Relativistic propagation of isolated states

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Outline

1 Introduction

1.1 Background

An isolated particle is distantly space-like separated from the influences of other particles. Isolated stable particles propagate along straight lines. In quantum mechanics, straight line propagation of isolated, stable states follows from the representation of states by functions undergoing interaction-free unitary time evolution. Straight line propagation corresponds with the classical mechanics of isolated particles. A necessary property of realizations of relativistic quantum physics is that isolated stable states exhibit no interaction.

An argument of a state describing function depicts a particle-like state if the dominant support of the argument is isolated. Elementary particles correspond with the fields and form bound states. In the Wigner classification [13], representations of the Poincaré group describe the particles. Classically, particles are localized, however, only isolation and not localization of support is required to demonstrate straight line propagation. Isolation implies that the dominant support of the argument of interest is distantly space-like separated from the significant support of every other state describing function argument. The space-like separation must be sufficient that interaction is negligible. The cluster decomposition property of the vacuum expectation values (VEV) of fields together with Hamiltonians that satisfy Poincaré invariance of likelihoods result in straight line propagation of isolated state descriptions.

Demonstrably realizable constructions of relativistic quantum mechanics [4, 9] describe interaction with physically non-trivial VEV for the fields [7]. The scalar product is determined by the VEV, and from Born's rule, state transition likelihoods follow from the scalar products of initial and final state describing functions. Interaction is negligible when the contributions to scalar products from the $n > 2$, n-argument truncated (connected) VEV components are negligible. To satisfy relativity, observers in all inertial frames of reference must agree on the likelihoods of events. The state descriptions in alternative inertial frames of reference are related by Poincaré transformation. Then, from Born's rule for likelihoods, it follows that the scalar product is a Poincaré invariant and consequently, the Poincaré group is unitarily represented in the Hilbert space of state describing functions [3, 13]. A Poincaré invariance-compliant Hamiltonian is the Hermitian generator of temporal displacement in the representation of the Poincaré group. These Hamiltonians are physically "trivial." This contrasts with the approach in relativistic quantum field theory (RQFT) [1, 5, 10]. In the Dyson series development of the Feynman rules for scattering amplitudes, the VEV of fields are free field VEV and interaction is described by interaction Hamiltonians expressed in free field operators. Physically non-trivial interaction Hamiltonians are a familiar method from non-relativistic quantum mechanics, but the Haag (Haag-Hall-Wightman-Greenberg) theorem [3] demonstrates that quantum mechanics can not be fully realized with this approach. Neither Poincaré covariant description for the finite interval propagation of general state descriptions [2], nor straight line propagation of isolated stable states follow in RQFT that manifest interaction. Consideration of relativistic invariance and the propagation of isolated states illustrates the futility of realizing relativistic quantum mechanics with interaction Hamiltonians. Section 2 of this note includes a demonstration that isolated state descriptions propagate along straight lines in the constructed realizations of relativistic quantum physics. Examples of finite interval evolution of isolated state describing functions are introduced in section 1.3 before their development in section 3.

Association of the interactions exhibited in the constructions with classical description of interactions is a conspicuous, outstanding effort for the constructed realizations of relativistic quantum physics. The association of the VEV of fields with classical equations of motion and field equations is obscured by the scope of non-classical effects: localized and widely distributed state descriptions, position and momentum uncertainties, indistinguishable particles, particle creation-annihilation, and entanglements. Long range interactions describe deviations from straight line propagation exhibited by non-isolated state descriptions. Suggestive non-relativistic approximations [7] are consistent with gravity and electromagnetism but incomplete. Conversely, plane wave scattering amplitudes directly follow from the VEV but apply when state descriptions spatially overlap. For spatially overlapping state descriptions, short range interactions dominate, and constructed scattering amplitudes approximate RQFT scattering amplitudes [7, 8]. The approximation of RQFT scattering amplitudes associates the dominant short range interactions with the interaction Hamiltonians from RQFT. The descriptions of long range interactions are generally distinct from the description of interactions that determine the plane wave scattering amplitudes.

For convenience, demonstrations below are limited to the example of a single, neutral scalar field associated with a finite mass m.

1.2 The spatial support of state descriptions

In quantum mechanics, the descriptions of nature are functions over spacetime. These functions are elements of Hilbert spaces [4, 9]. The spatial configurations of natural objects are perceived and labeled by times, and Poincaré transformations relate perceptions from alternative inertial frames of reference. Time is not an observable of natural objects: time labels changes to perceived spatial configurations. Time is generally kept externally to the object under observation. In the constructions, a state describing function in the one-argument subspace $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$ of the Hilbert space of states $\mathbf{H}_{\mathcal{P}}$ [7] with Fourier transform domain representation

$$\tilde{\varphi}(p) = (p_0 + \omega)\tilde{f}(\mathbf{p}) \tag{1}$$

follows for every tempered function $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. The one-argument subspace consists of the descriptions of the unconfined elementary particles. The functions $\tilde{f}(\mathbf{p})$ describe the spatial support of the state describing function $\varphi(x)$ within constant time planes in the selected inertial reference frame with spacetime coordinates x. The mass shell energy

$$\omega = \omega(\|\mathbf{p}\|) = \sqrt{\lambda_c^{-2} + \mathbf{p}^2} \tag{2}$$

with $\mathbf{p}^2 = \|\mathbf{p}\|^2 = p_x^2 + p_y^2 + p_z^2$ and reduced Compton wavelength

$$\lambda_c = \frac{\hbar}{mc}$$

determined by the mass m, speed of light c and reduced Planck constant $\hbar$.

The constructed VEV [7] include a factor $\delta(p_j^2 - m^2)$ for each argument p_j in the Fourier transform domain. A consequence of the limitation of the support of the constructed VEV [7] to mass shells is that every one-argument state describing function is in an equivalence class of a state describing function of the form (1). Every state describing function $\varphi_1(x) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$ is in the Hilbert space norm completion of functions

$$\tilde{\varphi}_1(p) = (p_0 + \omega)\tilde{g}(p)$$

with $\tilde{g}(p) \in \mathbf{S}(\mathbb{R}^4)$. Due to the $\delta(p^2-m^2)$ factors in the VEV and the zeros $p_0+\omega$ from elements of $\mathbf{H}_{\mathcal{P}}$, the contribution of a state describing function to scalar products is evaluated at $p_0=\omega$, and then,

$$\tilde{g}(p) = \tilde{g}(\omega, \mathbf{p}).$$

With this equivalence, $\varphi(x)$ is in the form (1) with

$$\tilde{f}(\mathbf{p}) = \tilde{g}(\omega, \mathbf{p})$$

and $\tilde{f}(\mathbf{p})$ is in a completion of $\mathcal{S}(\mathbb{R}^3)$. Every state description is equivalent to a state description of the form (1).

The modulus of the state describing function $\varphi(x)$ provides one characterization of support. The modulus follows from the inverse Fourier transform of (1).

$$\begin{aligned}
\varphi(x) &= \int \frac{dp}{(2\pi)^2}\, e^{ipx}(p_0+\omega)\tilde{f}(\mathbf{p}) \\
&= \left(-i\frac{\partial}{\partial x_0} + \sqrt{\lambda_c^{-2} - \Delta}\right) \int \frac{dp_0}{(2\pi)^{\frac{1}{2}}}\, e^{ip_0(x_0-\lambda)} \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda}\tilde{f}(\mathbf{p}) \\
&= \left(-i\frac{\partial}{\partial x_0} + \sqrt{\lambda_c^{-2} - \Delta}\right) \delta(x_0-\lambda)\frac{\psi(\lambda,\mathbf{x})}{2\pi}.
\end{aligned} \tag{3}$$

with Lorentz energy-momentum 4-vectors designated $p = p_0, \mathbf{p}$, Euclidean momentum vectors $\mathbf{p} = p_x, p_y, p_z$, Lorentz invariant differential $dp = dp_0 dp_x dp_y dp_z$, Minkowski inner product $xp = x_0p_0 - \mathbf{x}\cdot\mathbf{p}$, Laplacian Δ for $\mathbf{x} \in \mathbf{R}^3$, and generalized functions $\delta(x_0)$ and $\delta'(x_0)$. The dominant spatial support of $\varphi(x)$ at the time λ is characterized by a function

$$\psi(\lambda, \mathbf{x}) = \int d\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda}\tilde{f}(\mathbf{p}). \tag{4}$$

The function $\psi(\lambda,\mathbf{x})$ over $\mathbf{x}$ is an element of $\mathcal{S}(\mathbb{R}^3)$ for each time $\lambda = ct \in \mathbb{R}$ when $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. The representation (3) is in a selected inertial frame of reference. More generally, the support of an argument of a multiple argument state describing function $\varphi_n((x)_n)$ follows by fixing all but one variable: $\varphi_1(x_j) = \varphi_n((x)_n)$ with x_j the variable and the remaining arguments $\{x_1, x_2, \dots x_n\}\backslash\{x_j\}$ fixed. The support generally depends on entanglement, on the values of x_k for $k \neq j$.

The physically relevant support of a state describing function $\varphi((x)_n) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$ follows from Born's rule that characterizes likelihoods of observations. The likelihood that an n-argument state describing function $\varphi_n((x)_n)$ is observed as k particles located near points $(\mathbf{y})_k = \mathbf{y}_1, \mathbf{y}_2, \dots \mathbf{y}_k \in \mathbb{R}^{3k}$ at a time λ is

$$\frac{|\langle d_k(\lambda, (\mathbf{y})_k)|\varphi_n\rangle|^2}{\|d_k\|^2\|\varphi_n\|^2}. \tag{5}$$

$\langle\varphi_k|\varphi_n\rangle$ is the scalar product in $\mathbf{H}_{\mathcal{P}}$, $\|\varphi_n\|^2=\langle\varphi_n|\varphi_n\rangle$, and $d_k(\lambda,(\mathbf{y})_k)\in\mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4k})$ designates a state describing function with point support at times $x_{j0}=\lambda$ for $j\in(1,k)$ and with spatial support dominantly within a small neighborhood of the points $(\mathbf{y})_k$. For example, in the Fourier transform domain,

$$\widetilde{d}_k((p)_k;\lambda,(\mathbf{y})_k)=\prod_{j=1}^{k}\left(\frac{p_{j0}+\omega_j}{2\omega_j}\right)e^{-ip_{j0}\lambda+i\mathbf{p}_j\cdot\mathbf{y}_j}e^{-\epsilon\mathbf{p}_j^2}$$

with arbitrarily small $\epsilon\in\mathbb{R}_{>0}$ and $\omega_j=\omega(\|\mathbf{p}_j\|)$.

VEV determine the scalar products in (5). The n-argument VEV expand in a cluster expansion of products of $\ell\le n$, ℓ-argument truncated functions [3, 7]. If all arguments $(\mathbf{x})_n$ of φ_n are isolated, the contributions to scalar products from the $\ell>2$, ℓ-argument truncated (connected) VEV are negligible and

$$\mathcal{W}_{k,n}((x)_{2n})\approx\begin{cases}\sum\limits_{pairs}\prod\limits_{\ell=1}^{n}W_2(x_\ell,x_{i_\ell}) & \text{if } k=n\\ 0 & \text{if } k\neq n\end{cases}$$

in evaluation of the scalar product $\langle d_k|\varphi_n\rangle$. Without isolation of supports, the VEV exhibit interaction, for example, d_k and φ_n are generally not orthogonal for $k\neq n$. In the constructions, the scalar field two point is a Pauli-Jordan function with the Fourier transform

$$\widetilde{W_2}(p_1,p_2)=\delta(p_1+p_2)\theta(p_{02})\delta(p_2^2-m^2).$$

The free field two-point function characterizes propagation of isolated states.[a] The spatial support of an isolated argument is sufficiently greatly space-like separated from the dominant spatial supports of the remaining arguments of the state describing function that the $\ell>2$, ℓ-argument truncated contributions to VEV are negligible. Truncated VEV decline to zero as their arguments are greatly space-like separated. The lack of interaction exhibited by isolated state descriptions originates from this decline. The cluster decomposition property of VEV provides that only two-point functions provide a significant contribution to the scalar product when all arguments have isolated support.

For product state describing functions

$$\varphi_n((x)_n)=\prod_{j=1}^{n}\varphi_{1,j}(x_j),\tag{6}$$

the support of each argument is independent of the values of the other arguments. For product state describing functions with isolated arguments, cluster decomposition provides that

$$\langle d_n(\lambda,(\mathbf{y})_n)|\varphi_n\rangle\approx\prod_{j=1}^{n}\langle d_1(\lambda,\mathbf{y}_j)|\varphi_{1,j}\rangle$$

[a]For a confined species, the two-point function is zero and the support of arguments can not be isolated [7].

when the dominant spatial support of each argument $\mathbf{x}_j$ includes $\mathbf{y}_j$. From isolation and with $y_j = \lambda, \mathbf{y}_j$, $W_2(y_j,x) \approx 0$ if $\mathbf{y}_j$ lies significantly outside the region of dominant support of the isolated argument $\mathbf{x}$ with $x_0 = \lambda$. The scalar product within the one-argument subspace $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$ is

$$\begin{aligned}\langle\psi_1|\varphi_1\rangle &= \int dx_1dx_2\,\overline{\psi_1(x_1)}\,W_2(x_1,x_2)\,\varphi_1(x_2)\\ &= \int dp_1dp_2\,\overline{\widetilde{\psi}_1(-p_1)}\,\widetilde{W}_2(p_1,p_2)\,\widetilde{\varphi}_1(p_2)\end{aligned}$$

from the Parseval's equality definition of the Fourier transform of generalized functions. The factors in the scalar product (5) for product (6) state describing functions $\varphi_n((x)_n)$ are

$$\langle d_1(\lambda,\mathbf{y})|\varphi_{1,j}\rangle = \int dp_1dp_2\,\frac{-p_{01}+\omega_1}{2\omega_1}\,e^{-ip_{10}\lambda}e^{i\mathbf{p}_1\cdot\mathbf{y}}\delta(p_1+p_2)\theta(p_{02})\delta(p_2^2-m^2)(p_{02}-\omega_2)\tilde{f}(\mathbf{p}_2)$$

with $\varphi_{1,j}$ characterized by the function $\tilde{f}(\mathbf{p})$ and $\mathbf{y} = \mathbf{y}_j$. The limitation of VEV support to mass shells sets each $p_0 = \omega$ due to the zero introduced by the factor $p_0+\omega$ in (1). The delta function-like limit of the complex conjugate of the Fourier transform of $d_1(\lambda,\mathbf{y})$ is

$$\overline{\widetilde{d}_1(-p;\lambda,\mathbf{y})} = \frac{-p_0+\omega}{2\omega}\,e^{-ipy}$$

with $y_0 = \lambda$. The delta function-like limit[b] of $d_1(\lambda,\mathbf{y})$ lies outside of $\mathbf{H}_{\mathcal{P}}$. The delta function-like limit of $d_1(\lambda,\mathbf{y})$ simplifies the scalar product to

$$\begin{aligned}\langle d_1(\lambda,\mathbf{y})|\varphi_1\rangle &\approx \int d\mathbf{p}\,e^{-i\mathbf{p}\cdot\mathbf{y}}e^{i\omega\lambda}\tilde{f}(\mathbf{p})\\ &= \psi(\lambda,\mathbf{y})\end{aligned} \tag{7}$$

with the same $\psi(\lambda,\mathbf{y})$ from both the state describing function (3) and (7). The isolated argument $\mathbf{y}$ describes a freely propagating state at time λ.

With interest limited to one isolated argument $\mathbf{x}_j$ of a product (6) state describing function $\varphi_n((x)_n)$, approximation of the expectation value of a localized one particle operator A associated by isolation with observation of one function factor $\varphi_{1,j}(\lambda,\mathbf{x}_j)$ of $\varphi_n((x)_n)$ is

$$\begin{aligned}E[A;\varphi_n] &= \frac{\langle\varphi_{1,j}|A\varphi_{1,j}\rangle}{\langle\varphi_{1,j}|\varphi_{1,j}\rangle}\\ \langle\varphi_{1,j}|A\varphi_{1,j}\rangle &= \int dp_1dp_2\,(-p_{01}+\omega_1)\overline{\tilde{f}(-\mathbf{p}_1)}\delta(p_1+p_2)\theta(p_{02})\delta(p_2^2-m^2)A(p_{02}+\omega_2)\tilde{f}(\mathbf{p}_2)\\ &= \int d\mathbf{p}\,\overline{\tilde{f}(\mathbf{p})}A\,2\omega\tilde{f}(\mathbf{p}).\end{aligned} \tag{8}$$

[b]The Hilbert space norm of the delta function-like limit diverges: the likelihood of observation at a single point is zero. Likelihoods of observation within finite volume neighborhoods are finite.

The isolated argument associated with the localized A is described by the function $\tilde{f}(\mathbf{p})$. Localized observation is associated with an isolated argument $\mathbf{x}_j$ when observation of A is constrained to exclude the dominant support of every argument except the dominant support of the isolated $\mathbf{x}_j$. From the Hamiltonian in the n-argument subspace of $\mathbf{H}_{\mathcal{P}}$

$$H = \prod_{j=1}^{n} e^{-i\omega\lambda_j} : \quad |\varphi_n\rangle \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n}) \mapsto |H\varphi_n\rangle \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$$

[7], it follows that for product (6) state describing functions with isolated arguments and for localized A, expectations simplify to

$$\begin{aligned} E[A; e^{-iH\lambda}\varphi_n] &= \frac{\langle e^{-i\omega\lambda}\varphi_{1,j}|Ae^{-i\omega\lambda}\varphi_{1,j}\rangle}{\langle\varphi_{1,j}|\varphi_{1,j}\rangle} \\ \langle e^{-i\omega\lambda}\varphi_{1,j}|Ae^{-i\omega\lambda}\varphi_{1,j}\rangle &= \int d\mathbf{p}\; e^{i\omega\lambda}\overline{\tilde{f}(\mathbf{p})}A\, 2\omega e^{-i\omega\lambda}\tilde{f}(\mathbf{p}) \end{aligned}$$

and generally, A does not commute with ω.

Poincaré transformations relate state descriptions in alternative inertial frames of reference. The description of spatial support that follows from an $\tilde{f}(\mathbf{p})$ in (3) or (7) applies within a constant time slice of spacetime in one inertial reference frame. Nevertheless, the Poincaré invariant scalar product of $\mathbf{H}_{\mathcal{P}}$ and the invariant limitation of the support of VEV to mass shells imply a description of the spatial support in any inertial frame of reference [7]. Poincaré transforms map representations of the form (1) from one inertial frame of reference to any other inertial frame of reference. The scalar product is invariant

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle \underline{g}|\underline{f}\rangle$$

and the state describing functions are covariant with Poincaré transformations. The transformstions of state describing functions are

$$\begin{aligned} (a,\Lambda)\varphi_n((x)_n) &= \varphi_n((\Lambda^{-1}(x-a))_n) \\ (a,\Lambda)\tilde{\varphi}_n((p)_n) &= \prod_{k=1}^{n} e^{-ip_k a}\tilde{\varphi}_n((\Lambda^{-1}p)_n) \end{aligned}$$

in the case with a single neutral scalar field. The Poincaré transform (a,Λ) consists of a proper orthochronous Lorentz transformation Λ determined by an $A \in \mathrm{SL}(2,\mathbb{C})$, and a spacetime translation $a \in \mathbb{R}^4$. The representation of the Poincaré group is unitarily implemented [3],

$$(a,\Lambda)^* = (a,\Lambda)^{-1} = (-\Lambda^{-1}a, \Lambda^{-1}).$$

Poincaré transformations of a state describing function of the form (1) results in

$$(a,\Lambda)(p_0+\omega)\tilde{f}(\mathbf{p}) = ((\Lambda^{-1}p)_0 + \omega')e^{-ipa}\tilde{f}((\Lambda^{-1}p)_x, (\Lambda^{-1}p)_y, (\Lambda^{-1}p)_z)$$

with $\omega' = \omega(\|((\Lambda^{-1}p)_x, (\Lambda^{-1}p)_y, (\Lambda^{-1}p)_z)\|)$. Designating $p' = \Lambda^{-1}p$, Lorentz invariance provides that

$$\begin{aligned} p^2 - m^2 &= (p_0 + \omega)(p_0 - \omega) \\ &= p'^2 - m^2 \\ &= (p'_0 + \omega')(p'_0 - \omega'). \end{aligned}$$

Then, if $p'_0 \neq \omega'$,

$$p'_0 + \omega' = (p_0 + \omega)\frac{p_0 - \omega}{p'_0 - \omega'}$$

and the factor

$$\frac{p_0 - \omega}{p'_0 - \omega'} = \frac{p'_0 + \omega'}{p_0 + \omega}$$

is regular at $p'_0 = \omega'$. Finally,

$$(a, \Lambda)(p_0 + \omega)\tilde{f}(\mathbf{p}) = (p_0 + \omega)e^{-ipa}\,\frac{\omega}{\omega'}\,\tilde{f}((\Lambda^{-1}p)_x, (\Lambda^{-1}p)_y, (\Lambda^{-1}p)_z). \tag{9}$$

The limitation of the support of VEV to mass shells and the zeros on negative energy mass shells contributed by functions in $\mathbf{H}_{\mathcal{P}}$ provide that scalar products are evaluated with energies $p_0 = \omega$. Setting $p_0 = \omega$ except for the p_0 in the factor $p_0 + \omega$ that introduces the zero on negative mass shells, the Poincaré transformed (9) is a state describing function in the form (1). $\tilde{f}(\mathbf{p})$ describes the spatial support on a constant time slice in an original, inertial reference frame. (9) provides the description of the state after a composition of boosts, rotations, and translations with respect to the original frame of reference. The description (9) is in the coordinates of an observer in the initial frame.

1.3 Approximations for $\psi(\lambda, \mathbf{x})$

The function $\psi(\lambda, \mathbf{x})$ that describes the support of one argument of a state describing function $\varphi_n((x)_n) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$ is an inverse Fourier transform (4). For $\lambda \neq 0$ and the Gaussian $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ considered in this note, this Fourier transform is not elementary. However, for Gaussian $\tilde{f}(\mathbf{p})$ and limited intervals λ, familiar non-relativistic approximations [6] result in Gaussian quadrature to approximate $\psi(\lambda, \mathbf{x})$. For sufficiently limited times λ and Gaussian $\tilde{f}(\mathbf{p})$ dominantly supported over non-relativistic momenta $\lambda_c^2 \mathbf{p}^2 \ll 1$,

$$\omega\lambda \approx \frac{\lambda}{\lambda_c}\,(1 + \frac{1}{2}\lambda_c^2\mathbf{p}^2 - \frac{1}{8}(\lambda_c^2\mathbf{p}^2)^2 + \ldots)$$

from binomial series expansion and then for rotationally invariant Gaussians

$$\tilde{f}(\mathbf{p}) = e^{-\sigma^2\mathbf{p}^2}, \tag{10}$$

the description of support is

$$
\begin{aligned}
\psi(\lambda,\mathbf{x}) &= \int d\mathbf{p}\; e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda} e^{-\sigma^2\mathbf{p}^2} \\
&\approx \int d\mathbf{p}\; e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i(1+\frac{1}{2}\lambda_c^2\mathbf{p}^2)\frac{\lambda}{\lambda_c}} e^{-\sigma^2\mathbf{p}^2} \\
&= e^{i\frac{\lambda}{\lambda_c}} \prod_{\nu=1}^{3} \left(\int dp_\nu\; e^{-ip_\nu x_\nu} e^{i\frac{1}{2}\lambda_c p_\nu^2\lambda} e^{-\sigma^2 p_\nu^2} \right) \\
&= \frac{\pi^{\frac{3}{2}} e^{i\frac{\lambda}{\lambda_c}}}{(\sigma^2 - \frac{i}{2}\lambda_c\lambda)^{\frac{3}{2}}}\; e^{-\dfrac{\mathbf{x}^2}{4\sigma^2 - 2i\lambda_c\lambda}}.
\end{aligned}
\tag{11}
$$

The approximation for $\psi(\lambda,\mathbf{x})$ is a Gaussian of fixed width in momentum and spreading support over space as time λ increases.

$$
|\psi(\lambda,\mathbf{x})| \approx \frac{\pi^{\frac{3}{2}}}{(\sigma^4 + \frac{1}{4}\lambda_c^2\lambda^2)^{\frac{3}{4}}}\; e^{-\dfrac{\mathbf{x}^2}{4\sigma^2 + \frac{1}{\sigma^2}\lambda_c^2\lambda^2}}.
$$

The approximation (11) applies if

$$
\left(\lambda_c^{-1} + \frac{1}{2}\lambda_c\mathbf{p}^2 - \omega(\mathbf{p}) \right) \lambda \ll \pi \tag{12}
$$

within the dominant support of the integrand, $\sigma^2\mathbf{p}^2 < K^2$ with $K \sim 3$. Within this dominant support, the momenta must be non-relativistic, $\lambda_c^2\mathbf{p}^2 \ll 1$, and as a consequence, $\lambda_c^2 \ll \sigma^2$.

A non-classical consideration in quantum mechanical free propagation is the spread of momentum and position support described in the Heisenberg uncertainty principle. The dominant momentum support of an isolated state describing function remains constant with time and the initial spread of momenta provides that the spatial support broadens with increased time.

For the Gaussian $\tilde{f}(\mathbf{p})$, the expectation of the location is zero and the square of the non-relativistic approximation for location follows from

$$
\begin{aligned}
E[X^2; e^{-i\omega\lambda}\varphi_1] &\approx -\frac{2\lambda_c^{-1}}{\|\varphi_1\|^2} \int d\mathbf{p}\; e^{(i\frac{1}{2}\lambda_c\lambda - \sigma^2)\mathbf{p}^2} \left(\sum_\nu \frac{\partial^2}{\partial p_\nu^2}\; e^{(-i\frac{1}{2}\lambda_c\lambda - \sigma^2)\mathbf{p}^2} \right) \\
&= \frac{2\lambda_c^{-1}}{\|\varphi_1\|^2} \int d\mathbf{p} \sum_\nu \left(2(\sigma^2 - i\tfrac{1}{2}\lambda_c\lambda) - 4(\sigma^2 - i\tfrac{1}{2}\lambda_c\lambda)^2 p_\nu^2 \right) e^{-2\sigma^2\mathbf{p}^2} \\
&= \sum_\nu \left(2(\sigma^2 - i\tfrac{1}{2}\lambda_c\lambda) - \frac{(\sigma^2 - i\frac{1}{2}\lambda_c\lambda)^2}{\sigma^2} \right) \\
&= 3(\sigma^2 + \frac{\lambda_c^2\lambda^2}{4\sigma^2})
\end{aligned}
$$

with

$$\|\varphi_1\|^2 \approx 2\lambda_c^{-1} \int d\mathbf{p}\, e^{-2\sigma^2 \mathbf{p}^2}.$$

The spatial spread of support of the freely propagating particle increases with time. The expectations of momentum and momentum squared do not vary with time as a consequence of the commutation of p_ν with ω. Gaussian $\tilde{f}(\mathbf{p})$ initially meet the Heisenberg uncertainty principle lower bound. The Gaussian quadrature follow from

$$\int_{-\infty}^{\infty} dx\, (A+Bx+Cx^2)\, e^{-\alpha x^2+\beta x} = \sqrt{\frac{\pi}{\alpha}} \left(A+B\,\frac{\beta}{2\alpha}+C\left(\frac{1}{2\alpha}+\frac{\beta^2}{4\alpha^2}\right)\right) e^{\frac{\beta^2}{2\alpha}} \tag{13}$$

for $\alpha, \beta \in \mathbb{C}$ and $\mathbb{R}e(\alpha) > 0$.

A relativistic approximation for the function (4) with rotationally invariant Gaussian $\tilde{f}(\mathbf{p})$ is developed in section 3.

$$\psi(\lambda, \mathbf{x}) \approx \frac{\pi^{\frac{3}{2}} e^{i\alpha_0\lambda}}{(\sigma^2 - i\lambda\sigma_s)^{\frac{3}{2}}} \exp\left(-\frac{\mathbf{x}^2}{4\,(\sigma^2 - i\lambda\sigma_s)}\right) \tag{14}$$

with σ_s from (34).

$$\sigma_s^2 \int d\mathbf{p}\; 4\mathbf{p}^2\omega\, e^{-2\sigma^2\mathbf{p}^2} = \int d\mathbf{p}\; \frac{\mathbf{p}^2}{\omega}\, e^{-2\sigma^2\mathbf{p}^2}.$$

α_0 is an undetermined constant and ω is from (2). If σ is sufficiently large that the dominant support of (10) is non-relativistic, $\sigma^2 > K^2\lambda_c^2$, then $\omega \approx \lambda_c^{-1}$ and

$$\sigma_s \approx \frac{\lambda_c}{2}.$$

Then, with

$$\alpha_0 = \lambda_c^{-1},$$

(14) coincides with (11) in non-relativistic instances. The approximation (14) coincides with (11) in non-relativistic instances but the limitations to non-relativistic momenta and to λ that satisfy (12) are removed in (14). For relativistic momenta, $2\sigma_s < \lambda_c$ and the spatial spreading with increased λ is less than would be extrapolated from the non-relativistic approximation.

2 Corresponding classical trajectories

The quantum mechanical description of linear motion is that the expectation of location executes linear motion and the expectation of momentum is constant with time. Then, the support of the isolated argument exhibits no interaction and this lack of interaction persists as long as supports remain isolated. In this section, it is demonstrated that isolated states exhibit straight line, free particle propagation in the constructed realizations of relativistic quantum physics [7]. Free particle propagation follows from the cluster decomposition property of VEV and Poincaré invariance-compliant Hamiltonians. Interaction is exhibited if the supports of the arguments of the state descriptions are not sufficiently isolated. Without isolation, scalar products include non-negligible contributions from the $n>2$, n-argument truncated VEV. Exhibitions of interaction include the plane wave limit scattering amplitudes.

For Poincaré invariance-compliant Hamiltonians [13], the Hamiltonian commutes with the momentum operators, $[H,P_\nu]=0$, $\nu=x,y,z$, and as a consequence, the momenta are constant over time as long as arguments are isolated. Constant momenta describes the time evolution of free particles. Momenta P_ν are observables, self-adjoint operators, and the expectations of the momenta for isolated particle-like state descriptions are constant with time as long as the support of the state describing function arguments are sufficiently space-like separated that contributions of the $n>2$, n-argument truncated VEV are negligible. Propagation remains free until the particle encounters additional particles, until the isolation of support fails. If the support of an argument of a state describing function is isolated, the state described by the argument is identifiable from its isolated location.

With interest limited to one isolated argument $\mathbf{x}$ described by $\varphi_1(x)$ at time $x_0=\lambda$ from a product (6) state describing function $\varphi_n((x)_n)$, the time evolution of the expectation of a local observable A includes the factor

$$\begin{aligned}
\langle e^{-i\omega\lambda}\varphi_1|Ae^{-i\omega\lambda}\varphi_1\rangle &= \langle e^{-i\omega\lambda}\varphi_1|\left(e^{-i\omega\lambda}A+[A,e^{-i\omega\lambda}]\right)\varphi_1\rangle\\
&= \langle\varphi_1|\left(A+e^{i\omega\lambda}[A,e^{-i\omega\lambda}]\right)\varphi_1\rangle\\
&= \langle\varphi_1|A\varphi_1\rangle+\langle\varphi_1|e^{i\omega\lambda}[A,e^{-i\omega\lambda}]\varphi_1\rangle\\
&= \int d\mathbf{p}\;\overline{\tilde{f}(\mathbf{p})}A\,2\omega\tilde{f}(\mathbf{p})+\overline{\tilde{f}(\mathbf{p})}e^{i\omega\lambda}[A,e^{-i\omega\lambda}]\,2\omega\tilde{f}(\mathbf{p}).
\end{aligned}$$

A is a local observable perceived within a volume that overlaps the isolated support of the selected argument $\mathbf{x}$ and is disjoint from the dominant support of all other arguments of $\varphi_n((x)_n)$. Isolation enables the identification of an observation of A with the state describing function $\varphi_1(x)$. From (1) and (8), expectations of the state describing function $\varphi_1(x)$ are described by $\tilde{f}(\mathbf{p})$. The expectations include an initial value plus a term that is zero if $\lambda=0$ or $[A,\omega]=0$. However, location and momentum operators are not localized operators, and the discussion applies to spatially limited operators that approximate the location and momentum operators within the dominant support of $\varphi_1(x)$.

If the observable is $A\approx P_\nu$ or P_ν^2, one of three components of momentum, $\nu=x,y,z$, then the commutator vanishes. As a consequence, the expectations for the momentum are constant with time as long as the dominant support of the argument of interest remains isolated.

For isolated, particle-like state descriptions, there is no interaction and the description of the momentum is constant over time.

Relativistic location operators [12] have frequency domain representations

$$\hat{X}_\nu = i\,\omega^{\frac{1}{2}} \frac{\partial}{\partial p_\nu}\,\omega^{-\frac{1}{2}}.$$

If the observable is $A \approx \hat{X}_\nu$ or $\hat{X}_\nu^2$, then the commutators with time translation do not vanish but $[\omega^{-\frac{1}{2}}, e^{-i\omega\lambda}] = 0$. Expectation values (8) are

$$\langle e^{-i\omega\lambda}\varphi_1 | (\hat{X}_\nu)^k e^{-i\omega\lambda}\varphi_1\rangle = 2i^2 \int dp\, \overline{e^{-i\omega\lambda}\omega^{\frac{1}{2}}\tilde{f}(\mathbf{p})}\, \frac{\partial^k}{\partial p_\nu^k}\, e^{-i\omega\lambda}\omega^{\frac{1}{2}}\tilde{f}(\mathbf{p})$$

from

$$e^{i\omega\lambda}(\hat{X}_\nu)^k e^{-i\omega\lambda} = i^k \omega^{\frac{1}{2}} e^{i\omega\lambda} \frac{\partial^k}{\partial p_\nu^k} e^{-i\omega\lambda}\omega^{-\frac{1}{2}}.$$

For the expectation of $\hat{X}_\nu$, the chain rule and expansion provides

$$\begin{aligned} e^{i\omega\lambda}[\hat{X}_\nu, e^{-i\omega\lambda}] &= ie^{i\omega\lambda}\omega^{\frac{1}{2}}[\frac{\partial}{\partial p_\nu}, e^{-i\omega\lambda}]\omega^{-\frac{1}{2}} \\ &= \frac{p_\nu}{\omega}\lambda \\ &= v_\nu t \end{aligned}$$

from the temporal distance parameter $\lambda = ct$ and designation of velocity $\mathbf{v} = v_x, v_y, v_z$. Velocity and momentum are related [11]

$$\frac{v_\nu}{c} = \frac{p_\nu}{\omega}$$

or

$$\hbar p_\nu = \frac{m v_\nu}{\sqrt{1 - \frac{v^2}{c^2}}}$$

in the units of this note [7].

For the expectation of $\hat{X}_\nu^2$, the chain rule, addition of terms and the commutativity of p_ν

with ω provides

$$
\begin{aligned}
e^{i\omega\lambda}\,[\hat{X}_\nu^2, e^{-i\omega\lambda}]\,\omega &= -\omega^{\frac{1}{2}}\,e^{i\omega\lambda}\,[\frac{\partial^2}{\partial p_\nu^2}, e^{-i\omega\lambda}]\,\omega^{\frac{1}{2}} \\
&= -\omega^{\frac{1}{2}}\,e^{i\omega\lambda}\left(\left(\frac{\partial^2 e^{-i\omega\lambda}}{\partial p_\nu^2}\right) + 2\left(\frac{\partial e^{-i\omega\lambda}}{\partial p_\nu}\right)\frac{\partial}{\partial p_\nu}\right)\omega^{\frac{1}{2}} \\
&= -\omega^{\frac{1}{2}}\,e^{i\omega\lambda}\left(-i\lambda\left(\frac{\partial}{\partial p_\nu}\frac{p_\nu}{\omega}e^{-i\omega\lambda}\right) - 2i\lambda\frac{p_\nu}{\omega}e^{-i\omega\lambda}\frac{\partial}{\partial p_\nu}\right)\omega^{\frac{1}{2}} \\
&= -\omega^{\frac{1}{2}}\left(-i\frac{\lambda}{\omega} + i\lambda\frac{p_\nu^2}{\omega^3} - \lambda^2\frac{p_\nu^2}{\omega^2} - 2i\lambda\frac{p_\nu}{\omega}\frac{\partial}{\partial p_\nu}\right)\omega^{\frac{1}{2}} \\
&= i\lambda - i\lambda\frac{p_\nu^2}{\omega^2} + \lambda^2\frac{p_\nu^2}{\omega} + 2i\lambda\omega^{-\frac{1}{2}}p_\nu\frac{\partial}{\partial p_\nu}\omega^{\frac{1}{2}} \\
&= i\lambda + \lambda^2\frac{p_\nu^2}{\omega} + 2i\lambda p_\nu\frac{\partial}{\partial p_\nu} \\
&= 2i\lambda\sqrt{p_\nu}\frac{\partial}{\partial p_\nu}\sqrt{p_\nu} + \lambda^2\frac{p_\nu^2}{\omega}.
\end{aligned}
$$

From the expectation (8), time evolution of the means and squares of momentum and position are

$$
\begin{aligned}
E[\mathbf{P}; e^{-i\omega\lambda}\phi_1] &= E[\mathbf{P}; \varphi_1] \\
E[\hat{\mathbf{X}}; e^{-i\omega\lambda}\varphi_1] &= E[\hat{\mathbf{X}}; \varphi_1] + \lambda E[\tfrac{1}{\omega}\mathbf{P}, ; \varphi_1] \\
E[\mathbf{P}^2; e^{-i\omega\lambda}\varphi_1] &= E[\mathbf{P}^2; \varphi_1] \\
E[\hat{\mathbf{X}}^2; e^{-i\omega\lambda}\varphi_1] &= E[\hat{\mathbf{X}}^2; \varphi_1] + 2\lambda\sum_\nu E[i\sqrt{p_\nu}\frac{\partial}{\partial p_\nu}\frac{\sqrt{p_\nu}}{\omega}; \varphi_1] + \lambda^2\sum_\nu E[\frac{p_\nu^2}{\omega^2}; \varphi_1].
\end{aligned}
\tag{15}
$$

Functions $\psi(\lambda, \mathbf{x})$ describe the support and isolated expectation values for each isolated argument of state describing functions $\varphi_n((x)_n)$. Symmetries of these functions $\psi(\lambda, \mathbf{x})$ follow from (4). $\psi(\lambda, \mathbf{x})$ is a Fourier transform of $e^{i\omega\lambda}\tilde{f}(\mathbf{p})$. The complex conjugate is

$$
\begin{aligned}
\overline{\psi(\lambda, \mathbf{x})} &= \int d\mathbf{p}\; e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\omega\lambda}\overline{\tilde{f}(\mathbf{p})} \\
&= \psi_c(-\lambda, -\mathbf{x})
\end{aligned}
$$

with $\psi_c(\lambda, \mathbf{x})$ designating the function of the form (4) obtained by substitution of the complex conjugate $\overline{\tilde{f}(\mathbf{p})}$ for $\tilde{f}(\mathbf{p})$. For real functions $\tilde{f}(\mathbf{p})$,

$$\psi(-\lambda, -\mathbf{x}) = \psi(\lambda, \mathbf{x}).$$

Propagation for increasingly negative λ is in the opposite direction in space, but from (15), the support still spreads as $|\lambda|$ increases. Then, $\psi(|\lambda|, \mathbf{x})$ describes a state propagating forward in time from an initial state described by $\psi(0, \mathbf{x})$ regardless of whether the displacement in time λ is positive or negative.

The results of this section establish that in the constructed realizations of relativistic quantum physics [7], the position and momentum expectation values (15) of isolated systems propagate along straight lines. The spread of spatial support with time is also characterized in (15).

3 Approximation of relativistic free propagation

In section 1.2, functions $\psi(\lambda, \mathbf{x})$ of the form (4) characterize the physically relevant support of each isolated argument of state describing functions $\varphi_n((x)_n) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$. In this section, approximations are developed to evaluate $\psi(\lambda, \mathbf{x})$ for states that include relativistic momenta. Example approximate $\psi(\lambda, \mathbf{x})$ are developed for rotationally invariant Gaussian $\tilde{f}(\mathbf{p})$. The approximate $\psi(\lambda, \mathbf{x})$ provide examples of the straight line motion and support spread of isolated state relativistic evolution.

For rotationally invariant functions $\tilde{f}(\mathbf{p})$, a rotation aligns $\mathbf{p}$ with any selected spatial vector $\mathbf{x}$ without modification of $\tilde{f}(\mathbf{p})$. This alignment reduces the 3 dimensional integral representation (4) for $\psi(\lambda, \mathbf{x})$ to a one dimensional quadrature. For rotationally invariant functions, a change of summation variables to polar coordinates results in

$$\psi(\lambda, \mathbf{x}) = \int_0^\infty \rho^2 d\rho \int_0^\pi \sin\phi \, d\phi \int_0^{2\pi} d\theta \;\; e^{-i\rho r \cos\phi} e^{i\omega(\rho)\lambda} \tilde{f}(\mathbf{p})$$

with

$$\mathbf{p} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix} \tag{16}$$

and

$$\begin{aligned} \mathbf{p}\cdot\mathbf{x} &= \rho r \cos\phi \\ \rho^2 &= \mathbf{p}^2 \\ r^2 &= \mathbf{x}^2. \end{aligned}$$

The angle summations are elementary leaving the ρ summation to evaluate (4).

$$\psi(\lambda, \mathbf{x}) = -\frac{2\pi i}{r} \int_0^\infty \rho d\rho \;\left(e^{i\rho r} - e^{-i\rho r}\right) e^{i\omega\lambda} \tilde{f}(\mathbf{p}).$$

For even $\tilde{f}(\mathbf{p})$, $\tilde{f}(-\mathbf{p}) = \tilde{f}(\mathbf{p})$, the two terms add.

$$\psi(\lambda, \mathbf{x}) = -\frac{2\pi i}{r} \int_{-\infty}^{\infty} \rho d\rho \; e^{i\rho r} e^{i\omega\lambda} \tilde{f}(\mathbf{p}). \tag{17}$$

Rotationally invariant $\tilde{f}(\mathbf{p})$ are even functions.

The examples selected to approximate $\psi(\lambda, \mathbf{x})$ are the Gaussian functions (10). Gaussian (10) are rotationally invariant. Gaussian functions are the most classical-like states in the sense that the Heisenberg lower bound on simultaneous knowledge of momentum and location is met for $\lambda = 0$. At $\lambda = 0$, the Gaussian $\tilde{f}(\mathbf{p})$ matches as well as possible a classical description of the state. The classical state is characterized solely by location and momentum.

With $\tilde{f}(\mathbf{p}) = \exp(-\sigma^2\rho^2)$ from (10) and for $\lambda \neq 0$, the summation (17) is not elementary. In this section, approximations expressed in elementary functions are developed for the relativistic propagation of Gaussian $\tilde{f}(\mathbf{p})$.

3.1 Integral representation of time translation

An integral representation of the unitary time translation operator facilitates approximation. For $a_o, \lambda_o \in \mathbb{R}_{>0}$, the Gaussian quadrature (13) provides

$$\begin{aligned}
e^{i\omega\lambda - a_o^2} &= \sqrt{\frac{\lambda_o^2}{\pi}} \int_{-\infty}^{\infty} ds \; e^{-\lambda_o^2 s^2 + 2\lambda_o b(\rho) s} \\
&= \sqrt{\frac{\lambda_o^2}{\pi}} \int_{-\infty}^{\infty} ds \; e^{-\lambda_o^2 (s - \frac{b(\rho)}{\lambda_o})^2 + b(\rho)^2} \\
&= e^{b(\rho)^2}.
\end{aligned}$$

Equality is satisfied if

$$b(\rho)^2 = i\omega\lambda - a_o^2.$$

The exponential $e^{-\lambda_o^2 s^2 + 2\lambda_o b(\rho) s}$ is an entire function of s and then the Cauchy integral formula, lack of singularities and exponential decline at large real values of s provides that the integral around any closed contour is zero. Consequently, the summation of a real s from $-\infty$ to ∞ equals the summation with s translated by any finite, complex constant $b(\rho)/\lambda_o$.

$$b(\rho) = (-a_o^2 + i\omega\lambda)^{\frac{1}{2}}. \tag{18}$$

The principal root from the two solutions for $b(\rho)$ is selected below. The real part of $b(\rho)^2$ is negative and the imaginary part is positive (second complex quadrant) and consequently, the principal root of $b(\rho)^2$ has positive real and imaginary components (first quadrant). Then, given the principal root $b(\rho)$ and for any $a_o, \lambda_o \in \mathbb{R}_{>0}$,

$$e^{i\omega\lambda} = \frac{\lambda_o e^{a_o^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds\; e^{-\lambda_o^2 s^2 + 2\lambda_o b(\rho) s}. \tag{19}$$

A convenient expression for the principal root of $b(\rho)^2$ is

$$\begin{aligned} b(\rho) &= b_R(\rho) + i b_Q(\rho) \\ b_R(\rho) &= \sqrt{\frac{\sqrt{a_o^4 + \omega^2\lambda^2} - a_o^2}{2}} \\ b_Q(\rho) &= \frac{\omega\lambda}{2 b_R(\rho)}. \end{aligned} \tag{20}$$

The triangle inequality and the monotone increase of the square root provide convenient bounds

$$\begin{aligned} \omega &= \sqrt{\lambda_c^{-2} + \rho^2} \le \lambda_c^{-1} + |\rho| \\ |b(\rho)| &= \sqrt{|b(\rho)^2|} = (a_o^4 + \omega^2\lambda^2)^{\frac{1}{4}} \le \sqrt{a_o^2 + \omega|\lambda|} \\ &\le a_o + \sqrt{\tfrac{|\lambda|}{\lambda_c}} + \sqrt{|\lambda\rho|} \\ 0 &\le b_R(\rho) \\ &\le \sqrt{\tfrac{1}{2}(a_o^2 + \omega|\lambda| - a_o^2)} \\ &\le \sqrt{\tfrac{|\lambda|}{2\lambda_c}} + \sqrt{\tfrac{1}{2}|\lambda\rho|} \end{aligned} \tag{21}$$

with $a_o, \lambda_c \in \mathbb{R}_{>0}$ and $\lambda, \rho \in \mathbb{R}$.

If a_o is selected sufficiently large to satisfy $a_o > 1$ and $a_o \gg \omega|\lambda|$ when ρ is limited, $|\rho| < \rho_x$ with a limitation to finite ρ_x justified below in section 3.2, then binomial series expansion yields

$$\begin{aligned} b_R(\rho) &= \sqrt{\frac{a_o^2\sqrt{1 + \frac{\omega^2\lambda^2}{a_o^4}}) - a_o^2}{2}} \\ &\approx \sqrt{\frac{\omega^2\lambda^2}{4a_o^2}\left(1 - \frac{\omega^2\lambda^2}{4a_o^4}\right)} \\ &\approx \frac{\omega\lambda}{2a_o}\left(1 - \frac{\omega^2\lambda^2}{8a_o^4}\right) \\ b_Q(\rho) &\approx a_o\left(1 + \frac{\omega^2\lambda^2}{8a_o^4}\right). \end{aligned} \tag{22}$$

To leading orders for sufficiently large a_o, the real part of $b(\rho)$ is much less than unity and the imaginary part is nearly constant. The variation of $b(\rho)$ with ρ follows from the infinitely differentiable

$$\omega \approx \begin{cases} \lambda_c^{-1}\left(1+\frac{1}{2}\lambda_c^2\rho^2\right) & \text{if } \lambda_c^2\rho^2 \ll 1, \text{ non-relativistic momenta} \\ \rho\left(1+\dfrac{1}{2\lambda_c^2\rho^2}\right) & \text{if } \lambda_c^2\rho^2 \gg 1, \text{ very relativistic momenta.} \end{cases} \tag{23}$$

Substitution of the integral representation (19) into the one dimensional quadrature representation (17) for $\psi(\lambda,\mathbf{x})$ provides

$$\begin{aligned} \psi(\lambda,\mathbf{x}) &= -\frac{2\pi i}{r}\int_{-\infty}^{\infty}\rho d\rho\; e^{i\rho r}e^{i\omega\lambda}\;\tilde{f}(\mathbf{p}) \\ &= -\frac{2\sqrt{\pi}\,i}{r}\lambda_o e^{a_o^2}\int_{-\infty}^{\infty}\rho d\rho\; e^{i\rho r}\tilde{f}(\mathbf{p})\int_{-\infty}^{\infty}ds\; e^{-\lambda_o^2 s^2+2\lambda_o b(\rho)s}. \end{aligned} \tag{24}$$

Inspiration for the integral representation (19) is attenuation of the contribution from the real $b_R(\rho)$ and the nearly constant imaginary $b_Q(\rho)$ in the approximation (22).

3.2 Relativistic approximation for $\psi(\lambda,\mathbf{x})$

It is demonstrated in this section that for Gaussian $\tilde{f}(\mathbf{p})$, the summation (24) effectively truncates to a summation over a finite, rectangularly bounded area, and then approximations result in Gaussian quadrature for $\psi(\lambda,\mathbf{x})$.

The integrand in (24) is

$$e^{-\rho^2\sigma^2-\lambda_o^2 s^2+2b(\rho)\lambda_o s+ir\rho}$$

with $\rho\times s\in(-\infty,\infty)\times(-\infty,\infty)$. This integrand is absolutely Lebesgue integrable. The exponential is an entire function, and Lebesgue integrability reduces to sufficient decline as $|\rho|$ and $|s|$ increase. From the definition (20) of $b_R(\rho)$ and the bound (21), the absolute value of the integrand in (24) is a function of exponential decline for large $|\rho|$ or $|s|$.

$$\begin{aligned} |e^{-\rho^2\sigma^2-\lambda_o^2 s^2+2b(\rho)\lambda_o s+ir\rho}| &\le e^{-\sigma^2\rho^2-\lambda_o^2 s^2+2b_R(\rho)\lambda_o s} \\ &= e^{-\sigma^2\rho^2-(\lambda_o s-b_R(\rho))^2+b_R(\rho)^2}. \end{aligned} \tag{25}$$

A finite region within $(-\infty,\infty)\times(-\infty,\infty)$ dominates the contributions to the summation (24). The region of dominant contribution to (24) is within a finite rectangle

$$\rho\times s\in(-\rho_x,\rho_x)\times(-s_x,s_x) \tag{26}$$

and the integrand is negligible and declining exponentially beyond the rectangle. Let z designate the negative of the exponent in the integrand of (24).

$$z = \lambda_o^2 s^2 + \sigma^2 \rho^2 - 2b(\rho)\lambda_o s - ir\rho. \tag{27}$$

The modulus (25) of the integrand achieves maxima at points designated s_o, ρ_o. Due to the exponential decline of $|e^{-z}|$ for large $|s|$ and $|\rho|$, the dominant contribution is from s, ρ that satisfy

$$\lambda_o^2 s^2 + \sigma^2 \rho^2 - 2b_R(\rho)\lambda_o s \leq K^2$$

for $K \sim 3$. The upper bound defines the boundary of the region of dominant contribution. The finiteness of the region of significant contribution is established by setting $\sigma\rho = R(\theta)\cos\theta$ and $\lambda_o s = R(\theta)\sin\theta$. Then, the R and θ along the boundary of the region of dominant contribution satisfy

$$\begin{aligned} \lambda_o^2 s^2 + \sigma^2 \rho^2 - 2b_R(\rho)\lambda_o s &= R^2 + 2b_R(\tfrac{1}{\sigma} R\cos\theta) R\sin\theta \\ &= K^2. \end{aligned}$$

A finite solution $R = R(\theta)$ to $\mathbb{R}e(z) = K^2$ for each θ is ensured by the continuity of $\mathbb{R}e(z)$ and that $\mathbb{R}e(z) < K^2$ for $R = 0$ and $\mathbb{R}e(z) > K^2$ for $R^2 \gg K^2$ given the bound (21) on $b_R(\rho)$. The minimal enclosing rectangle defines s_x and ρ_x.

From the interchange of summation order justified by Fubini's theorem given Lebesgue integrability,

$$\psi(\lambda, \mathbf{x}) = -\frac{2\sqrt{\pi}\, i}{r} \lambda_o e^{a_o^2} \int_{-\infty}^{\infty} ds\; e^{-\lambda_o^2 s^2} \int_{-\infty}^{\infty} \rho d\rho\; e^{-\rho^2\sigma^2 + 2b(\rho)\lambda_o s + ir\rho}. \tag{28}$$

To achieve Gaussian forms that approximate $\psi(\lambda, \mathbf{x})$, the exponent of the integrand of (28) is approximated with a polynomial,

$$2b(\rho) \approx \beta_0 + \beta_2 \rho^2, \tag{29}$$

within the finite region $(-s_x, s_x) \times (-\rho_x, \rho_x)$. A quadratic approximation for $b(\rho)$ leads to a Gaussian integrand. The quadratic approximation must lack a term linear in ρ. With a $\beta_1 \rho$ term, $\tilde{f}(\mathbf{p})$ is not even, and the simplification (17) that results in an elementary Gaussian quadrature is not valid. The coefficient β_0 is determined in section 1.3 to match non-relativistic approximations and β_2 is determined in section 3.3 to match the expectation of the square of relativistic location over time. It is convenient to express the coefficients with designations α_0, α_2.

$$\begin{aligned} \beta_0 &= 2ia_o + i\frac{\alpha_0}{a_o}\lambda \\ \beta_2 &= \frac{\alpha_2}{a_o}\,\lambda \end{aligned} \tag{30}$$

with $\alpha_0, \alpha_2 \in \mathbb{C}$. α_0, α_2 are independent of the summation variables ρ, s.

Substitution of the approximation (29) for $b(\rho)$ into (28) provides that the ρ summation is approximated by a Gaussian quadrature (13) that converges if $\sigma^2 > \mathbb{R}e(\beta_2)\lambda_o s_x$ with s_x from (26). Evaluation of the ρ summation provides

$$\begin{aligned}
\psi(\lambda, \mathbf{x}) &= -\frac{2\sqrt{\pi}\, i}{r}\lambda_o e^{a_o^2}\int_{-\infty}^{\infty} ds\; e^{-\lambda_o^2 s^2}\int_{-\infty}^{\infty}\rho d\rho\; e^{-\rho^2\sigma^2+2b(\rho)\lambda_o s+ir\rho} \\
&\approx -\frac{2\sqrt{\pi}\, i}{r}\lambda_o e^{a_o^2}\int_{-s_x}^{s_x} ds\; e^{-\lambda_o^2 s^2} e^{\beta_0\lambda_o s}\int_{-\infty}^{\infty}\rho d\rho\; e^{-(\sigma^2-\beta_2\lambda_o s)\rho^2+ir\rho} \\
&= -\frac{2\sqrt{\pi}\, i}{r}\lambda_o e^{a_o^2}\int_{-s_x}^{s_x} ds\; e^{-\lambda_o^2 s^2} e^{\beta_0\lambda_o s}\sqrt{\frac{\pi}{\sigma^2-\beta_2\lambda_o s}}\;\left(\frac{ir}{2(\sigma^2-\beta_2\lambda_o s)}\right)\; e^{\frac{-r^2}{4(\sigma^2-\beta_2\lambda_o s)}} \\
&= \pi\lambda_o e^{a_o^2}\int_{-s_x}^{s_x} ds\; e^{-\lambda_o^2 s^2} e^{\beta_0\lambda_o s}\frac{1}{(\sigma^2-\beta_2\lambda_o s)^{\frac{3}{2}}}\; e^{\frac{-r^2}{4(\sigma^2-\beta_2\lambda_o s)}}.
\end{aligned}$$

Substitution of binomial series approximations accurate if $\sigma^2 \gg |\beta_2|\lambda_o s_x$ result in Gaussian quadrature over s. Satisfaction of $\sigma^2 \gg |\beta_2|\lambda_o s_x$ and $\sigma^2 > \mathbb{R}e(\beta_2)\lambda_o s_x$ follow if

$$\lambda_o \ll \frac{a_o\sigma^2}{\lambda s_x|\alpha_2|}$$

from the expression (30) for β_2. The binomial series approximations result in

$$\begin{aligned}
\psi(\lambda, \mathbf{x}) &\approx \frac{\pi}{\sigma^3}\lambda_o e^{a_o^2}\int_{-s_x}^{s_x} ds\; e^{-\lambda_o^2 s^2} e^{\beta_0\lambda_o s}\left(1+\frac{3\beta_2\lambda_o s}{2\sigma^2}\right)\; e^{\frac{-r^2}{4\sigma^2}(1+\frac{\beta_2\lambda_o s}{\sigma^2})} \\
&\approx \frac{\pi}{\sigma^3}\lambda_o e^{a_o^2}\int_{-\infty}^{\infty} ds\; e^{-\lambda_o^2 s^2} e^{\beta_0\lambda_o s}\left(1+\frac{3\beta_2\lambda_o s}{2\sigma^2}\right)\; e^{\frac{-r^2}{4\sigma^2}(1+\frac{\beta_2\lambda_o s}{\sigma^2})} \\
&= \frac{\pi}{\sigma^3}e^{\frac{-r^2}{4\sigma^2}}\lambda_o e^{a_o^2}\sqrt{\frac{\pi}{\lambda_o^2}}\left(1+\frac{3\beta_2\lambda_o}{2\sigma^2}\frac{(\beta_o\lambda_o-\frac{\beta_2 r^2\lambda_o}{4\sigma^4})}{2\lambda_o^2}\right)\exp\!\left(\frac{\lambda_o^2(\beta_o-\frac{\beta_2 r^2}{4\sigma^4})^2}{4\lambda_o^2}\right) \\
&= \frac{\pi^{\frac{3}{2}}}{\sigma^3}e^{\frac{-r^2}{4\sigma^2}}e^{a_o^2}\left(1+\frac{3\beta_2(\beta_o-\frac{\beta_2 r^2}{4\sigma^4})}{4\sigma^2}\right)\exp\!\left(\tfrac{1}{4}(\beta_o-\tfrac{\beta_2 r^2}{4\sigma^4})^2\right).
\end{aligned}$$

This follows from the negligible contribution to the integral from $|s| > s_x$, and the Gaussian quadrature (13).

The exponent includes

$$\begin{aligned}
\tfrac{1}{4}(\beta_o-\frac{\beta_2 r^2}{4\sigma^4})^2 &= \tfrac{1}{4}\left(2ia_o+i\frac{\alpha_0}{a_o}\lambda+\frac{\lambda r^2}{4a_o\sigma^4}\,\alpha_2\right)^2 \\
&\approx -a_o^2+i\alpha_0\lambda+i\frac{\alpha_2 r^2}{4\sigma^4}\,\lambda
\end{aligned}$$

for $a_o \gg \max(1, \omega(\rho_x)|\lambda|, |\alpha_2 \lambda| r^2/\sigma^4)$ with the substitutions (30) for β_0, β_2. β_2 is negligible with respect to β_0 for sufficiently large a_o. Similarly,

$$\frac{3\beta_0\beta_2}{4\sigma^2} \approx i\frac{3\alpha_2}{2\sigma^2}\lambda.$$

Simplification results in

$$\psi(\lambda, \mathbf{x}) \approx \frac{\pi^{\frac{3}{2}} e^{i\alpha_0\lambda}}{\sigma^3}\left(1 + i\frac{3\lambda}{2\sigma^2}\alpha_2\right)\exp\left(-\frac{r^2}{4\sigma^2}\left(1 + i\frac{\lambda}{\sigma^2}\alpha_2\right)\right)$$

with no dependence on the values of the parameters a_o, λ_o from the integral representation (19).

The approximation

$$\psi(\lambda, \mathbf{x}) \approx \frac{\pi^{\frac{3}{2}} e^{i\alpha_0\lambda}}{(\sigma^2 - i\lambda\sigma_s)^{\frac{3}{2}}}\exp\left(-\frac{r^2}{4\,(\sigma^2 - i\lambda\sigma_s)}\right)$$

follows from definition of

$$\sigma_s = \frac{\sigma^2\alpha_2}{\sigma^2 + i\alpha_2\lambda} \tag{31}$$

and a convenient, finite function normalization for $\tilde{f}(\mathbf{p})$. This is the approximation for the function $\psi(\lambda, \mathbf{x})$ presented in (14). The constant $\alpha_0 = \lambda_c^{-1}$ is determined from non-relativistic approximation in section 1.3, and consequently $|\alpha_0\lambda| \leq \omega(\rho_x)|\lambda|$.

The approximation (14) of $\psi(\lambda, \mathbf{x})$ from (4) with Gaussian $\tilde{f}(\mathbf{p})$ from (10) rests on the fit (29) to $b(\rho)$ and the selection of parameters a_o, λ_o in the integral representation of time translation (19). The parameters are selected to provide an approximation of the spatial dependence of $\psi(\lambda, \mathbf{x})$ at each λ. The parameters a_o, λ_o are selected to satisfy

$$a_o^2 \gg \max(|\alpha_0\lambda|, \frac{|\alpha_2\lambda| r^2}{\sigma^4}) \quad \text{and} \quad \lambda_o \ll \frac{a_o\sigma^2}{s_x|\lambda\alpha_2|}.$$

The conditions are satisfied for sufficiently large a_o and small λ_o given finite $\lambda, r, \sigma, \alpha_2$ and s_x.

3.3 Determination of σ_s

The coefficient α_2 of the fit (29) is set to match the expectation value (15) of the square of relativistic location. The means of location and momentum are zero for both $\psi(\lambda, \mathbf{x})$ and the approximation (14), and, from the lack of temporal variation, the expectations of the squares of momentum for $\psi(\lambda, \mathbf{x})$ and the approximation (14) are equal. A match of the square of relativistic location then sets the four expectations (15) for $\psi(\lambda, \mathbf{x})$ and the approximation (14) equal.

Expectation values (8) conveniently follow from the Fourier transform of the approximation (14). For rotationally invariant functions $\psi(\lambda, \mathbf{x})$,

$$\begin{aligned}\int d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}\psi(\lambda,\mathbf{x}) &= \int_0^\infty r^2 dr \int_0^\pi \sin\phi\, d\phi \int_0^{2\pi} \frac{d\theta}{(2\pi)^3}\ e^{i\rho r\cos\phi}\psi(\lambda,\mathbf{x}) \\ &= \frac{i}{(2\pi)^2\rho}\int_{-\infty}^{\infty} rdr\ e^{-i\rho r}\psi(\lambda,\mathbf{x})\end{aligned}$$

with the definitions above (16) and from the development of (17). The Fourier transform of the approximation (14) is a Gaussian quadrature (13). With $\widehat{\psi}(\lambda, \mathbf{x})$ designating the approximation (14) of $\psi(\lambda, \mathbf{x})$,

$$\begin{aligned}\int d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}\widehat{\psi}(\lambda,\mathbf{x}) &= \frac{ie^{i\alpha_0\lambda}}{4\pi^{\frac{1}{2}}\rho\,(\sigma^2 - i\lambda\sigma_s)^{\frac{3}{2}}}\int_{-\infty}^{\infty} rdr\ e^{-i\rho r}\ \exp\left(-\frac{r^2}{4\,(\sigma^2 - i\lambda\sigma_s)}\right) \\ &= \frac{ie^{i\alpha_0\lambda}}{4\pi^{\frac{1}{2}}\rho\,(\sigma^2 - i\lambda\sigma_s)^{\frac{3}{2}}}\sqrt{4\pi}\,\left(\sigma^2 - i\lambda\sigma_s\right)^{\frac{3}{2}}(-2i\rho)e^{-r^2(\sigma^2 - i\lambda\sigma_s)} \\ &= e^{i\alpha_0\lambda}e^{-\rho^2(\sigma^2 - i\lambda\sigma_s)}.\end{aligned} \tag{32}$$

The approximate $\widehat{\psi}(\lambda, \mathbf{x})$ is a Gaussian over space and applies without limitations on the duration of the period λ nor the momentum support of $\tilde{f}(\mathbf{p})$. The coefficients β_0, β_2 of the fit (29) are determined: to set $\widehat{\psi}(\lambda, \mathbf{x})$ equal to $\psi(\lambda, \mathbf{x})$ in non-relativistic instances; and to match the square of relativistic location.

$$E[\hat{\mathbf{X}}^2; \psi(\lambda,\mathbf{x})] = E[\hat{\mathbf{X}}^2; \widehat{\psi}(\lambda,\mathbf{x})]. \tag{33}$$

In Born's rule for expectations (8), the norms in (33) are the same for $\psi(\lambda, \mathbf{x})$ and the approximate (14).

$$\begin{aligned}\|\psi\| &= \int d\mathbf{p}\ 2\omega \left|e^{i\omega\lambda}e^{-\sigma^2\mathbf{p}^2}\right|^2 \\ \|\widehat{\psi}\| &= \int d\mathbf{p}\ 2\omega \left|e^{i\alpha_0\lambda}e^{-\mathbf{p}^2(\sigma^2 - i\lambda\sigma_s)}\right|^2.\end{aligned}$$

Then, the integration by parts,

$$-\sum_\nu \int d\mathbf{p}\ \omega^{\frac{1}{2}}\overline{e^{g(\mathbf{p})}}\left(\frac{\partial^2}{\partial p_\nu^2}\ \omega^{\frac{1}{2}}e^{g(\mathbf{p})}\right) = \sum_\nu \int d\mathbf{p}\ \left|\frac{\partial}{\partial p_\nu}\ \omega^{\frac{1}{2}}e^{g(\mathbf{p})}\right|^2$$

reduces the equality (33) to

$$\sum_\nu \int d\mathbf{p}\ \left|\frac{\partial}{\partial p_\nu}\ \omega^{\frac{1}{2}}e^{g_1(\mathbf{p})}\right|^2 = \sum_\nu \int d\mathbf{p}\ \left|\frac{\partial}{\partial p_\nu}\ \omega^{\frac{1}{2}}e^{g_2(\mathbf{p})}\right|^2$$

with

$$g_1(\mathbf{p}) = i\omega\lambda - \sigma^2\mathbf{p}^2$$

from (4) with (10) and

$$g_2(\mathbf{p}) = i\alpha_0\lambda - \mathbf{p}^2(\sigma^2 - i\lambda\sigma_s)$$

from the Fourier transform (32). The integration by parts displays that $E[\hat{\mathbf{X}}^2;\varphi_1]$ is real and nonnegative. The product and chain rules provide

$$\frac{\partial}{\partial p_\nu}\,\omega^{\frac{1}{2}}e^{g(\mathbf{p})} = \left(\frac{p_\nu}{2\omega^{\frac{3}{2}}} + \omega^{\frac{1}{2}}g'(\mathbf{p})\right)e^{g(\mathbf{p})}$$

with prime indicating derivative with respect to p_ν. From substitution of the two $g(\mathbf{p})$, equality is

$$\sum_\nu \int d\mathbf{p}\left|\left(\frac{p_\nu}{2\omega^2} - 2p_\nu(\sigma^2 - i\lambda\sigma_s)\right)\right|^2 \omega\, e^{-2\sigma^2\mathbf{p}^2}$$
$$= \sum_\nu \int d\mathbf{p}\left|\left(\frac{p_\nu}{2\omega^2} + i\frac{p_\nu}{\omega}\lambda - 2\sigma^2 p_\nu\right)\right|^2 \omega\, e^{-2\sigma^2\mathbf{p}^2}.$$

Simplification results in the expression for σ_s that satisfies (33).

$$\sigma_s^2\int d\mathbf{p}\; 4\mathbf{p}^2\omega\, e^{-2\sigma^2\mathbf{p}^2} = \int d\mathbf{p}\;\frac{\mathbf{p}^2}{\omega}\,e^{-2\sigma^2\mathbf{p}^2}. \tag{34}$$

The positive root of (34) is σ_s. σ_s is real, independent of λ and varies with σ.

The Fourier transforms of $\psi(\lambda,\mathbf{x})$ and $\widehat{\psi}(\lambda,\mathbf{x})$,

$$e^{i\omega(\rho)\lambda}e^{-\sigma^2\rho^2} \quad\text{and}\quad e^{i\alpha_0\lambda}e^{-\rho^2(\sigma^2 - i\lambda\sigma_s)},$$

respectively, are approximately pointwise equal if momenta are non-relativistic. The discussion of section 1.3 identifies that $\alpha_0 = \lambda_c^{-1}$ and from (34),

$$\sigma_s \approx \frac{1}{2}\lambda_c$$

in non-relativistic momenta when $\omega \approx \lambda_c^{-1}$. For non-relativistic momenta, the Fourier transform of the approximation reproduces the first two terms of the binomial series for $\omega(\rho)$.

$$\begin{aligned}\omega(\rho) &\approx \alpha_0 + \sigma_s\rho^2\\ &= \lambda_c^{-1} + \tfrac{1}{2}\lambda_c\rho^2.\end{aligned}$$

For relativistic momenta (23), $\alpha_0 + \rho^2\,\sigma_s$ is not an accurate pointwise approximation for ω except in neighborhoods of selected ρ, but pointwise approximation of functions ($\mathcal{L}^\infty$ almost everywhere) is not required to achieve pointwise approximation of their Fourier transforms. The equality of moments (15), and comparison of $b(\rho)$ with quadratic fits (29) within the dominant support of the integrand in (28) confirms a similarity of the Gaussian approximation (14) and $\psi(\lambda, \mathbf{x})$ from (4) with (10). The Gaussian form (14) follows from selection of the quadratic approximation (29) for $b(\rho)$.

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