\right)\frac{Q_F(1)}{\lambda_c} + \frac{\lambda_c}{4}\left(\frac{r_2 + r_4}{4\sigma_R^2}\right)^2 Q_C(1)$$

within the non-relativistic and large r_j approximations.

8.17 Essentially local functions

The physical understanding of state descriptions in these notes uses that there are functions arbitrarily dominantly supported within finite volumes among the anti-local functions $\mathcal{P}(\mathbb{R}^4)$. These functions are essentially localized. There are no strictly localized functions in $\mathcal{P}(\mathbb{R}^4)$, [33].

H. Reeh and S. Schlieder [45, 53] demonstrated that the operation $(a^2 - \Delta)^{\frac{1}{2}}$ over $\mathbb{R}^3$ has the *anti-local property*: if both f and $(a^2 - \Delta)^{\frac{1}{2}} f$ vanish within some finite volume of $\mathbb{R}^3$, then $f \in \mathcal{L}^2(\mathbb{R}^3)$ is identically zero. Δ the Laplacian for $\mathbf{x} \in \mathbb{R}^3$. $\varphi \in \mathcal{P}(\mathbb{R}^4)$ has a Fourier transform of the form

$$\tilde{\varphi}(p) = (p_0 + \omega)\,\tilde{g}(p)$$

with $g \in \mathcal{S}(\mathbb{R}^4)$ and $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$ from (10). Then

$$\varphi(x) = -i\frac{dg(x)}{dx_0} + (\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g(x)$$

and φ vanishing in a finite volume provides that both $\dot{g}$ and $(\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g$ vanish in the volume [33]. Anti-locality can be motivated by the observation that if $\varphi(x)$ was supported solely in a finite volume of $\mathbf{x} \in \mathbb{R}^3$, then the Fourier transform would be an entire analytic function of $\mathbf{p} \in \mathbb{R}^3$, but both $p_0\,\tilde{g}(p)$ and $\omega\,\tilde{g}(p)$ can not both be entire due to the cut line of $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$.

One example suffices to demonstrate the existence of essentially localized functions within $\mathcal{P}(\mathbb{R}^4)$. The selected example φ has a Fourier transform

$$\tilde{\varphi}(p) = (p_0 + \omega)\tilde{g}(p_0)\exp(-\alpha\mathbf{p}^2)$$

with $g \in \mathcal{S}(\mathbb{R})$. Then

$$\varphi(x) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\ \left(-ig'(x_0) + \omega g(x_0)\right)\exp(-\alpha\mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x}) \tag{294}$$

from the properties (24) of the Fourier transform (22).

From the Gaussian summation (281) from appendix 8.15,

$$\begin{aligned}
\int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\ \exp(-\alpha\mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x}) &= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\phi d\phi\ \exp(-\alpha\rho^2 - i\rho r\sin\phi) \\
&= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{1}{2}}}\ \exp(-\alpha\rho^2)\frac{\left(e^{-i\rho r} - e^{i\rho r}\right)}{-i\rho r} \\
&= \frac{i}{r}\int_{-\infty}^\infty \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}}\ \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr}\int_{-\infty}^\infty \frac{d\rho}{(2\pi)^{\frac{1}{2}}}\ \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr}\frac{1}{\sqrt{2\alpha}}\exp\left(-\frac{r^2}{4\alpha}\right) \\
&= (2\alpha)^{-\frac{3}{2}}\exp\left(-\frac{r^2}{4\alpha}\right)
\end{aligned}$$

using the spherical symmetry, change to polar coordinates with z-axis aligned with $\mathbf{x} = (0, 0, r)$,

$$\mathbf{p} = (\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, \rho\sin\phi)$$

and ρ, r are the positive roots of the Euclidean lengths, $\rho^2 = \mathbf{p}^2$ and $r^2 = \mathbf{x}^2$, respectively. The change of variable $\rho' = -\rho$ in the second term in the third line and simplification results in the fourth line. In this case, the summation is the product of three one-dimensional Gaussian summations (281) from appendix 8.15 but the development in polar coordinates is preparation for a second required summation. Similarly,

$$\begin{aligned}
\left|\int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\ \omega\exp(-\alpha\mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x})\right| &= \left|\frac{i}{r}\int_{-\infty}^\infty \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}}\ \omega\exp(-\alpha\rho^2 - i\rho r)\right| \\
&< \frac{1}{r}\int_{-\infty}^\infty \frac{|\rho| d\rho}{(2\pi)^{\frac{1}{2}}}\ \omega\exp(-\alpha\rho^2) \\
&= \frac{a}{r}
\end{aligned}$$

with the indicated constant a finite and independent of r. While apparently a loose upper bound, this bound suffices to demonstrate that the dominant support of $\varphi(x)$ from (294) lies

within finite spheres: the likelihood per unit volume decreases as $1/r$ with expanding distance r from the center of support. $\varphi(x)$ is finite for $r = 0$ and

$$\varphi(x) < \frac{|g'(x_0)|}{(2\alpha)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4\alpha}\right) + |g(x_0)|\,\frac{a}{r}$$

for every $g \in \mathcal{S}(\mathbb{R})$ with g from (294). Translations, dilations and any spherically symmetric $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ with essentially localized support substituted for the Gaussian function also have dominant support within a finite volume.

8.18 Illustrative relative states

A measurement process is illustrated in this appendix. The example illustrates interpretations of projection operators, unitary time translations, and relative states discussed in [12] and appendix 8.2.8. The simplified illustration is not derived from the realizations constructed in section 3.

Consider an observer with three orthogonal states of interest interacting with an observed characterized by two orthogonal states. The three states of the observer are designated "no observation," "observed state 1" and "observed state 2." Designate the two observed states as "up" and "down" regardless of whether they are states characterized by a spin. The two states could describe any observed quantity characterized by two possibilities, for example: located within detector A or B; Schrödinger's live or dead cat; spin up or down. Then, there are six states of interest within the Hilbert space if the observer is described independently of the observed, for example, when the observer is distantly space-like separated from the observed and not entangled. Designate these states using "no" for "no observation," "sup" for "observed state 1" and "sdn" for "observed state 2" with "up" and "dn" for the two observed states. For an ideal measurement, $|\text{sdn}, \text{up}\rangle$ and $|\text{sup}, \text{dn}\rangle$ never appear in nature and these states are not coupled to the states of interest. These two states need not be considered further. This leaves four states of interest.

$$|\text{no}, \text{up}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\text{no}, \text{dn}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\text{sup}, \text{up}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\text{sdn}, \text{dn}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{295}$$

Presumably, these states represent orthogonal subspaces of states $|\underline{f}\rangle$ labeled by particular function sequences $\underline{f}$, and their interaction is described by one of the VEV constructed in section 3.3. The observation is conceived as a scattering event: initially independently described observed and observer are spatially distant, they approach and strongly interact, and then entangled observed with observer states propagate away and cease to interact. An effective Hamiltonian is described below. The Hamiltonian is selected for simplicity.

The evolution of an initial state described by the four orthogonal states (295) is described

$$|v(\lambda)\rangle = U(\lambda)|v_o\rangle$$

with a unitary, 4×4 block diagonal

$$U(\lambda) = \begin{pmatrix} U_2(\lambda) & 0 \\ 0 & U_2(\lambda) \end{pmatrix}$$

using 2×2 unitary

$$U_2(\lambda) = \begin{pmatrix} e^{i\eta}\cos\theta & -e^{i\phi-i\rho}\sin\theta \\ e^{i\rho}\sin\theta & e^{i\phi-i\eta}\cos\theta \end{pmatrix}.$$

The four parameters, θ, ϕ, η, ρ, are real. To simplify this example, the two unitary block submatrices are set equal with $\eta = \rho = \phi = 0$ and

$$\theta(\lambda) = \frac{\pi}{2}\frac{1}{I_\infty}\int_{-\infty}^{\lambda} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}}.$$

The normalization I_∞ is a beta function valid if $\epsilon > 0$.

$$I_\infty = \int_{-\infty}^{\infty} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}} = \frac{\Gamma(\frac{1}{2})\Gamma(\epsilon)}{\Gamma(\frac{1}{2}+\epsilon)}.$$

$\theta(\lambda)$ is absolutely continuous with $\theta(-\infty) = 0$ and $\theta(\infty) = \pi/2$.

The Hamiltonian generates time translation.

$$\begin{aligned} H(\lambda)v(\lambda) &= i\frac{dv(\lambda)}{d\lambda} \\ &= i\dot{U}(\lambda)v_o \\ &= i\dot{U}(\lambda)U^{-1}(\lambda)v(\lambda) \\ &= i\dot{U}(\lambda)U^{*}(\lambda)v(\lambda) \end{aligned}$$

in units of inverse length. For the 2×2 blocks of the Hamiltonian,

$$\begin{aligned} H_2(\lambda) &= i\dot{U}_2(\lambda)U_2^*(\lambda) \\ &= \begin{pmatrix} -i\dot{\theta}\sin\theta & -i\dot{\theta}\cos\theta \\ i\dot{\theta}\cos\theta & -i\dot{\theta}\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i\dot{\theta}(\lambda) \\ i\dot{\theta}(\lambda) & 0 \end{pmatrix} \end{aligned}$$

with

$$\dot{\theta}(\lambda) = \frac{\pi}{2}\,\frac{1}{I_\infty}\,\frac{a^{2\epsilon}}{(a^2+\lambda^2)^{\frac{1}{2}+\epsilon}}.$$

The Hamiltonian is Hermitian as a consequence of the unitarity of $U_2(\lambda)$. The evolution of the initial state v_o occurs most rapidly near $\lambda = 0$. The parameter a is a length characterizing the effective closest approach of observed to observer.

An initial state

$$|v_o\rangle = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and normalization $|\alpha|^2 + |\beta|^2 = 1$ evolves to

$$|v(\lambda)\rangle = U(\lambda)|v_o\rangle = \begin{pmatrix} \alpha\cos\theta(\lambda) \\ \alpha\sin\theta(\lambda) \\ \beta\cos\theta(\lambda) \\ \beta\sin\theta(\lambda) \end{pmatrix} \tag{296}$$

with $\cos\theta(-\infty) = 1$, $\sin\theta(-\infty) = 0$ and $\cos\theta(\infty) = 0$, $\sin\theta(\infty) = 1$. Initially, no observation has occurred and as time evolves, the observed and observer approach and the likelihood of state transitions increases. Eventually, the observer and observed separate and likelihoods stabilize with the relevant observer states entangled with observed states.

The selected form for $U(\lambda)$ couples the observer with the observed and lacks any evolution of "up" and "dn" observed states, e.g., evolution of live into dead cats. A more general unitary transformation implements changes to the likelihoods of live or dead with time, for example, the composition of a rotation of $v_o = (1,0,0,0)$ to $(\cos\vartheta, 0, \sin\vartheta, 0)$ with $U(\lambda)$. If the transition is internal to the observed, more general effective interactions apply than if the transition is due to interaction with the observer. In the case with the transition due to interaction with the observer, $\vartheta(\lambda)$ would have similar properties to $\theta(\lambda)$ except the transition need not be from one certain extreme to the other. $\vartheta(\lambda)$ is constrained by physical considerations, e.g., dead cats do not become live again. For this example composition, the Hamiltonian is time-dependent,

$$H(\lambda) = \begin{pmatrix} 0 & -i\dot{\theta} & -i\dot{\vartheta}\cos^2\theta & -i\dot{\vartheta}\cos\theta\sin\theta \\ i\dot{\theta} & 0 & -i\dot{\vartheta}\cos\theta\sin\theta & -i\dot{\vartheta}\sin^2\theta \\ i\dot{\vartheta}\cos^2\theta & i\dot{\vartheta}\cos\theta\sin\theta & 0 & -i\dot{\theta} \\ i\dot{\vartheta}\cos\theta\sin\theta & i\dot{\vartheta}\sin^2\theta & i\dot{\theta} & 0 \end{pmatrix}$$

and reproduces the result (296) but with $\alpha = \cos\vartheta(\lambda)$ and $\beta = \sin\vartheta(\lambda)$.

The projections onto the perceived observed states are

$$P_{\rm up} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{\rm dn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the projections onto the possible states of the observer are

$$E_{\rm no} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\rm sup} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\rm sdn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These projections commute, $[E_{\rm xx}, P_{\rm yy}] = 0$. This commutation implies the decomposition into relative states discussed in [12] and appendix 8.2.8. The projections onto perceived observed states, $P_{\rm yy}$, commute with time translation, $[U(\lambda), P_{\rm yy}] = 0$, providing that the likelihoods of the two observed states do not vary with time. The projections onto the possibilities for observer state, $E_{\rm xx}$, do not commute with time translation and state of the observer evolves with time. Likelihoods are

$$\text{up likelihood } = E[P_{\rm up}] = \langle v(\lambda)|P_{\rm up}v(\lambda)\rangle = |\alpha|^2$$

and

$$\text{dn likelihood } = E[P_{\rm dn}] = \langle v(\lambda)|P_{\rm dn}v(\lambda)\rangle = |\beta|^2$$

both independently of λ in the first example. The likelihoods of the observer having the indicated perceived history of observations are

$$\text{likelihood of no observation} = E[E_{\rm no}] = \langle v(\lambda)|E_{\rm no}v(\lambda)\rangle = \cos^2\theta(\lambda) \xrightarrow{\lambda\to\infty} 0$$
$$\text{likelihood that up observed} = E[E_{\rm sup}] = \langle v(\lambda)|E_{\rm sup}v(\lambda)\rangle = |\alpha|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\alpha|^2$$
$$\text{likelihood that dn observed} = E[E_{\rm sdn}] = \langle v(\lambda)|E_{\rm sdn}v(\lambda)\rangle = |\beta|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\beta|^2.$$

From (223) and for the state density operator $\rho = |v(\lambda)\rangle\langle v(\lambda)|$ for the pure state $|v(\lambda)\rangle$, the relative state density operators,

$$\rho^{xx} = \frac{E_{xx}\rho E_{xx}}{E[E_{xx}]},$$

are

$$\rho^{\rm no} = \begin{pmatrix} |\alpha|^2 & 0 & \alpha\overline{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ \overline{\alpha}\beta & 0 & |\beta|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\rm sup} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\rm sdn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These are unit trace matrices operating in orthogonal subspaces, that is, with ranges in the null spaces of the other state density operators. $\rho^{xx}\rho^{xx'} = 0$ unless xx'=xx.

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