Relativistic Quantum Physics

Glenn Eric Johnson
Corolla, NC
relquantphys.org

November 10, 2026

Keywords: Foundations of quantum mechanics, relativistic quantum field theory, constructive quantum field theory.

Contents

1 Introduction **4**

2 Realization of relativistic quantum physics **6**

3 Constructions of relativistic quantum mechanics **13**
3.1 Quantum mechanical description of state 13
3.1.1 The scalar product 17
3.1.2 The quantum field and VEV 19
3.1.3 The adjoint of the field and Hermiticity 20
3.1.4 Likelihoods: Born's rule 22
3.2 Relativistic free fields 23
3.2.1 Free field VEV 23
3.2.2 Example $M(p)$, D and $S(A)$ 25
3.3 Physically nontrivial relativistic fields 30
3.3.1 Connected functions 30
3.3.2 Split signed symmetric functions 32
3.3.3 Physically nontrivial VEV functions 34
3.3.4 Cluster expansion of VEV 37
3.3.5 Non-zero expectation value fields 40
3.4 The basis function spaces 41
3.4.1 The basis function spaces $\underline{\mathcal{P}}$ 41
3.4.2 An alternative formulation based on $\underline{\mathcal{S}}$ 43

4 Characteristics of the relativistic fields **44**
4.1 Axioms for the VEV of relativistic fields . . . 44
4.2 Satisfaction of the axioms . . . 52
4.2.1 Summary of constructions . . . 52
4.2.2 Nonnegative $\circ$-products . . . 53
4.2.3 Nonnegativity of $\underline{\mathcal{U}}$ . . . 55
4.2.4 Poincaré invariance . . . 60
4.2.5 Cluster decomposition . . . 61
4.2.6 Regularity . . . 64
4.3 Relation to free fields . . . 69
4.4 The Hamiltonian and Hilbert space field operators . . . 70
4.5 Scattering likelihoods . . . 72

5 Non-relativistic quantum-classical correspondences **75**
5.1 The support of state describing functions . . . 78
5.2 Properties of states with classical descriptions . . . 80
5.3 Correspondence from the evolution of support along trajectories . . . 85
5.4 Two body correspondence . . . 91
5.4.1 Non-relativistic approximation . . . 93
5.4.2 Brief interval and limited acceleration approximation . . . 98
5.4.3 Energy correspondence and $-g/r$ potentials . . . 104
5.5 Extended interval propagation . . . 106

6 Bound states **108**
6.1 Preliminaries . . . 108
6.2 The state describing functions of bound states . . . 110
6.2.1 A relativistic description of bound states . . . 110
6.2.2 The support of bound state describing functions . . . 112

7 Technical concerns with canonical quantization **119**

8 Appendices **124**
8.1 The Dirac-von Neumann axioms for quantum mechanics . . . 124
8.2 Hilbert spaces and quantum mechanics . . . 127
8.2.1 Hilbert spaces . . . 127
8.2.2 Rigged Hilbert spaces . . . 128
8.2.3 The Cauchy-Schwarz-Bunyakovsky inequality . . . 129
8.2.4 Entanglement . . . 130
8.2.5 Hilbert space operators . . . 131
8.2.6 Operators in quantum mechanics . . . 132

8.2.7 Interpretation of the functions $f_n((x)_n)_{(\kappa)_n}$ in $\underline{f}$ . . . 133
8.2.8 Observation in Hilbert spaces . . . 136
8.2.9 Localized observables, separation and independence . . . 143
8.2.10 Tensor products of linear vector spaces . . . 147
8.3 Translations, location operators and relativistically invariant localized states . . . 148
8.4 Inconsistency of the classical description with nature . . . 155
8.5 The Einstein-Podolsky-Rosen paradox . . . 157
8.6 Heisenberg uncertainty . . . 158
8.7 Equivalence of the cluster and operator expansions for free field VEV . . . 162
8.8 Realizations of $Q_{2,2}$ . . . 166
8.9 Feynman-Dyson series and the constructions . . . 169
8.10 The generalized function $\delta(H_B(\mathbf{q}_S, \mathbf{q}_D))$ . . . 175
8.11 Center of momentum frames and energy-momentum conservation . . . 178
8.12 Similarity transforms and compositions of $M(p), D, S(A)$ triples . . . 180
8.13 Two body classical trajectories . . . 186
8.14 Non-relativistic energy approximations . . . 188
8.15 Gaussian quadratures . . . 189
8.16 Expected value of the energy of $\varphi_2(0)$ . . . 192
8.17 Essentially local functions . . . 194
8.18 Illustrative relative states . . . 196

1 Introduction

> It seems clear that the present quantum mechanics is not in its final form. Some further changes will be needed, just about as drastic as the changes made in passing from Bohr's orbit theory to quantum mechanics. Some day a new quantum mechanics, a relativistic one, will be discovered, in which we will not have these infinities occurring at all. – Paul Dirac, in *Albert Einstein : Historical and Cultural Perspectives: The Centennial Symposium in Jerusalem*, edited by Gerald James Holton and Yehuda Elkana, 1979, p. 85.

The absence of physically nontrivial realizations of relativistic quantum mechanics suggests that either quantum mechanics is insufficient to describe nature, or there are unnecessary considerations within relativistic quantum field theory (RQFT) [7, 23, 24, 54, 63] that preclude realization. Although the Feynman rules employ concepts and methods from quantum mechanics, finite period evolution of general state descriptions is not realized [8, 9, 58]. From Born's rule, transition likelihoods are squared magnitudes of scalar products of initial and final state descriptions. States are described by functions that label elements in a Hilbert space. From the principle of relativity [15], likelihoods are the same for all inertial observers. Unitary realization of the Poincaré group provides relativistic invariance of the scalar product in compliance with relativity [66].[a] The generator of time translations is the Hamiltonian, and the Hamiltonian commutes with the generators of spatial translations, the momenta, in a realization of the Poincaré group. In a RQFT development, a Hamiltonian that commutes with momenta is physically trivial. Description of interaction in relativistic quantum mechanics becomes realizable if the Hamiltonian is from a representation of the Poincaré group and if interaction is described by the vacuum expectation values (VEV) of the fields, section 2. Constructions of relativistic quantum physics that exhibit interaction are presented in this note, section 3. The realizations are finite and in four dimensional spacetime, and scattering likelihoods include approximations to Feynman series scattering amplitudes, appendix 8.9 and [36].

The elevation[b] of position to densely defined Hermitian operator fails in relativistic quantum physics, [44], section 7 and appendix 8.3. The Hilbert spaces of relativistic quantum physics lack orthogonal projection operators onto spacetime points [68]. With the counterexample to elevation, quantum fields are not assumed to be Hermitian in the constructions. The constructions display VEV that exhibit interaction and provide the scalar product in the Hilbert space

[a] The Poincaré group is also commonly designated as the inhomogeneous Lorentz group. The Poincaré group includes Minkowski space transformations that leave the spacetime interval between events invariant.

[b] An elevation of a classical dynamical variable pairs the variable q with a densely defined Hilbert space operator Q implemented $Q\psi(q) = q\psi(q)$ with quantum state descriptions $\psi(q)$ that are functions over the domain of q. An elevation Q is Hermitian in the Hilbert space scalar product: all expectation values $\langle\psi|Q\psi\rangle$ are real if and only if Q is Hermitian, appendix 8.2.6. The (generalized) eigenfunctions of Q, $Q\psi_{q_o}(q) = q_o\psi_{q_o}(q)$, are Dirac delta functions for observables with continuous values, or Kronecker delta functions for observables with discrete values.

of state descriptions. It is subsequently demonstrated that quantum fields are Hilbert space operators, section 4.4, but the quantum fields are not densely defined Hermitian operators when interaction is exhibited, section 2.

The constructed VEV satisfy the principles of quantum mechanics and relativity [9, 13, 62, 63] described in axioms A.1-7 of section 4.1. In particular, the constructions satisfy the Wightman axioms [58, 64] after technical conditions that implement Hermitian fields are removed.

The lack of interaction Hamiltonians and non-Hermitian fields are extensive deviations from established characterizations of the association of classical dynamics with quantum dynamics. The constructions provide finite realizations of relativistic quantum physics in four dimensional spacetime, but lose the established correspondences with classical descriptions of interaction. Nevertheless, quantum-classical correspondences with established descriptions remain. The difficulty that follows is that, without the elevation of a classical Hamiltonian, the correspondences of the quantum dynamics with classical dynamical models are more obscure, and richer. This note includes initial studies of the classical dynamics that correspond with a construction. Correspondences include: scattering likelihoods; non-relativistic propagation of the support of classical particle-like state descriptions; and bound states of the elementary particles. Scattering likelihoods follow closely from the VEV. Asymptotic times imply non-overlap of state describing functions and then cluster decomposition provides a correspondence with classical particles. The approximations of Feynman series scattering amplitudes associate constructions with the interaction Hamiltonians of RQFT. The classical dynamics associated with finite interval transitions for particle-like states and bound states have far richer correspondences. Multiple classical Hamiltonians apply to a single construction of relativistic quantum physics: the corresponding interactions vary with properties of the state describing functions. Anticipation is that an as yet unidentified principle in addition to axioms A.1-7 determines correspondences with the physically relevant ones. The principle would determine the strength g of the $-g/r$ potentials that correspond with the quantum dynamics of non-relativistic particle-like states. The $-g/r$ interaction, section 5, suggests classical electrostatics or approximations to gravity. Presumably, classical electrodynamics and geometrodynamics emerge from more precise approximations over greater intervals. The constructions include bound states of the elementary particles, section 6, and the additional principle would determine the set of observed, discrete bound state masses.

Quantum-classical correspondences are derived from the temporal evolution of regions of isolated support of the state describing functions. Isolated support, section 5.2, enables association of arguments of state describing functions with classical particles. The adopted relativistic quantum-classical correspondence is to conditionally represent the isolated dominant supports of state describing functions[c] with classical dynamical variables. In quantum mechanics, the

[c]The support of a function is commonly defined as the set of points in the domain where the function is not zero. Here, for location and momentum, the dominant support of a function refers to Lebesgue summations over finite volumes that include a dominant contribution to the mean squared amplitude of the function. The dominant supports of interest are discussed further in section 5.1 and [31]. Relatively negligible mean squared

likely perceptions are within the dominant support of state describing functions.

Discussion emphasizes the principles of relativistic quantum physics, establishment of example realizations, and a physical understanding of the development. Associated phenomenology is relatively unexplored other than to establish connections with Feyman series and classical dynamics. New constructions, matured from [32, 34, 36, 38], are developed in these notes. A stronger cluster decomposition property than the uniqueness of the vacuum condition used in [32, 34, 38] provides that truncated functions [9] are connected functions. The stronger cluster decomposition axiom: provides a replacement for formal Hermiticity in demonstrations that the quantum fields (3) are Hilbert space operators; implies that states with sufficiently isolated and space-like separated dominant supports are described by free particles; provides the essential independence of the local observables of non-entangled, spatially distant bodies; and implies a unique vacuum. Primary references are Borchers' [10] and Wightman's [58, 64] developments, Bogolubov, Logunov and Todorov's review [9], Weinberg's RQFT text [63], and mathematical background includes [3, 12, 19, 20, 21, 24, 26, 41, 47, 49, 62]. Realization of relativistic quantum physics is introduced in section 2. Section 3 provides a more comprehensive description of VEV and presents realizations with multiple constituent fields. Properties of the constructions including axioms A.1-7 and scattering likelihoods are discussed in section 4. The quantum-classical correspondences for state describing functions with isolated, non-relativistic support are considered in section 5. Bound states are displayed in section 6, and limitations of canonical quantization are discussed in section 7.

2 Realization of relativistic quantum physics

In quantum mechanics, the descriptions of nature are elements of a Hilbert space [13, 62]. The descriptions of nature are sequences of complex-valued functions over three-dimensional space. The evolution of these functions is parameterized by time. Elementary particles are described in the Wigner classification with irreducible, nonnegative energy representations of the Poincaré group [9, 63].

Sequences of state describing functions over a 3+1 spacetime

$$\underline{f} = (f_0, f_1(x_1)_1, \ldots f_1(x_1)_{\kappa_1}, \ldots f_2(x_1, x_2)_{\kappa_1,\kappa_2}, \ldots) \tag{1}$$

label elements $|\underline{f}\rangle$ of a rigged (equipped) Hilbert space $\mathbf{H}_{\mathcal{P}}$, [9, 10, 32, 34, 38, 58], section 3 and appendix 8.2. These function sequences include descriptions of the vacuum, single particle, multiple particle, wave-like, and composite states. Each $x_j \in \mathbb{R}^4$, $j, \kappa_j \in \mathbb{N}$, $\kappa_j \in \{1, N_c\}$ with N_c the number of constituent fields. $f_0 \in \mathbb{C}$ is the component of the state in the vacuum state characterized by the sequence

$$\Omega = (1, 0, 0 \ldots).$$

amplitudes over finite volumes are taken as zero to apply the more standard definition of function support.

The N_c functions $f_1(x)_\kappa \in \mathbf{H}_\mathcal{P}(\mathbb{R}^4)_\kappa$ describe single elementary particle states with masses m_κ. Understanding of the $(N_c)^n$ functions $f_n((x)_n)_{(\kappa)_n} \in \mathbf{H}_\mathcal{P}(\mathbb{R}^{4n})_{(\kappa)_n}$ distinguishes this approach to RQP from RQFT developments, section 3.1 and appendix 8.2.7 contrasted with [23, 63]. In the constructions, $f_n((x)_n)_{(\kappa)_n}$ describes n particles in the absence of significant interaction, for example, if the dominant support of each argument is distantly space-like isolated from the dominant support of any other state describing function argument, but $f_n((x)_n)_{(\kappa)_n}$ can also describe $k \neq n$ particles of varied species, section 3.1. $\mathbf{H}_\mathcal{P}(\mathbb{R}^{4n})_{(\kappa)_n}$ is an n-argument subspace of $\mathbf{H}_\mathcal{P}$ described by masses m_{κ_j}. A convenient representation expands the description of state (1) as a linear combination of products of a field $\Phi(x)_\kappa$ applied to the vacuum with coefficients that are the functions $f_n((x)_n)_{(\kappa)_n}$.

$$|\underline{f}\rangle = \sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \Omega\rangle \tag{2}$$

with

$$(x)_n = x_1, x_2 \ldots x_n \in \mathbb{R}^{4n}$$

the designation for a sequence of n arguments $x_j \in \mathbb{R}^4$. The conditional particle interpretation of the $n \geq 2$, n-argument functions in (1) follows for physically nontrivial vacuum expectation values of the fields $\Phi(x)_\kappa$, section 3. The product of quantum fields $\prod_k \Phi(x_k)_{\kappa_k}$ provides that the composition

$$\sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \underline{g}\rangle = |\underline{f} \times \underline{g}\rangle \tag{3}$$

is a multiplication $\underline{f} \times \underline{g}$ of function sequences [10]. The $\times$-multiplication of function sequences is

$$\underline{f} \times \underline{g} = (f_0 g_0, \ldots, \sum_{\ell=0}^{n} f_\ell(x_1, \ldots x_\ell)_{(\kappa)_{1,\ell}} \, g_{n-\ell}(x_{\ell+1}, \ldots x_n)_{(\kappa)_{\ell+1,n}}, \ldots). \tag{4}$$

The vacuum Ω is the identity for the $\times$-multiplication and then, in (2), the quantum fields map

$$|\Omega\rangle \in \mathbf{H}_\mathcal{P} \quad \mapsto \quad |\underline{f}\rangle \in \mathbf{H}_\mathcal{P}$$

from $\underline{f} \times \Omega = \underline{f}$. Notation is discussed further in section 3.1 and section 4.4 includes a demonstration that fields (3) are Hilbert space operators.

Vacuum expectation values (VEV) of the quantum fields $\Phi(x)_\kappa$,

$$\langle \Phi(x_k)_{\kappa_k} \ldots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \ldots \Phi(x_n)_{\kappa_n} \Omega\rangle \in \mathcal{S}'(\mathbb{R}^{4n}) \tag{5}$$

are generalized functions (distributions) in the dual $\mathcal{S}'(\mathbb{R}^{4n})$ to the Schwartz tempered test functions $\mathcal{S}(\mathbb{R}^{4n})$ [20]. The VEV together with a basis space of function sequences $\underline{\mathcal{P}} \subseteq \underline{\mathcal{S}}$

determine the Hilbert space realization. The VEV determine the scalar product of Hilbert space elements.

$$\langle \underline{f}|\underline{g}\rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\, \langle \Phi(x_1)_{\kappa_1} \dots \Phi(x_n)_{\kappa_n}\Omega | \Phi(x_{n+1})_{\kappa_{n+1}} \dots \Phi(x_{n+m})_{\kappa_{n+m}}\Omega\rangle \\ \times \overline{f_n}(x_1, \dots x_n)_{\kappa_1 \dots \kappa_n}\, g_m(x_{n+1}, \dots x_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}} \tag{6}$$

with formal summation notation [19] for the generalized functions.

From Born's rule, the magnitudes squared of scalar products in $\mathbf{H}_{\mathcal{P}}$ define likelihoods. Consequently, the VEV (5) must satisfy the physical properties of quantum mechanics and relativity, properties such as Poincaré invariance, nonnegative energies, and Bose-Einstein or Fermi-Dirac statistics. Axioms A.1-7 in section 4.1 present these properties. The physical conditions from the Wightman axioms [9, 10, 58, 64] plus cluster decomposition and regularity are included in axioms A.1-7. Significantly, technical assumptions within the Wightman axioms that implement Hermitian fields are discarded. The constructions exhibit VEV and function sequences $\underline{f} \in \mathcal{P}$ compliant with axioms A.1-7.

The state descriptions $|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$ are labeled by representatives $\underline{f}$ of equivalence classes within the Hilbert space norm completion of $\underline{\mathcal{P}}$ [9, 10, 12, 58]. In the emphasized development of the constructions, the selected basis spaces of functions $\underline{\mathcal{P}}$ are subspaces of the space of Schwartz tempered functions $\underline{\mathcal{S}}$ [20]. In an alternative development, the basis space of functions is $\underline{\mathcal{S}}$. Every $\underline{f} \in \underline{\mathcal{P}}$ is a terminating sequence of functions of the form

$$\tilde{f}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n} (p_{j0} + \omega_j)\, \tilde{h}_n((p)_n)_{(\kappa)_n} \tag{7}$$

with the $h_n \in \mathcal{S}(\mathbb{R}^{4n})$ and the wave number

$$\omega_j = \omega(\mathbf{p}_j) = \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}. \tag{8}$$

The reduced Compton wavelength for the mass m_{κ_j} associated with a constituent field $\Phi(x_j)_{\kappa_j}$ is

$$\lambda_{cj} = \frac{\hbar}{m_{\kappa_j} c}. \tag{9}$$

There is a distinct subspace $\mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$ for each distinct sequence of masses $m_{\kappa_1}, m_{\kappa_2}, \dots m_{\kappa_n}$, section 3.4. The selection (7) of basis function space $\underline{\mathcal{P}}$ limits the support of state descriptions to nonnegative energies, section 4.1. For the constructed VEV, $\mathbf{H}_{\mathcal{P}}$ includes elements described by generalized functions with point support over time, functions used by Lehmann, Symanzik and Zimmermann (LSZ) to describe scattering, [9] and section 4.5.

Hermitian field operators $\Phi(x)_\kappa$ have not been realized in RQFT that exhibit interaction. One motivation for Hermiticity has been to implement quantum fields (2) that elevate classical

fields. Elevations are Hermitian. Hermitian quantum field operators (2) are a consequence of the original Wightman and similar axioms for RQFT [9, 10, 58, 59, 64]. Fields are densely defined Hermitian if

$$\langle \underline{h}|\Phi(\underline{f})^*\underline{g}\rangle = \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle \tag{10}$$

for "real" function sequences $\underline{f} = \underline{f}^*$ with the Hermitian adjoint

$$\langle \Phi(\underline{f})^*\underline{h}|\underline{g}\rangle = \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle$$

for the sum of the component fields

$$\Phi(\underline{f}) = \sum_{\kappa=1}^{N_c} \int dx\ \Phi(x)_\kappa f(x)_\kappa.$$

Hermiticity applies on a dense domain $|\underline{h}\rangle, |\underline{g}\rangle \in \mathcal{D}_\Phi \subset \mathbf{H}_\mathcal{P}$. However, the example of relativistic position, section 1, demonstrates that not all classical dynamical variables have elevations in relativistic physics, and that a classical dynamical variable may be observable without Hermiticity of its elevation.

To emphasize significant differences between the constructions and Wightman's original development [9, 10, 58, 64], the Wightman functions $W_n((x)_n)_{(\kappa)_n}$ and the more general VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are introduced. A $*$-dual function sequence $\underline{f}^* \in \underline{\mathcal{S}}$ is defined for each $\underline{f} \in \underline{\mathcal{S}}$ by argument transposition, complex conjugation and linear transformation D.

$$\widetilde{f_n^*}((p)_n)_{(\kappa)_n} = (D^T\cdot)_n\overline{\tilde{f}_n}(-p_n, \ldots -p_1)_{\kappa_n\ldots\kappa_1} \tag{11}$$

with $\tilde{f}(p)$ the Fourier transform (21) of $f(x)$, $\overline{z}$ the complex conjugate of $z \in \mathbb{C}$, and in a matrix notation with $(D^T)_{ij} = D_{ji} \in \mathbb{C}$,

$$(D\cdot)_n V_{(\kappa)_{n+m}} = \sum_{\ell_1=1}^{N_c}\sum_{\ell_2=1}^{N_c}\ldots\sum_{\ell_n=1}^{N_c} D_{\kappa_1\ell_1}D_{\kappa_2\ell_2}\ldots D_{\kappa_n\ell_n}V_{\ell_1\ldots\ell_n\kappa_{n+1}\ldots\kappa_{n+m}}. \tag{12}$$

The Fourier transform of $\overline{f(x)}$ is $\overline{\tilde{f}(-p)}$. D is determined in a representation of the Lorentz group and example D are provided in section 3.2.2. With this notation, the scalar product (6) is

$$\begin{aligned}\langle \underline{f}|\underline{g}\rangle &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(x)_{n+m}\ (D\cdot)_n\mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times\overline{f_n}(x_n,\ldots x_1)_{\kappa_n\ldots\kappa_1}\ g_m(x_{n+1},\ldots x_{n+m})_{(\kappa)_{n+1,n+m}} \\ &= \mathcal{W}(\underline{f}^*,\underline{g})\end{aligned} \tag{13}$$

with VEV functions substituted for the VEV (5). The VEV functions follow from the VEV of fields (6).

$$(D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} = \langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle \tag{14}$$

in the matrix notation (12). The original Wightman axioms assert that the basis space of function sequences is the $*$-involutive[d] algebra of tempered functions $\underline{\mathcal{S}}$, and that the scalar product (6) simplifies to

$$\begin{aligned} \langle \underline{f} | \underline{g} \rangle &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, \langle \Omega | \Phi(x_1)_{\kappa_1} \dots \Phi(x_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ &\qquad \times f_n^*(x_n, \dots x_1)_{\kappa_n \dots \kappa_1} \, g_m(x_{n+1}, \dots x_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}} \\ &= \underline{W}(\underline{f}^* \times \underline{g}) \end{aligned} \tag{15}$$

with the function sequence $\underline{f} \times \underline{g}$ the same product of function sequences (4) as in the definition (3) of field [10]. The simplified scalar product (15) follows from the scalar product (6) if the VEV (5) satisfy

$$\begin{aligned} W_n((x)_n)_{(\kappa)_n} &= \langle \Omega | \Phi(x_1)_{\kappa_1} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle \\ &= (D^{-1}\cdot)_k \langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle \end{aligned} \tag{16}$$

for any k, $0 \leq k \leq n$. D is from the $*$-dual (11) and is non-singular, $D^{-1} = \overline{D}$ from (26) in section 3.1.1. If (16) is satisfied and the basis function space of function sequences is the $*$-involutive $\underline{\mathcal{S}}$, the field is Hermitian, $\Phi(\underline{f})^* = \Phi(\underline{f})$ for real function sequences $\underline{f}^* = \underline{f}$ [9, 10, 58, 64]. A (16) compliant sequence of VEV is a Wightman functional $\underline{W}$, the sequence (1) of Wightman functions $W_n((x)_n)_{(\kappa)_n}$. However, despite determined efforts [2, 4, 9, 30, 40, 42, 58], no physically nontrivial realization for a Wightman functional $\underline{W}$ has been discovered. Physically trivial free field VEV satisfy the Wightman axioms, but here, to construct physically nontrivial fields, the simplification of the scalar product (15) and $*$-involutivity of the basis space of function sequences are not assumed. Below, the condition (16) that implies the simplification (15) of the scalar product (6) is denoted formal Hermiticity. One or both of $*$-involutivity and formal Hermiticity are violated if VEV exhibit interaction, section 3. In the constructions, Hermiticity of the field operators $\Phi(x)_\kappa$ is inconsistent with interaction. If the quantum field operator (3) were unitarily similar to a free field, then the demonstrable Hermiticity of a free field would imply Hermiticity of the field. As a consequence, and consistently with the Haag (Haag-Hall-Wightman-Greenberg) theorem [9, 58], unitary similarity of interacting and free field operators is abandoned for fields that exhibit interaction. Hermiticity is unnecessary to realize even physically trivial RQFT [32].

[d] Here, an algebra of function sequences $\underline{\mathcal{A}}$ is denoted $*$-involutive if $\underline{f}^* \in \underline{\mathcal{A}}$ for every $\underline{f} \in \underline{\mathcal{A}}$, $(\underline{f}^*)^* = \underline{f}$, $(\underline{f} + \underline{g})^* = \underline{f}^* + \underline{g}^*$ and $(\underline{f} \times \underline{g})^* = \underline{g}^* \times \underline{f}^*$ for the mapping (11) and product (4).

VEV that exhibit interaction include $n \geq 4$, n-argument connected functions. For example, the n-argument VEV

$$\langle \tilde{\Phi}(p_1) \dots \tilde{\Phi}(p_k)\Omega | \dots \tilde{\Phi}(p_n)\Omega \rangle = \dots + c_n \delta(p_1 + p_2 + \dots + p_n) \prod_{j=1}^{n} \delta(p_j^2 - m^2c^2/\hbar^2) \tag{17}$$

exhibits interaction in the construction with a single neutral scalar field $\Phi(x)$, section 3.3 and [32, 34]. In (17), $N_c = 1$, $n \geq 4$, $n-2 \geq k \geq 2$, $\tilde{\Phi}(p)$ designates the Fourier transform (22) of $\Phi(x)$, and the contributions of the fewer than n-point connected functions are understood. VEV are cluster expansions of connected functions, section 3.3.4. The last term in (17) is Poincaré invariant, connected, and supported only on mass m shells. However, the energy-momentum support includes both positive and negative energies. The selection (7) of basis function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ limits the states (2) in $\mathbf{H}_{\mathcal{P}}$ to positive energies. With the limitation of the support of constructed VEV to mass shells, state descriptions are limited to nonnegative energies by the zeros (7) on negative energy mass shells from functions in $\underline{\mathcal{P}}$: field component κ_j is associated with mass m_{κ_j}, section 3.4.

$$\delta(p_j^2 - m_{\kappa_j}^2 c^2/\hbar^2)\theta(-p_{j0})\tilde{f}_n((p)_n)_{(\kappa)_n} = 0 \tag{18}$$

if $f_n \in \underline{\mathcal{P}}$ and $1 \leq j \leq n$. This follows from (7) with $p_j^2 - \lambda_{cj}^{-1} = (p_{j0} - \omega_j)(p_{j0} + \omega_j)$, (8) and (9).

$$\left(\frac{\delta(p_{j0} - \omega_j) + \delta(p_{j0} + \omega_j)}{2\omega_j} \right) \theta(-p_{j0})(p_{j0} + \omega_j)\, \tilde{h}_n((p)_n)_{(\kappa)_n} = 0.$$

There is no negative energy contribution to the scalar product (6) from a state describing function (7). For positive energies,

$$\left(\frac{\delta(p_{j0} - \omega_j) + \delta(p_{j0} + \omega_j)}{2\omega_j} \right) \theta(p_{j0})(p_{j0} + \omega_j)\, \tilde{h}_n((p)_n)_{(\kappa)_n} = \tilde{h}_n((p)_{j-1}, (\omega_j, \mathbf{p}_j), (p)_{j+1,n})_{(\kappa)_n}$$

and $\tilde{h}_n((p)_n)_{(\kappa)_n}$ in (7) reduces to a function solely of the momenta, $\tilde{h}_n((\omega, \mathbf{p})_n)_{(\kappa)_n} \in \mathcal{S}(\mathbb{R}^{3n})$. Due to this limited support property (18) and the inclusion of $n \geq 4$, n-argument connected functions in the VEV (5), interpretation of the state describing functions $f_n((x)_n)_{(\kappa)_n}$ differs significantly from RQFT, section 3.1 and appendix 8.2.7. Physically nontrivial VEV (5) include n-argument connected functions with $n \geq 4$ and consequently states with different numbers and species of particles are generally not orthogonal. Since scalar products (6) of state describing functions with different numbers of arguments do not vanish, there are nonzero likelihoods of observing $k \neq n$ particles and distinct particle species for a state described by an n-argument function. But even with interaction, there are n-argument states well-represented as n classical particles. For example, n-argument functions $f_n((x)_n)_{(\kappa)_n}$ with widely space-like separated supports describe n particle states [32]. For the VEV (17) to be generalized functions and include massless particles, at least 3+1 dimensional spacetime is necessary, section 4.2.6.

The support constraint (18) for state describing functions of the form (7) also implies that there are no state describing functions with support strictly limited to bounded spatial volumes within $\mathbf{H}_{\mathcal{P}}$ [34]. Considered as functions over spacetime, state describing functions $f_n \in \mathcal{P}(\mathbb{R}^{4n})$ do not vanish within any finite spatial volume unless the function is identically zero: such functions are designated anti-local [55]. Nevertheless, comparing summations over equal finite volumes, anti-local functions include functions arbitrarily dominantly supported within finite spatial volumes: such functions are designated essentially localized here, appendix 8.17. The physically significant support of functions in $\underline{\mathcal{P}}$ may be local, section 5.2 and appendix 8.17.

Except for an unobservable[e] phase difference between forward and scattered contributions, plane wave scattering amplitudes resulting from VEV such as (17) coincide with first order terms from Feynman-Dyson series [32, 34] with, in examples with nonzero spin, corrections at very relativistic exchange momenta (small distances) [36]. For the neutral scalar field example (17), the scattering cross sections coincide with the first order contributions from $:P(\Phi)_4:$ interactions, appendix 8.9. The phase difference between (17) and the Feynman-Dyson series is necessary for nonnegativity of the scalar product (6). Connected functions (17) introduce interaction to the constructions, but preclude Hermitian field operators. Even though (17) satisfies formal Hermiticity, the limitation (18) on the support of the functions in $\underline{\mathcal{P}}$ eliminates $*$-involutivity of the algebra of function sequences.

Equivalently, the constructions may be considered to have the $*$-involutive basis function space $\underline{\mathcal{S}}$ with VEV modified from formally Hermitian forms such as (17), section 3.4.2. In this alternative development, the VEV (17) are modified to implement positive energies.

$$\begin{aligned}\langle\tilde{\Phi}(p_1)\dots\tilde{\Phi}(p_k)\Omega|\dots\tilde{\Phi}(p_n)\Omega\rangle = \dots + c_n\delta(p_1+p_2+\dots+p_n)\\ \times\prod_{j=1}^{k}\theta(-p_{j0})\delta(p_j^2-m^2c^2/\hbar^2)\prod_{\ell=k+1}^{n}\theta(E_\ell)\delta(p_\ell^2-m^2c^2/\hbar^2).\end{aligned} \tag{19}$$

The tempered functions $\underline{\mathcal{S}}$ include dense sets of real functions with bounded support. The development based on the tempered functions $\underline{\mathcal{S}}$ follows from the development based on support limited functions $\underline{\mathcal{P}}$ by an equivalence

$$\frac{\omega_j \pm p_{j0}}{2\omega_j} = \theta(\pm p_{j0}) \tag{20}$$

as multipliers [20] of generalized functions supported solely on mass shells. The alternative RQP constructions present the same puzzlement as the Reeh-Schlieder theorem in RQFT [46], section 3.4.2: although the labels on states include functions of localized support, the spatial support of quantum fields is global in relativistic quantum physics. The lack of formal Hermiticity of the VEV is evident in (19).

[e] Unobservable in the scattering limit, and RQFT methods do not provide estimates for finite interval transition likelihoods.

Realizations of relativistic quantum physics include: the VEV (5) that determine the scalar product (6); the basis space $\underline{\mathcal{P}}$ of test function (7) sequences; and a physical understanding of the state descriptions $|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$ (2).

3 Constructions of relativistic quantum mechanics

> In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization. – David Hilbert, in *Bulletin of the American Mathematical Society* 8 (1902), 437-479.

VEV (5) that realize relativistic quantum physics are constructed in this section. These VEV generalize the single neutral scalar field example (17) introduced in section 2 to multiple constituent fields. Function sequences $\underline{f}$ in basis function spaces $\underline{\mathcal{P}}$ that satisfy (7) describe nature and label elements of rigged Hilbert spaces $\mathbf{H}_{\mathcal{P}}$. The Hilbert space is the completion of basis function spaces $\underline{\mathcal{P}}$ with a scalar product that follows from the constructed VEV. Hilbert space realizations of relativistic quantum physics are discussed in section 3.1. The well-established VEV of relativistic free fields are discussed within the current context in section 3.2. The physically nontrivial VEV are constructed in section 3.3. Prospective axioms A.1-7 that state the principles of quantum mechanics and relativity are presented in section 4.1. Satisfaction of the axioms is demonstrated in sections 4.2.2-4.2.6. The basis spaces $\underline{\mathcal{P}}$ are discussed in section 3.4. Also in section 3.4, the constructions are shown to be equivalent to constructions based on the basis function space $\underline{\mathcal{S}}$ with positive energy support constrained, non-Hermitian VEV. The approach based upon support constrained functions $\underline{\mathcal{P}}$ is emphasized in these notes. These notes are also limited primarily to the technically simpler finite mass elementary particles. Zero mass elementary particles include additional considerations, [37] and section 4.2.6. The Hamiltonian is evaluated in section 4.4 and scattering likelihoods are illustrated in section 4.5. Additional properties of the constructions are discussed in sections 4.4, 5, and 6, and appendices.

3.1 Quantum mechanical description of state

Key considerations for realization of relativistic quantum physics include:

1. functions are the description of nature. State describing functions over spacetime and quantum numbers label elements of rigged Hilbert spaces, and the Fourier transforms of these functions describe momenta

2. the Hamiltonian generates time translation. To comply with relativity, the Hamiltonian must be a generator of a Hilbert space realization of the Poincaré group.

In the classical concept, observables are classical dynamical variables that describe geometric bodies. The classical concept includes an observer who can describe the evolution of each body, and each body is identifiable. The description of state is considered independently of observation, or perhaps more precisely, the state is considered determined and observation need not disturb that state. This geometric description with an omniscient observer is what is meant by classical physics in these notes. Quantum mechanics supersedes, not "quantizes," classical descriptions. The quantum description of nature is fundamentally incompatible with classical description despite the often excellent approximations to quantum dynamics provided by classical dynamics. A description of nature as elements of rigged Hilbert spaces implements the discrete line spectra of atomic emissions, the quantized energy of photons, the entanglement of states necessary to consistent description of quantized conserved quantities, the indistinguishability of similarly described bodies that provides an extensive entropy, particle creation and annihilation, and manifests the observed wave-particle duality. When the dominant supports of state describing functions are well-represented by a single location and momentum, and isolated, the description provided by the state describing function is classical particle-like. For these particular state descriptions, there is a close correspondence of quantum and classical descriptions, section 5. More generally, description is inherently quantum mechanical. For significantly overlapping descriptions, or when entanglement applies, or when non-commuting observables are considered, or on small spatial scales, or for relativistic collisions, classical description is contradicted. For finite mass particles, the scale for "small" is generally set by the Compton wavelength (9). Trajectories of distinguishable bodies do not apply in quantum mechanics except approximately as long as a correspondence with classical bodies applies. The likelihoods of perceptions of the location of natural objects are described by functions in a Hilbert space, and the likelihoods of perceptions of momentum are described by Fourier transforms of the state describing functions. Classical dynamics must approximate "macroscopic" quantum state descriptions, those perceived as classically described, but the approximation need not be arbitrarily precise and need not apply to any state. The quantum description of state precludes classical description; states of nature are never exactly classically described. The Heisenberg uncertainty principle follows from the quantum description of state and is characteristic of the quantum description of nature, appendix 8.6. There are no functions with a single point of support with Fourier transforms exhibiting a single point of support. The intrinsic understanding of quantum mechanics is the relative state (Everett-Wheeler-Graham) interpretation, [11] and appendix 8.2.8. The resulting understanding of observation explains how common perceptions of nature differ from its description, and resolves the Einstein-Podolosky-Rosen (EPR) [16], Schrödinger's cat [53] and Wigner's friend [67] paradoxes associated with earlier understandings of quantum mechanics. Quantum mechanics contradicts classical description and in this sense the concepts are disconcerting. But, quantum mechanics is necessitated to describe

nature. The classical description is a perception facilitated by often excellent approximation, and a concept developed over centuries of study. The correspondence of classical and quantum descriptions of nature is discussed further in section 5 and appendices.

The physically significant features of state describing functions follow from Lebesgue summations over measurable subsets of $\mathbb{R}^3$. Nature is described by equivalence classes of functions. Any correspondence of Lebesgue measurable subsets with classical dynamical variables $\mathbf{x}$ or $\mathbf{p}$, points in $\mathbb{R}^3$, is necessarily inexact. Classical dynamical variables are representatives for the support of appropriate state describing functions.

Introduced in Section 2, the elements $|\underline{f}\rangle$ of the rigged Hilbert spaces of interest $\mathbf{H}_{\mathcal{P}}$ are described by sequences of functions $\underline{f}$ from within completions of basis spaces $\underline{\mathcal{P}}$ [9, 10, 12, 58].

$$|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$$

is described by any representative state describing function sequence (1) from the equivalence class of $\underline{f}$, equivalent in the Hilbert space norm. Perception of these states and their temporal evolution are the concerns of mechanics. This Hilbert space description fulfills essential properties of nature, [13, 62] and appendix 8.2.

Due to the $n \geq 4$, n-argument connected VEV in the physically nontrivial constructions, functions $f_n((x)_n)_{(\kappa)_n}$ and $f_k((x)_k)_{(\kappa)_k}$ with $n \neq k$ are generally not orthogonal. Then, from Born's rule, there is a finite likelihood that the state labeled by an $f_n((x)_n)_{(\kappa)_n}$ is perceived as a state described by an $f_k((x)_k)_{(\kappa)_k}$: particle numbers are generally not conserved. Nevertheless, $f_n((x)_n)_{(\kappa)_n}$ and $f_k((x)_k)_{(\kappa)_k}$ become orthogonal as their dominant supports are widely spacelike separated. In this event and due to connectivity of the VEV, only the free field VEV contribute to the scalar product and the scalar product vanishes unless $n = k$.

The labels on arguments generally do not identify particular elementary particles. Gibb's paradox is resolved by the indistinguishability of similarly described particles. For arguments x_k, κ_k of state describing functions $f_n((x)_n)_{(\kappa)_n}$ labeled by the same mass, charge and polarization, if the spatial supports of these similarly labeled arguments overlap, then the results of observation are not reliably associated with the support of any single argument. Isolation of spatial support enables association of a volume of space with the state description provided by a particular argument: the volume of space reliably associates with the isolated support of an argument and this support associates with perceptions interpreted as properties of a classically described particle. The evolution of a perceived object is described as a trajectory and this description persists until significant support overlap occurs. The association of classical particles with state describing functions is developed further in section 5. Indistinguishability is implemented as split signed symmetry of the VEV with interchange of argument labels, section 3.3.2.

To describe momentum, functions in the basis function spaces $\underline{\mathcal{P}}$ are required to have Fourier transforms. $\tilde{f}_n((p)_n)_{(\kappa)_n}$ denotes the Fourier transform of $f_n((x)_n)_{(\kappa)_n}$. The Fourier transform adopted here applies in four-dimensional spacetime and is the evident multiple argument ex-

tension of

$$\tilde{\psi}(p) = \int \frac{dx}{(2\pi)^2}\, e^{-ipx}\psi(x) \tag{21}$$

using wave number p, and the Lorentz invariants $px = p_0ct - \mathbf{p}\cdot\mathbf{x}$ and four dimensional spacetime volume element $dx = dx_0dx_1dx_2dx_3$.[f] Mass m is in natural units and a length associated with a mass m is the reduced Compton wavelength λ_c from (9). The Fourier transforms of generalized functions are defined [19] to satisfy Parseval's equality

$$\tilde{T}(\tilde{\psi}) = T(\psi). \tag{22}$$

As a consequence and when applicable, the Fourier transforms of generalized functions are

$$\tilde{T}(p) = \int \frac{dx}{(2\pi)^2}\, e^{ipx}T(x)$$

with the sign reversal relative to (21). The Fourier transform is invertible [49].

$$\psi(x) = \int \frac{dp}{(2\pi)^2}\, e^{ipx}\tilde{\psi}(p).$$

Among the properties of Fourier transforms are the Fourier transform pairs

$$\begin{array}{rcl} \psi(\Lambda(x-a)) & \leftrightarrow & e^{-ipa}\widetilde{\psi}(\Lambda p) \\ \dfrac{d\psi(x)}{dx} & \leftrightarrow & igp\widetilde{\psi}(p) \\ e^{-\alpha\|x\|^2} & \leftrightarrow & \dfrac{1}{4\alpha^2}e^{-\|p\|^2/(4\alpha)} \end{array} \tag{23}$$

with $\|x\|^2$ the Euclidean (sum of squares) length squared of the Lorentz 4-vector x, Λ a Lorentz transformation, a a spacetime translation, and α a complex parameter with $\Re e(\alpha) > 0$. g is the Minkowski signature matrix,

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{24}$$

and then $gp = (p_0, -\mathbf{p})$. The third property in (23) illustrates that functions $\psi(x)$ with broad spacetime support have Fourier transforms with concentrated energy-momentum support, and

[f] Spacetime vectors $x = (x_0, \mathbf{x}) \in \mathbb{R}^4$ are lengths with $x_0 = ct$ and spatial vectors $\mathbf{x} = x, y, z \in \mathbb{R}^3$. c is the speed of light and t is time. Energy-momenta have the units of inverse length and are wave numbers designated $p = (p_0, \mathbf{p}) \in \mathbb{R}^4$ with energies $E = \hbar cp_0$ and momentum vectors $\hbar\mathbf{p} \in \mathbb{R}^3$, section 3.1. $\hbar$ is Planck's constant h divided by 2π. The three Cartesian components of momentum vectors $\mathbf{p}$ are designated p_ν, $\nu = x, y, z$.

vice versa. Indeed, in general, the standard errors of $|\psi(x)|^2$ in spacetime and of $|\tilde{\psi}(p)|^2$ in energy-momenta considered as probability distributions satisfy $\sigma_x\sigma_p \geq \frac{1}{2}$ in each dimension. $\sigma_x\sigma_p \geq \frac{1}{2}$ illustrates the Heisenberg uncertainty principle discussed in appendix 8.6. More precise knowledge of the location of a body implies degraded knowledge of the time rate of change of the location, and this effect is more pronounced for low mass bodies than for heavy bodies due to $\hbar\mathbf{p} \approx m\mathbf{v}$ at non-relativistic velocities $\mathbf{v}$. Then

$$\sigma_x\sigma_v \geq \frac{\hbar}{2m}$$

in each of the three dimensions.

3.1.1 The scalar product

The Hilbert spaces of interest have a complex scalar product,

$$\langle \underline{f}|\underline{g}\rangle \in \mathbb{C},$$

for every pair of elements $|\underline{f}\rangle, |\underline{g}\rangle \in \mathbf{H}_{\mathcal{P}}$ described by state describing function sequences $\underline{f}, \underline{g}$. The scalar product provides the norm,

$$\|\underline{f}\| = \sqrt{\langle \underline{f}|\underline{f}\rangle}. \tag{25}$$

To achieve Poincaré invariance of likelihoods and state descriptions supported solely on nonnegative energies, a degenerate scalar product of function sequences $\underline{f}, \underline{g}$ is implemented with generalized functions, the VEV functions[g]

$$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \in \mathcal{S}'(\mathbb{R}^{4n})$$

[9, 20, 58]. The degenerate scalar product is

$$\begin{aligned}\mathcal{W}(\underline{f}^*, \underline{g}) &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \mathcal{W}_{n,m}(f^*_{n,(\kappa)_n}\, g_{m,(\kappa)_{n+1,n+m}})_{(\kappa)_{n+m}} \\ &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\ (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}\, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned}$$

from (13) with formal summation notation for generalized functions [19] in the last line. Each spacetime Lorentz vector x_k is summed over $\mathbb{R}^4$ and each $\kappa_j \in \mathbb{N}$ is summed from 1 to N_c. The indices n and m are summed over the nonnegative integers $\mathbb{N}_0$. These VEV functions

[g] From theorems due to Wightman and Wizimirski, the VEV of fields must be generalized functions [9].

$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ generalize the Wightman functions $W_n((x)_n)_{(\kappa)_n}$ [9, 10, 58, 64]. Multiple arguments are denoted

$$(x)_{j,k} = x_j, x_{j+1}, \dots x_k \in \mathbb{R}^{4(k-j+1)}$$

in the ascending case, $(x)_{j,k} = x_j, x_{j-1}, \dots x_k$ otherwise and $(x)_n = (x)_{1,n}$. The $*$-dual $\underline{f}^*$ of a function sequence $\underline{f}$ uses complex conjugation, argument transpositions, and the nonsingular $N_c \times N_c$ linear transformation D from (11). The $*$-dual functions are

$$f_n^*((x)_n)_{(\kappa)_n} = (D^T\cdot)_n\overline{f_n}(x_n, \dots x_1)_{\kappa_n \dots \kappa_1}$$

using the matrix notation (12) and then the Fourier transform[h] of $f_n^*((x)_n)_{(\kappa)_n}$ is (11). D, designated here as Dirac conjugation, is determined by representation of the Lorentz group and satisfies

$$\overline{D}D = \mathbb{I}_{N_c} \tag{26}$$

with $\mathbb{I}_{N_c}$ the $N_c \times N_c$ identity and as a consequence, the $*$-dual satisfies

$$\begin{aligned} \underline{f}^{**} &= \underline{f} \\ (\underline{f} + \underline{g})^* &= \underline{f}^* + \underline{g}^* \\ (\underline{g} \times \underline{f})^* &= \underline{f}^* \times \underline{g}^*. \end{aligned}$$

The linear space of function sequences $\underline{\mathcal{P}}$ becomes an algebra with the $\times$-product (4). The $*$-dual would be an involution of the algebra of function sequences if it were an automorphism. The $*$-dual maps $\underline{\mathcal{S}} \mapsto \underline{\mathcal{S}}$ but, the $*$-dual is not an automorphism for the energy support constrained functions $\underline{\mathcal{P}}$: $\underline{\mathcal{P}}^* \neq \underline{\mathcal{P}}$ due to the nonnegative energy constraint (7). From the representation of the $*$-dual (11), $\underline{\mathcal{P}} \cup \underline{\mathcal{P}}^* \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{c\Omega\}$ with $c \in \mathbb{R}$ and Ω the vacuum. The $*$-dual maps zeros on negative energy mass shells to positive energy mass shells, sections 2 and 3.4.1.

A sequence of generalized functions is denoted

$$\underline{\mathcal{W}} = (\mathcal{W}_0, \mathcal{W}_{1,0}, \mathcal{W}_{0,1}, \dots, \mathcal{W}_{n,0}, \mathcal{W}_{n-1,1} \dots, \mathcal{W}_{0,n}, \dots). \tag{27}$$

The arguments of $\mathcal{W}_{k,n-k}$ are $(x)_n, (\kappa)_n$ and similarly to (1), there are $(N_c)^n$ functions in the subsequence $\mathcal{W}_{k,n-k}$ distinguished by $(\kappa)_n$. From the scalar product (13), the n spacetime arguments of $\mathcal{W}_{k,n-k}$ are designated as k $*$-dual arguments $x_1, \dots x_k$ and $n-k$ function arguments $x_{k+1}, \dots x_n$. The VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ in the scalar product (13) follow from the VEV of fields (6) in (14).

The scalar product in $\mathbf{H}_{\mathcal{P}}$ results from the isometry

$$\langle \underline{f} | \underline{g} \rangle = \mathcal{W}(\underline{f}^*, \underline{g})$$

[h] $\int dx\ e^{-ipx}\overline{\psi(x)} = \overline{\int dx\ e^{ipx}\psi(x)} = \overline{\tilde{\psi}(-p)}$.

that associates elements of the Hilbert space with equivalence classes of function sequences [12]. Then the norm (25) is

$$\|\underline{f}\| = (\mathcal{W}(\underline{f}^*, \underline{f}))^{1/2}.$$

An abbreviated notation $\langle f_n | g_m \rangle$ is used for a scalar product when the constituent j-argument functions $f_j = 0$ for all $j \neq n$ and k-argument functions $g_k = 0$ for all $k \neq m$ in function sequences $\underline{f}$ and $\underline{g}$.

$$\begin{aligned} \langle f_n | g_m \rangle \quad = \quad & \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ & \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1} \, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned} \tag{28}$$

The alternative developments with state describing functions $\underline{f}$ derived from either $\underline{\mathcal{P}}$ or $\underline{\mathcal{S}}$, sections 2 and 3.4, illustrate an ambiguity in relativistic physics. With the scalar products necessary for relativistic physics (13), an assignment of multiplier functions to the VEV functions $\underline{\mathcal{W}}$ or state describing functions $\underline{f}$ is ambiguous. If eigenvectors $\underline{e}_q$ are determined given the elevated operator Q for a classical dynamical variable q, then, state transition likelihoods $|\langle \underline{f} | \hat{\underline{e}}_g \rangle|^2$ generally vary with the assignment of multiplier functions to $\underline{\mathcal{W}}$.

3.1.2 The quantum field and VEV

The quantum field (3) is a Hilbert space operator in the Hilbert space with scalar product (6) and elements labeled by function sequences $\underline{f}$ in the completion of $\underline{\mathcal{P}}$.

The quantum field is multiplication (4) in the algebra of function sequences. Borchers' [10] definition is

$$\Phi(\underline{f}) \, \underline{g} = \underline{f} \times \underline{g} \tag{29}$$

but in this note, the discussion of fields $\Phi(\underline{f})$ includes only sequences $\underline{f}$ that consist of single argument functions $f_1(x)_\kappa$. For the convention in these notes, if f_k is nonzero and $f_k \in \underline{f}$ in $\Phi(\underline{f})$, then $k = 1$. The Schwartz kernel theorem [21] provides the more general definition. With this convention, the quantum field consists of N_c constituent fields $\Phi(x)_\kappa$

$$\begin{aligned} \Phi(\underline{f}) \quad &= \sum_{\kappa=1}^{N_c} \Phi(f_\kappa)_\kappa \\ &= \sum_{\kappa=1}^{N_c} \int dx \; \Phi(x)_\kappa f_1(x)_\kappa. \end{aligned}$$

with the constituent field $\Phi(x)_\kappa$ associated with mass m_κ and formal summation notation [19] for the operator-valued distributions [9].

For the constructions, the multiplication (4) preserves Hilbert space norm-equivalence classes of function sequences, and the definition (29) of field as multiplication in the algebra of function sequences elevates to Hilbert space operators. Demonstrated in section 4.4, the constructed quantum fields are unbounded Hilbert space operators. If physically nontrivial, the constructed quantum fields are not Hermitian, section 2. For physically trivial free fields, Wick monomials and generalized free fields, the basis function space extends to $\underline{\mathcal{S}}$ and Hermitian field operators are realized.

There are no eigenfunctions of the field $\Phi(\underline{f})$ that elevate a classical field $A(\underline{f})$. If $\underline{e}$ labels an eigenfunction sequence of the field, then for a sequence of single argument functions $\underline{f}$,

$$|\Phi(\underline{f})\underline{e}\rangle = A(\underline{f})|\underline{e}\rangle$$

with $A(\underline{f}) = \sum_\kappa \int dx\ A(x)_\kappa f(x)_\kappa$. The $A(x)_\kappa$ are solutions to the corresponding classical field equations. For fields with a zero expectation value,

$$\begin{aligned}\Phi(\underline{f})\,\underline{e} &= \underline{f} \times \underline{e} \\ &= (0, \underline{f}_1, 0, \ldots) \times (\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3 \ldots) \\ &= (0, \underline{f}_1 \times \underline{e}_0, \underline{f}_1 \times \underline{e}_1, \underline{f}_1 \times \underline{e}_2 \ldots) \\ &= (A(\underline{f})\underline{e}_0, A(\underline{f})\underline{e}_1, A(\underline{f})\underline{e}_2, A(\underline{f})\underline{e}_3 \ldots)\end{aligned}$$

from the $\times$-product (4) and the definition of field (29) with $\underline{e}_n$ the sequence consisting of only n-argument functions. If $\underline{e}$ describes an eigenfunction of $\Phi(\underline{f})$ with a finite eigenvalue $A(\underline{f})$, then $A(\underline{f})\underline{e} - \underline{f} \times \underline{e}$ is a sequence that labels an element in the null space of $\mathbf{H}_\mathcal{P}$. Selection of the sequence of zeroes to represent $A(\underline{f})\underline{e} - \underline{f} \times \underline{e}$ results in the recursive $A(\underline{f})\underline{e}_{n+1} = \underline{f}_1 \times \underline{e}_n$. Then $A(\underline{f})\underline{e}_0 = 0$ with $A(\underline{f}) \neq 0$ provides that $\underline{e} = 0$.

3.1.3 The adjoint of the field and Hermiticity

The canonical quantization of a real classical field would be Hermitian. A definition of Hermiticity suitable for complex and multiple constituent fields is to denote a field $\Phi(x)_\kappa$ as densely defined Hermitian if the composite fields (29)

$$\Phi(\underline{f}) = \sum_{\kappa=1}^{N_c} \Phi(f_\kappa)_\kappa$$

satisfy

$$\langle \underline{h}|\Phi(\underline{f})^* \underline{g}\rangle = \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle$$

for real function sequences $\underline{f} = \underline{f}^*$ and with

$$\langle \Phi(\underline{f})^* \underline{h}|\underline{g}\rangle = \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle$$

on a dense domain $|\underline{h}\rangle, |\underline{g}\rangle \in \mathcal{D}_\Phi \subset \mathbf{H}_\mathcal{P}$.

Real function sequences are dense in $*$-involutive spaces of function sequences such as the completion $\mathbf{H}_\mathcal{S}$ of $\underline{\mathcal{S}}$. The $*$-dual (11) function sequences $\underline{f}^*$ include an $N_c \times N_c$ linear transformation D from a representation of the Lorentz group, section 3.2.2. With complex coefficients, real functions are dense in $\underline{\mathcal{S}}$: every $\underline{h} \in \underline{\mathcal{S}}$ decomposes as

$$\underline{h} = \underline{h}_1 + i\underline{h}_2$$

for real $\underline{h}_1, \underline{h}_2 \in \underline{\mathcal{S}}$.

$$\begin{aligned} 2\underline{h}_1 &= \underline{h} + \underline{h}^* \\ 2\underline{h}_2 &= -i(\underline{h} - \underline{h}^*). \end{aligned}$$

For VEV and function sequences that satisfy Wightman's axioms, the field is densely defined and Hermitian. However, for the support limited non-$*$-involutive basis space $\underline{\mathcal{P}}$, there are no nonzero real function sequences. Every element $\underline{g} \in \underline{\mathcal{P}}$ has an adjoint $\underline{g}^* \in \underline{\mathcal{P}}^* \subset \mathcal{S}$ defined by (11), but as a result of the support constraint (7), $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{(c, 0, 0 \ldots)\}$ with $c \in \mathbb{R}$.

Hermitian quantum fields (29) follow if the basis space of function sequences is $*$-involutive and the VEV (5) satisfy formal Hermiticity (15). For a densely defined Hermitian field,

$$\begin{aligned} \langle \Phi(\underline{f})\underline{h}|\underline{g}\rangle &= \langle \underline{h}|\Phi(\underline{f})^*\underline{g}\rangle \\ &= \langle \underline{h}|\Phi(\underline{f})\underline{g}\rangle \end{aligned}$$

with $\underline{f}, \underline{g}$ labeling elements from a common, dense domain $\mathcal{D}_\Phi$. For VEV (5) that satisfy formal Hermiticity (15),

$$\begin{aligned} \langle \Phi(\underline{f})\underline{h}|\underline{g}\rangle &= \mathcal{W}((\underline{f} \times \underline{h})^*, \underline{g}\rangle \\ &= \mathcal{W}(\underline{h}^*, \underline{f} \times \underline{g}) \\ &= \mathcal{W}(\underline{h}^* \times \underline{f}^*, \underline{g}\rangle \end{aligned}$$

from the properties of the $*$-dual (11). For the of function sequences $\underline{h}_{n-1}, \underline{g}_m$ with only the functions with designated number of arguments not zero, if $\Phi(\underline{f})$ is Hermitian,

$$\begin{aligned} 0 &= \mathcal{W}(\underline{h}^*, \underline{f} \times \underline{g}) - \mathcal{W}(\underline{h}^* \times \underline{f}^*, \underline{g}\rangle \\ &= \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\, \overline{h_{n-1}}(x_{n-1}, \ldots x_1)_{\kappa_{n-1}\ldots\kappa_1}\, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}} \\ &\qquad \times (D\cdot)_n \left(\mathcal{W}_{n-1,m+1}((x)_{n+m})_{(\kappa)_{n+m}} f_1(x_n)_{\kappa_n} - \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} (D\overline{f_1}(x_n))_{\kappa_n} \right). \end{aligned}$$

This applies for any selection of $|\underline{f}_1\rangle \in \mathbf{H}_\mathcal{P}$ and all $|\underline{h}_{n-1}\rangle, |\underline{g}_m\rangle$ in the common, dense domain $\mathcal{D}_\Phi$. As a consequence,

$$\mathcal{W}_{n-1,m+1}((x)_{n+m})_{(\kappa)_{n+m}}\, f_1(x_n)_{\kappa_n} = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\, (D\overline{f_1}(x_n))_{\kappa_n} \tag{30}$$

if the field is densely defined Hermitian. If the Hilbert space includes a dense set of real functions $\underline{f} = \underline{f}^*$, then an

$$f_1(x_n)_{\kappa_n} = (D\overline{f_1}(x_n))_{\kappa_n}$$

is included and (30) is satisfied for formally Hermitian VEV (16). For formally Hermitian VEV and if the basis space of function sequences is $*$-involutive, then Hermitian fields follow. More generally, there are no nontrivial solutions to (30), and field operators are not densely defined Hermitian.

For VEV that satisfy formal Hermiticity (15) and function sequences that are $*$-involutive, the adjoint of the field is

$$\Phi(\underline{f})^* = \Phi(\underline{f}^*)$$

with $\underline{f}^*$ from (11). Indeed, in Wightman's development [9, 58], the definition of an adjoint results in the identification

$$\begin{aligned} \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle &= \underline{W}(\underline{h}^* \times \underline{f} \times \underline{g} \rangle \\ &= \underline{W}((\underline{f}^* \times \underline{h})^* \times \underline{g} \rangle \\ &= \langle \Phi(\underline{f}^*) \underline{h} | \underline{g} \rangle \\ &= \langle \Phi(\underline{f})^* \underline{h} | \underline{g} \rangle. \end{aligned} \tag{31}$$

This follows for the scalar product (15), the quantum field (29), the product of function sequences (4), the properties of the $*$-dual of sequences (11), and the definition of adjoint. However, for the physically nontrivial constructions, either the VEV do not satisfy formal Hermiticity, or spectral support A.4 is not satisfied without the support limitation (7) on the basis function space $\underline{\mathcal{P}}$. With the support limitation, $\underline{f}^* \notin \underline{\mathcal{P}}$ if $\underline{f} \in \underline{\mathcal{P}}$ and the basis function space is not $*$-involutive.

3.1.4 Likelihoods: Born's rule

Physical understanding follows from Born's rule that determines the likelihoods of perceptions of the state describing functions. Born's rule provides that the likelihood of observing the state described by $|g\rangle$ for a state initially described by $|\psi\rangle$ is the squared magnitude of the scalar product,

$$\text{likelihood} = |\langle g | \psi \rangle|^2 \leq 1$$

for normalized state descriptions, $\|g\| = \|\psi\| = 1$. Born's rule requires no additional assumptions for the forms or properties of operators to evaluate likelihoods: a scalar product and the description (1) of states $|\underline{f}\rangle$ are inherent to the separable Hilbert space.

A separable Hilbert space has a denumerable basis of orthonormal elements that can be designated $|\underline{e}_n\rangle$ [3, 41]. These elements provide a resolution of the identity operator in $\mathbf{H}_{\mathcal{P}}$,

$$\mathbb{I} = \sum_n Q_n$$

with projection operators

$$Q_n = |\underline{e}_n\rangle\langle\underline{e}_n|$$

in bra-ket notation. Then the expansion of any state in this selected basis is

$$|\underline{f}\rangle = \sum_n \langle\underline{e}_n|\underline{f}\rangle \; |\underline{e}_n\rangle.$$

The projection operators Q_n are the elementary propositions [6, 62] associated with observation. For normalized states, $\|\underline{f}\| = 1$, the squared magnitudes $|\langle\underline{e}_n|\underline{f}\rangle|^2$ are the likelihoods that proposition n is answered affirmatively. The elementary propositions are the queries "will the state described by $\underline{f}$ be perceived as the state described by $\underline{e}_n$." This identification of likelihoods is Born's rule.

3.2 Relativistic free fields

3.2.1 Free field VEV

The well-known free field VEV satisfy both the original Wightman axioms [9, 10, 58, 64] and the axioms A.1-7. Free field VEV functions describe elementary particles and are included in the constructions. Also, the $n \geq 4$, n-argument connected VEV functions that characterize interactions are based on structures from the free field VEV functions. The physically trivial free field VEV satisfy formal Hermiticity W.a and involutivity W.b.

Satisfaction of the Wightman axioms suffices to determine that the neutral scalar field two-point function $W_2(x_1, x_2)$ is a Källén-Lehmann form, a summation over a mass spectrum of the Pauli-Jordan function [56]. The two-point functions used in the constructions are extensions of the Pauli-Jordan function to include nonzero spins. The two-point functions used in the constructions have Fourier transforms

$$\begin{aligned} \tilde{W}_2(p_1, p_2)_{\kappa_1\kappa_2} &= \delta(p_1 + p_2)\; \delta_2^+ \, M(p_2)_{\kappa_1\kappa_2} \\ &= \delta(\mathbf{p}_1 + \mathbf{p}_2)\; \delta_1^- \delta_2^+ \, 2\sqrt{\omega_1\omega_2}\, M(p_2)_{\kappa_1\kappa_2} \end{aligned} \tag{32}$$

with

$$\delta_k^\pm = \theta(\pm E_k)\delta(p_k^2 - \lambda_{ck}^{-2})$$

and λ_{ck} is the reduced Compton wavelength (9) for mass m_{κ_k}. These wavelengths λ_{ck} are functions of κ_k and the $\delta_k^\pm$ are functions of both p_k and κ_k. The supports of $\delta_k^\pm$ are on the positive $(+)$ or negative $(-)$ energy mass shells. A mass m_κ is associated with each constituent field $\Phi(x)_\kappa$. $M(p_2)_{\kappa_1\kappa_2}$ is an $N_c \times N_c$ array of multinomials in the components of the energy-momentum p_2.

The array $M(p)_{\kappa_1\kappa_2}$ in the two-point function (32) is used to construct the n-point connected functions. The elements of the $N_c \times N_c$ array $M(p)_{\kappa_1\kappa_2}$ are multinomials in the energy-momentum components [9] and in matrix notation, the array satisfies

$$DM(p) = C^*(p)C(p) \tag{33}$$

with D the Dirac conjugation matrix from (11) and $C^*(p)$ is the complex conjugate transpose of $C(p)$. The matrix nonnegativity [26] of $DM(p)$ provides the degenerate scalar product (13) [38]. Since $DM(p)$ is a nonnegative matrix, it is Hermitian and it follows from the condition (26) for D that

$$M(p)^* = DM(p)D^T. \tag{34}$$

Local commutativity A.5 is satisfied if the $M(p)$ are a direct sum decomposition into two components,

$$M(p) = \begin{pmatrix} B(p) & 0 \\ 0 & F(p) \end{pmatrix}, \tag{35}$$

that satisfy locality conditions

$$B(p)^T = B(-p), \quad \text{and} \quad F(p)^T = -F(-p). \tag{36}$$

The + sign applies for the boson constituent $B(p)$ and the − sign applies for the fermion constituent $F(p)$. (35) assigns a type, boson or fermion, to the value of each index $\kappa_j \in \{1, N_c\}$ for each j. From (35), the array $M(p)$ and consequently W_2 vanishes unless both indices κ_1, κ_2 are the same type, boson or fermion. Poincaré invariance of the scalar product is implied by two additional conditions on $M(p)$ and D.

$$\begin{aligned} S(A)M(p)S(A)^T &= M(\Lambda^{-1}p) \\ \overline{S(A)}D &= DS(A) \end{aligned} \tag{37}$$

with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz group (72). $A \in \mathrm{SL}(2, \mathbb{C})$, the group of determinant one, complex matrices. Poincaré invariance is demonstrated in section 4.2.4. Example $M(p)$, D and $S(A)$ that satisfy (26), (33), (35), (36) and (37) are illustrated in section 3.2.2.

W_2 exhibits local commutativity. Conditions (35) and (36) imply commutation or anti-commutation of free constituent fields. The two-point function (32), the Fourier transform convention (21), and translation invariance of the VEV express the local commutativity condition for free fields as

$$\begin{aligned} \langle\Omega|\left(\Phi_o(x)_{\kappa_1}\Phi_o(0)_{\kappa_2} \mp \Phi_o(0)_{\kappa_2}\Phi_o(x)_{\kappa_1}\right)\Omega\rangle &= \int dp\, \delta^+(p)\left(e^{-ipx}M(p)_{\kappa_1\kappa_2} \mp e^{ipx}M(p)_{\kappa_2\kappa_1}\right) \\ &= \int dp\, e^{-ipx}\left(\delta^+(p)M(p)_{\kappa_1\kappa_2} \mp \delta^-(p)M(-p)_{\kappa_2\kappa_1}\right) \\ &= \int dp\, e^{-ipx}M(p)_{\kappa_1\kappa_2}\left(\delta^+(p) - \delta^-(p)\right) \\ &= M(ig\tfrac{d}{dx})_{\kappa_1\kappa_2}\int dp\, e^{-ipx}\left(\delta^+(p) - \delta^-(p)\right) \end{aligned}$$

with $x = x_2 - x_1$ and $p = p_2$, after reflection of the summation variable $p \mapsto -p$ in the second term, substitution of (36), and the properties (23) of the Fourier transform applied to

the elements of the array $M(p)$ that are multinomials in the components of p. In $M(p)$, the components of $p = p_0, p_1, p_2, p_3 \in \mathbb{R}^4$ are replaced by

$$p_\nu \mapsto i g_{\nu\nu} \frac{\partial}{\partial x_\nu}$$

with g the Minkowski signature matrix (24). $\delta^{\pm}(p_k) = \delta_k^{\pm}$ from (32). If $x^2 < 0$, then there is a Lorentz transformation Λ with $\Lambda x = -x$ and as a consequence, the Pauli-Jordan commutator function vanishes for space-like x. It then follows that the commutator/anti-commutator of free constituent fields generally vanishes for space-like x. This local commutativity is satisfied for functions from $\underline{\mathcal{S}}$. $\underline{\mathcal{S}}$ includes functions with bounded support. $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}}$ includes no functions of bounded support.

If $f(x) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)_\kappa$, then the state described by $f(x)$ is supported only on positive energies for energy-momenta on the m_κ mass shell, and $\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa$ and $\Phi_o(f^*)_\kappa = \Phi_o^-(f^*)_\kappa$. For the two-point function (32) and scalar product (13), and because functions that label elements of $\mathbf{H}_{\mathcal{P}}$ lack support on the negative energy mass shells,

$$W_2(f^*\, g^*)_{\kappa_1\kappa_2} = W_2(f\, g)_{\kappa_1\kappa_2} = W_2(f\, g^*)_{\kappa_1\kappa_2} = 0 \tag{38}$$

and only $W_2(f^*\, g)_{\kappa_1\kappa_2}$ contributes if $f \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)_{\kappa_1}$ and $g \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)_{\kappa_2}$.

The free field VEV follow from the two-point VEV function (32), section 3.3.4 and appendix 8.7. Free field VEV result in scattering likelihoods with particle number conserved, and incoming and outgoing momenta equal in pairs, that is, no interaction. For the basis function space $\underline{\mathcal{S}}$, the free fields $\Phi_o(\underline{f})$ (29) are Hermitian Hilbert space operators that realize the Wightman and Gårding-Wightman axioms [9, 10, 58, 64]. For the basis function spaces $\underline{\mathcal{P}}$, the free fields $\Phi_o(\underline{f})$ are not Hermitian operators but an available extension of the basis function spaces from $\underline{\mathcal{P}}$ to $\underline{\mathcal{S}}$ achieves Hermitian free field operators [32]. Hermitian Hilbert space field operators appear to be peculiar to physically trivial realization. Physically trivial relativistic realizations include generalized free and Wick polynomial (monomial) fields [9, 58].

3.2.2 Example $M(p)$, D and $S(A)$

Realizations of free fields are available in [7, 9, 22, 23, 24, 54, 63]. These realizations provide Dirac conjugation D and two-point function arrays $M(p)$ with a realization of the Lorentz group $S(A)$, $A \in \mathrm{SL}(2, \mathbb{C})$. These $N_c \times N_c$ matrices $D, M(p), S(A)$ satisfy (26), (33), (35), (36), (37)

and the Lorentz group is realized (74):

$$\begin{aligned} \overline{D}D &= \mathbb{I}_{N_c} \\ DM(p) &= C^*(p)C(p) \\ M(\Lambda^{-1}p) &= S(A)M(p)S(A)^T \\ DS(A) &= \overline{S(A)}D \\ M(p) &= B(p) \oplus F(p) \\ B(p)^T &= B(-p) \\ F(p)^T &= -F(-p). \end{aligned} \tag{39}$$

with an $N_c \times N_c$ array $C(p)$ and $S(A)$ a representation of the Lorentz group (74). $N_c = N_B + N_F$ with $B(p)$ an $N_B \times N_B$ array of functions that describe bosons and $F(p)$ an $N_F \times N_F$ array of functions that describe fermions.

The conditions (39) are linear in $M(p)$ and therefore any nonnegative real multiple $|c|^2 M(p)$ also satisfies (39).

For a neutral scalar field, $N_c = 1$, the scalar representation of the Lorentz group applies and

$$M(p) = S(A) = D = 1.$$

A complex-valued (charged) scalar field has two constituent fields, $N_c = 2$.

$$\begin{aligned} M(p) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M(-p)^T \\ D &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S(A) &= \mathbb{I}_2 \end{aligned}$$

the 2×2 identity.

The charged scalar field has a symmetry associated with a charge, $S_\phi M(p) S_\phi^T = M(p)$ with

$$S_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$. The transformations S_ϕ commute with the Lorentz transformations $S(A)$ and the generators of translations,

$$[S_\phi, S(A)] = [S_\phi, \omega] = [S_\phi, \mathbf{p}] = 0$$

with ω from (8). S_ϕ introduces a symmetry in addition to Poincaré symmetry. The transformations S_ϕ leave two subspaces of one-argument state describing functions invariant,

$$S_\phi \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix} = e^{i\phi} \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix}, \text{ and } \quad S_\phi \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix}.$$

Invariance up to a phase under S_ϕ is physical equivalence and these functions are eigenfunctions of the Hermitian generator of S_ϕ. The commutation of S_ϕ with the generators of Poincaré transformations implies that labels assigned to state describing functions $f(x_1)_1, 0$ and $0, f(x_2)_2$ apply for all observers. The label is determined by classical correspondence and not axioms A.1-7. Labeling the two subspaces of states with the eigenvalues of the generator for S_ϕ, as charge 1 or -1, a charge operator

$$Q = -i \, \frac{d}{d\phi} S_\phi \Big|_{\phi=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{40}$$

results in

$$Q \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix} = \begin{pmatrix} f(x)_1 \\ 0 \end{pmatrix}, \text{ and } \quad Q \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix} = - \begin{pmatrix} 0 \\ f(x)_2 \end{pmatrix}.$$

If the spatial supports of each argument of the state describing functions are widely space-like separated and sufficiently localized at one time, a classical particle-like interpretation applies and the extension of the charge operator to the n-argument subspaces

$$Q f((x)_n)_{(\kappa)_n} = \sum_{j=1}^{n} (\delta_{1,\kappa_j} - \delta_{2,\kappa_j}) f((x)_n)_{(\kappa)_n}$$

results in a summation of the charges in the state described by $f((x)_n)_{(\kappa)_n}$. $\delta_{j,\ell}$ is the Kronecker delta. The extended Q commutes with the Hamiltonian $\omega_1 + \ldots \omega_n$ (103) in the n-particle subspaces and as a consequence, charge is conserved over time.

A massive, neutral vector boson field is realized with four constituent fields, $N_c = 4$. This spin-1 boson field uses a real 4×4 representation of the Lorentz group.

$$\begin{aligned} M(p)_{jk} &= \lambda_c^2 \, p_j p_k - g_{jk} \\ &= C^*(p) C(p) \\ D &= \mathbb{I}_4 \\ S(A) &= \Lambda^{-1} \end{aligned}$$

with the components of p denoted p_0, p_1, p_2, p_3 and $p^2 = \lambda_c^{-2}$. g is the 4×4 Minkowski signature matrix and Λ is a Lorentz transformation determined by $A \in \mathrm{SL}(2, \mathbb{C})$ (73). $M(p) = M(-p)^T$.

In four dimensions with the components of p denoted p_0, p_x, p_y, p_z,

$$C^*(p) = \begin{pmatrix} \lambda_c\sqrt{p_0^2 - \lambda_c^{-2}} & 0 & 0 & 0 \\ \lambda_c p_0 \frac{p_x}{\sqrt{p_0^2-\lambda_c^{-2}}} & \sqrt{\frac{p_y^2+p_z^2}{p_0^2-\lambda_c^{-2}}} & 0 & 0 \\ \lambda_c p_0 \frac{p_y}{\sqrt{p_0^2-\lambda_c^{-2}}} & \frac{-p_x p_y}{\sqrt{(p_y^2+p_z^2)(p_0^2-\lambda_c^{-2})}} & \frac{p_z}{\sqrt{p_y^2+p_z^2}} & 0 \\ \lambda_c p_0 \frac{p_z}{\sqrt{p_0^2-\lambda_c^{-2}}} & \frac{-p_x p_z}{\sqrt{(p_y^2+p_z^2)(p_0^2-\lambda_c^{-2})}} & \frac{-p_y}{\sqrt{p_y^2+p_z^2}} & 0 \end{pmatrix}$$

with λ_c^{-1} the inverse of the Compton wavelength $\lambda_c^{-1} = mc/\hbar$. For both the neutral and charged scalar fields, (74) is trivially satisfied and for the neutral vector boson, (74) is $\Lambda = \Lambda_2\Lambda_1$, the composition of Lorentz transformations.

A massive, charged fermion field is also realized with four constituent fields, $N_c = 4$. This spin-1/2 fermion field uses a 4×4 complex spinor representation of the Lorentz group.

$$\begin{aligned} M(p) &= \begin{pmatrix} 0 & \beta(p) \\ \beta(p)^T & 0 \end{pmatrix} \\ \beta(p) &= \begin{pmatrix} p_0 + p_z & p_x + ip_y \\ p_x - ip_y & p_0 - p_z \end{pmatrix} \\ D &= \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \\ S(A) &= \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \end{aligned}$$

with 2×2 $\beta(p) = \beta(p)^*$, $\mathbb{I}_2$ and $A \in \mathrm{SL}(2, \mathbb{C})$. $M(-p) = -M(p)^T$.

$$DM(p) = C^*(p)C(p) = \begin{pmatrix} c^T(p) & 0 \\ 0 & c^*(p) \end{pmatrix} \begin{pmatrix} \overline{c}(p) & 0 \\ 0 & c(p) \end{pmatrix}$$

is a nonnegative matrix with p, $p^2 = \lambda_c^{-2}$, denoted p_0, p_x, p_y, p_z within the forward cone and

$$c(p) = \begin{pmatrix} \sqrt{p_0 + p_z} & \frac{p_x+ip_y}{\sqrt{p_0+p_z}} \\ 0 & \frac{\lambda_c^{-1}}{\sqrt{p_0+p_z}} \end{pmatrix}.$$

For massive, charged fermion field, the group property and composition of Poincaré transformations (74) follows from $A = A_2A_1 \in SL(2, \mathbb{C})$ if $A_1, A_2 \in SL(2, \mathbb{C})$ as a consequence of the multiplication of determinants [26].

A massive, charged fermion field also has a symmetry $S_\phi M(p) S_\phi^T = M(p)$ associated with a charge,

$$S_\phi = \begin{pmatrix} e^{i\phi}\mathbb{I}_2 & 0 \\ 0 & e^{-i\phi}\mathbb{I}_2 \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$. Two invariant up to a phase subspaces are spanned by

$$S_\phi \begin{pmatrix} f(x)_1 \\ f(x)_2 \\ 0 \\ 0 \end{pmatrix} = e^{i\phi} \begin{pmatrix} f(x)_1 \\ f(x)_2 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \quad S_\phi \begin{pmatrix} 0 \\ 0 \\ f(x)_3 \\ f(x)_4 \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 0 \\ 0 \\ f(x)_3 \\ f(x)_4 \end{pmatrix}.$$

Similarly to the charged scalar field, there is a charge operator in the one-argument subspace of $\mathbf{H}_\mathcal{P}$,

$$Q = -i \left. \frac{d}{d\phi} S_\phi \right|_{\phi=0} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}.$$

Real ($\overline{V} = V$) and imaginary ($\overline{V} = -V$) orthogonal ($V^T = V^{-1}$) similarity transforms $VM(p)V^T, VS(A)V^T, VDV^T$ preserve satisfaction of (39) if there is no $B(p)$ or no $F(p)$ constituent in $M(p)$, appendix 8.12. The direct sum decomposition (35) and (36) are preserved if the similarity transform preserves superselection sectors,

$$V = V_B \oplus V_F$$

with V_B of the same dimension as B in the direct sum decomposition (35).

The physical characteristics of a representation $M(p), D, S(A)$ may be changed by similarity transformation V. For the example of the massive, charged spin-1/2 fermion field above, the 4×4 real, orthogonal transformation

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}$$

transforms the charged field to a neutral field. For the real and orthogonal V,

$$V^{-1} = V^T = V^*.$$

The transformed representation is

$$\begin{aligned} M(p) &= \frac{1}{2} \begin{pmatrix} -(\beta(p) + \beta(p)^T) & \beta(p) - \beta(p)^T \\ -(\beta(p) - \beta(p)^T) & \beta(p) + \beta(p)^T \end{pmatrix} \\ D &= \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \\ S(A) &= \frac{1}{2} \begin{pmatrix} A + \overline{A} & A - \overline{A} \\ A - \overline{A} & A + \overline{A} \end{pmatrix} \end{aligned}$$

with the same 2×2 $\beta(p)$ and $A \in \mathrm{SL}(2,\mathbb{C})$ as the massive, charged spin-1/2 fermion field. $\beta(p) + \beta(p)^T$ is real and $\beta(p) - \beta(p)^T$ is imaginary. The unitarily implemented symmetry S_ϕ associated with a conserved charge for the charged fermion field is transformed to

$$S_\phi = \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix}.$$

This S_ϕ indicates a symmetry that implies identity of the neutral spin-1/2 particle with its antiparticle. The neutral spin-1/2 field is associated with Majorana fermions [63].

Direct sum and Kronecker product compositions, appendix 8.12, of these example and additional representations of

$$M(p), D, S(A)$$

and subrepresentations provide a rich class of realizations of relativistic quantum physics.

3.3 Physically nontrivial relativistic fields

VEV that exhibit nontrivial relativistic physics are displayed in this section. The following sections describe: connectivity; split signed symmetry; representation of the $n \geq 4$, n-point connected VEV functions ${}^C\underline{\mathcal{U}}$ as standard functions, measures and elementary generalized functions; and cluster expansion. The VEV inherit their Poincaré transformation properties from a free field realization, section 3.2. Feynman series, [32, 36] and appendix 8.9, suggest forms of interest for the constructions.

3.3.1 Connected functions

Connected VEV functions include free field two-point functions (32) and $n \geq 4$, n-argument connected VEV functions illustrated in (17).

A connected function ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ attenuates with great space-like separation of its arguments. A connected function does not significantly contribute to scalar products if the support of an $f_n((x)_n) \in \mathcal{S}(\mathbb{R}^{4n})$ consists of two greatly space-like separated volumes. That is, ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ is connected if

$$\lim_{\rho\to\infty} {}^C\mathcal{W}_{k,n-k}(f_n(\rho))_{(\kappa)_n} = 0 \tag{41}$$

with

$$f_n(\rho) = f_n(x_1, \ldots x_j, x_{j+1} - \rho a, \ldots, x_n - \rho a).$$

The Lorentz vector a is space-like ($a^2 < 0$), $\rho \in \mathbb{R}$, and $1 \leq j < n$. (41) applies for all $(N_c)^n$ components labeled by the κ_j. In a scalar product (13),

$$f_n((x)_n) = h_k^*(((x)_k)\, g_{n-k}((x)_{k+1,n}) \in \mathcal{S}(\mathbb{R}^{4n})$$

with $h_k \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4k})_{(\kappa)_k}$ and $g_{n-k} \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4(n-k)})_{(\kappa)_{k+1,n}}$, and the $*$-dual from (11).

The sequence (27) of VEV functions $\underline{\mathcal{W}}$ are constructed from connected VEV functions. The connected VEV functions are designated

$$\begin{aligned} {}^{C}\mathcal{W}_{1,1}(x_1,x_2)_{\kappa_1\kappa_2} &= {}^{C}\mathcal{F}_{1,1}(x_1,x_2)_{\kappa_1\kappa_2} \\ &= W_2(x_1,x_2)_{\kappa_1\kappa_2} \\ {}^{C}\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} &= {}^{C}\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n} \\ &= c_n\delta(p_1+\ldots p_n)\prod_{j=1}^{n}\delta(p_j^2-\lambda_{cj}^{-2})\,Q_{k,n-k}((p)_n)_{(\kappa)_n} \end{aligned} \tag{42}$$

for $2 \le k \le n-2$ and $n = 2\ell \ge 4$ for $\ell \in \mathbb{N}$. W_2 is a free field two-point function (32) and ${}^{C}\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ designates the elements of the sequence of connected VEV functions for a free field.

$$ {}^{C}\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n} = 0$$

if $n \neq 2$ and $k \neq 1$. ${}^{C}\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$ designates the elements of the sequence of connected VEV functions that introduce interaction. The λ_{cj} are Compton wavelengths (9) for the finite elementary particle masses m_{κ_j} associated with the constituent fields $\Phi(x_j)_{\kappa_j}$. The array of functions $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ characterizes the interaction for multiple constituent fields. $Q_{k,2n-k}$ is constructed to be physically nontrivial, section 4.5 and appendix 8.8, satisfy Lorentz covariance, section 4.2.4, and exhibit normal statistics (44). From section 3.3.3, to satisfy Lorentz covariance A.3 for nonzero spins, odd orders of the VEV vanish.

$$\mathcal{W}_{k,2\ell-1-k}((x)_{2\ell-1})_{(\kappa)_{2\ell-1}} = 0$$

for all k. Nonzero spin elementary particles are created or annihilated in pairs. However, odd order VEV are included by many constructions that include scalar representations of the Lorentz group [32, 34]. To satisfy elemental stability A.7,

$$ {}^{C}\mathcal{U}_{0,k} = {}^{C}\mathcal{U}_{k,0} = {}^{C}\mathcal{U}_{1,k} = {}^{C}\mathcal{U}_{k,1} = 0$$

for $k \ge 3$ and

$$\mathcal{W}_{0,2} = \mathcal{W}_{2,0} = 0$$

for all κ_j. For the initial construction of VEV functions, constituent field means are set to zero,

$$\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0,$$

for all constituent fields κ without loss of generality since nonzero means are independently assigned, section 3.3.5. The remaining VEV function is

$$\mathcal{W}_{0,0} = 1,$$

the normalization of the vacuum.

The elements of the array $Q_{k,2n-k}$ are constructed in section 3.3.3 to be polynomial bounded growth, locally absolutely Lebesgue summable functions of the momenta $\mathbf{p}_j$ within the momentum domains included in the summations that evaluate the scalar products (13). After evaluation of the energy-momentum conservation and mass shell delta functions, the remaining Lebesgue summations that evaluate (41) include a factor $\exp(i\rho\,\mathbf{a}\cdot(\mathbf{p}_1+\dots\mathbf{p}_k))$ with $1\le k<n-1$. Then, the Riemann-Lebesgue lemma and Poincaré invariance of the scalar products demonstrate connectedness (41) for the connected VEV (42).

3.3.2 Split signed symmetric functions

The constructed VEV are split signed symmetric. The VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are signed symmetric with transpositions among either the k $*$-dual function arguments, or transpositions among the $n-k$ function arguments. If fields are not Hermitian operators, then a similar symmetry of the VEV with transposition of $*$-dual and function arguments is not necessary. Local commutativity of the field $\Phi(x)_\kappa$ follows from split signed symmetry of the VEV functions. Argument type, $*$-dual or function, follows from definition of the degenerate scalar product (13).

A convenient shorthand notation for VEV is to designate

$$\mathcal{W}_{k,n-k}(A)=\mathcal{W}_{k,n-k}((x)_A)_{(\kappa)_A} \tag{43}$$

for a set of n arguments with indices designated $A=i_1,\dots i_n$. That is, $(x)_A=x_{i_1},\dots x_{i_n}$ and similarly $(\kappa)_A=\kappa_{i_1},\dots\kappa_{i_n}$.

In the notation (43), a sequence (27) of generalized functions $\underline{\mathcal{V}}$ is designated split signed symmetric if the elements satisfy

$$\mathcal{V}_{k,n-k}(\pi_2(\pi_1(\{1,n\})))=\sigma(\pi_1,(\kappa)_n)\sigma(\pi_2,(\kappa)_n)\mathcal{V}_{k,n-k}(\{1,n\}) \tag{44}$$

for every k,n. $\pi_1(\{1,n\})$ designates one of the $k!$ permutations of the sequence $1,2\dots k$ of $*$-dual function arguments. $\pi_2(\{1,n\})$ designates one of the $(n-k)!$ permutations of the sequence $k{+}1,k{+}2\dots n$ of function arguments. In a notation with permutations denoted

$$\pi_\nu(A)=\pi_\nu(i_1),\pi_\nu(i_2),\dots\pi_\nu(i_n),$$

for $A=i_1,i_2\dots i_n$ and $\nu=1,2$,

$$\begin{aligned}\pi_1(\{1,n\}) &= \pi_1(1),\pi_1(2)\dots\pi_1(k),k+1,k+2,\dots n\\ \pi_2(\{1,n\}) &= 1,2,\dots k,\pi_2(k+1),\pi_2(k+2)\dots\pi_2(n).\end{aligned}$$

The signs $\sigma(\pi_\nu,(\kappa)_n)=\pm1$ are assigned to each permutation to satisfy local commutativity A.5 with normal commutation relations [9].

$$\begin{aligned}\sigma(\pi_1,(\kappa)_n)&=1 \qquad \text{if} \qquad \pi_1(\{1,n\})=\{1,n\}\\ \sigma(\pi_2,(\kappa)_n)&=1 \qquad \text{if} \qquad \pi_2(\{1,n\})=\{1,n\}.\end{aligned}$$

and

$$\sigma(\pi'_\nu,(\kappa)_n) = -\sigma(\pi_\nu,(\kappa)_n)$$

if the permutations π'_ν and π_ν differ by one transposition of two adjacent fermion indices, and

$$\sigma(\pi'_\nu,(\kappa)_n) = \sigma(\pi_\nu,(\kappa)_n),$$

if the one transposition is either two adjacent boson indices or transposition of a boson with an adjacent fermion index. In (35) of section 3.2, arguments x_j, κ_j are assigned as boson or fermion by the value of $\kappa_j \in \{1, N_c\}$. The sign associated with an ordering of indices $A = i_1, i_2, \ldots i_n = \pi_2(\pi_1(A_o))$ is with respect to a reference order $A_o = j_1, j_2, \ldots j_n$. The sign for an A with respect to A_o is accumulated over the sequence of adjacent index transpositions that result in the index order A starting from A_o. The sign is determined by the product of signs from the sequence of transpositions, and the sign is independent of the choice for a sequence of transpositions that result in A from A_o [9].

The split signed symmetrization $\underline{\mathcal{V}}$ of a sequence (27) of generalized functions $\underline{v}$ is

$$\mathcal{V}_{k,n-k}(A_o) = \sum_{\pi_1} \sigma(\pi_1,(\kappa)_{A_o}) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_{A_o})\, v_{k,n-k}(\pi_2(\pi_1(A_o))) \right). \tag{45}$$

The summations $\sum_{\pi_1}$ and $\sum_{\pi_2}$ include all $k!$ permutations of the indices $\{1,k\}$ and all $(n-k)!$ permutations of the indices $\{k+1,n\}$, respectively. The $\underline{\mathcal{V}}$ constructed from a generalized function $\underline{v}$ in (45) is split signed symmetric (44).

If $\underline{w}$ provides a degenerate scalar product, then its split signed symmetrization $\underline{\mathcal{W}}$ also provides a degenerate scalar product since

$$\mathcal{W}(\underline{f}^*, \underline{f}) = w(\underline{g}^*, \underline{g}) \geq 0 \tag{46}$$

with

$$\underline{g} = (f_0, f_1, \ldots \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) f_k(\pi_1(\{1,k\})), \ldots)$$

from (45) for the scalar product (13).

If $\underline{\mathcal{V}}$ provides a degenerate scalar product, then a scaled sequence of functions $\underline{\mathcal{V}_a}$ generated from $\underline{\mathcal{V}}$ as

$$\underline{\mathcal{V}_a} = (|a_0|^2, \overline{a}_1 a_0 \mathcal{V}_{1,0}, \ldots \overline{a}_k a_{n-k} \mathcal{V}_{k,n-k}, \ldots), \tag{47}$$

also provides a degenerate scalar product. Each $a_k \in \mathbb{C}$ with $|a_k| > 0$. That $\mathcal{V}_a(\underline{f}^*, \underline{f}) \geq 0$ follows from $\mathcal{V}(\underline{g}^*, \underline{g}) \geq 0$ by setting the sequence $\underline{f} = (g_0/a_0, g_1/a_1, \ldots) \in \underline{\mathcal{P}}$ in the scalar product (13).

3.3.3 Physically nontrivial VEV functions

Connected $n \geq 4$, n-argument VEV functions introduce interaction to the RQP constructions. The sequence of $n \geq 4$, n-argument connected functions is designated ${}^C\underline{\mathcal{U}}$ in (42). The constructed ${}^C\mathcal{U}_{k,n-k}$ are split signed symmetric (44), physically nontrivial, section 4.5 and appendix 8.8, Lorentz covariant, section 4.2.4 and nonnegative,

$$ {}^C\mathcal{U}(\underline{f}^*, \underline{f}) \geq 0$$

in the notation of the scalar product (13) with $\underline{\mathcal{W}} \mapsto {}^C\underline{\mathcal{U}}$. VEV are related (14) to VEV functions.

$$\langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle = (D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$$

with Dirac conjugation D introduced in the $*$-dual of function sequences (11). The matrix notation (12) applies.

The Fourier transforms of the connected VEV functions are

$${}^C\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n} = c_n \delta(p_1 + \dots p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_{cj}^{-2})\, Q_{k,n-k}((p)_n)_{(\kappa)_n}.$$

The support of these connected VEV functions over energy-momenta is the submanifold of $(p)_n \in \mathbb{R}^{4n}$ with energy-momentum conserved and each p_j on a mass shell, $p_j^2 = \lambda_{cj}^{-2}$. Energy-momentum conservation provides translation invariance of the scalar product (13). The ${}^C\underline{\mathcal{U}}$ use the triple of $N_c \times N_c$ matrices $D, M(p), S(A)$ from the free field two-point function $W_2(x_1, x_2)_{\kappa_1 \kappa_2}$ in the identification (42) of the connected VEV functions. The connected VEV functions ${}^C\mathcal{U}_{k,n-k}$ are zero for $n < 4$.

With the substitutions

$$\delta(p_1 + \dots p_n) = \int \frac{du}{(2\pi)^4} \prod_{j=1}^{n} e^{-ip_j u}$$

and the assignment

$$c_n = \int d\sigma(\lambda)\, \lambda^n, \tag{48}$$

the Fourier transforms of the connected functions ${}^C\mathcal{U}_{k,2n-k}$ are

$${}^C\widetilde{\mathcal{U}}_{k,2n-k}(\{1, 2n\}) = \iint \frac{d\sigma(\lambda)\, du}{(2\pi)^4}\, \lambda^{2n} \prod_{j=1}^{2n} e^{-ip_j u}\, \delta(p_j^2 - \lambda_{cj}^{-2})\, Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}} \tag{49}$$

in the shorthand notation (43). The summation du is over $\mathbb{R}^4$ and $d\sigma(\lambda)$ denotes a summation with nonnegative measure over $\mathbb{R}$. Then $c_{2n} \geq 0$. For example, $c_n = \rho^n$ for $d\sigma(\lambda) = \delta(\lambda - \rho) d\lambda$, or $c_n = n!/\rho^{n+1}$ for $d\sigma(\lambda) = \theta(\lambda) \exp(-\rho\lambda) d\lambda$ with $0 < \rho \in \mathbb{R}$.

The function arrays $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ that complete definition (42) of ${}^C\mathcal{U}_{k,n-k}$ is the split signed symmetrization (45) of function arrays $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$.

$$\begin{aligned} Q_{k,n-k}((p)_n)_{(\kappa)_n} &= \sum_{\pi_2} \sigma(\pi_2,(\kappa)_n) \left(\sum_{\pi_1} \sigma(\pi_1,(\kappa)_n)\ \mathfrak{q}_{k,n-k}(\pi_2(\pi_1(\{1,n\})))\right) \\ \mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n} &= \prod_{j=1}^{n} \left(\frac{d}{d\rho_j}\right) \exp\Big(\sum_{a,b\in\mathbb{J}_{k,n}} \rho_a\rho_b\, \mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}\Big) \end{aligned} \tag{50}$$

for $2 \leq k \leq n-2$ and with the $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$ evaluated at $(\rho)_n = 0$ after the differentiation. The $(\rho)_n \in \mathbb{R}^n$ are independent parameters. To eliminate a divergent term from extrapolation of (50) to $n = 2$,

$$Q_{k,2n-k}(\{1,2n\}) = 0 \qquad \text{if} \quad k = 0, 1, 2n-1, 2n.$$

The $\mathfrak{q}_{k,n-k}((p)_n)_{(\kappa)_n}$ are functions of products of pairs $\rho_a\rho_b$ with $a \neq b$ and as a consequence of the evaluation at $(\rho)_n = 0$,

$$Q_{k,2n+1-k}(\{1,2n+1\}) = 0$$

for $0 \leq k \leq 2n-1$. Split signed symmetry (44) of ${}^C\mathcal{U}_{k,n-k}(\{1,2n\})$ follows from the split signed symmetry of $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ together with invariance to argument transpositions of the preceding factors in (49). The summation over $a,b \in \mathbb{J}_{k,n}$ with $a,b \in \{1,n\} \subset \mathbb{N}$ is defined below in (53). There is significant freedom in selection of this summation over $a,b \in \mathbb{J}_{k,n}$. The $N_c \times N_c$ array of functions $\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}$ over energy-momenta p_j is described by two distinct $N_c \times N_c$ matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$ varying with the values of a,b.

$$\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b} = \begin{cases} \mathfrak{B}(\alpha_o p_a + p_b)_{\kappa_a\kappa_b} & \text{if both } a,b \in \{k+1,n\} \\ \mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a\kappa_b} & \text{if both } a,b \in \{1,k\} \\ \Upsilon(-p_a + p_b)_{\kappa_a\kappa_b} & \text{if } a \in \{1,k\} \text{ and } b \in \{k+1,n\} \\ 0 & \text{if } a \in \{k+1,n\} \text{ and } b \in \{1,k\}. \end{cases} \tag{51}$$

Considered as matrices, $\mathfrak{h} = \mathfrak{B}$ if arguments (p_a,κ_a) and (p_b,κ_b) are both $*$-dual function arguments or if both are function arguments, and $\mathfrak{h} = \Upsilon$ or zero if the arguments are split, one from a $*$-dual function and one from a function. Indices a,b are assigned as $*$-dual or function arguments in (50) from evaluation of a scalar product (13). $\alpha_o \geq 0$ is a real parameter that determines dynamics. Addressed in section 4.2.6, this form for $\mathfrak{h}(p_1,p_2)$ satisfies regularity A.1 for nonconstant $\mathfrak{h}(p_1,p_2)_{\kappa_1\kappa_2}$. Due to Lorentz covariance, nonconstant $\mathfrak{h}(p_1,p_2)_{\kappa_1\kappa_2}$ are necessarily singular, and (51) is selected to exclude the singularities from the submanifold of joint support of the VEV functions ${}^C\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n}$ and state describing functions from $\underline{\mathcal{P}}$.

The matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$ are Fourier or Laplace transforms of the array $M(p)$ from the free field two-point connected function (32) in the connected functions (42).

$$\begin{aligned} \mathfrak{B}(p) &= \int d\mu_{\mathfrak{B}}(s)\, e^{isp} M(s) \\ \Upsilon(p) &= \int d\mu_{\Upsilon}(s)\, e^{-sp} M(s) \end{aligned} \tag{52}$$

with $s \in \mathbb{R}^4$, and for Υ, s_0 limited to the Poincaré invariant $s_0 > 0$. $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$ are real-valued Lorentz invariant measures, $d\mu_{\Upsilon}(s)$ is a nonnegative measure, and $sp = s_0 p_0 - \mathbf{s}\cdot\mathbf{p}$ is a Lorentz invariant. The measures are limited to achieve polynomial bounded growth, locally absolutely Lebesgue summable functions $\mathfrak{B}(p)$ and $\Upsilon(p)$ within the domains included in the summations that evaluate the scalar products (13) [56]. An evaluation of $Q_{2,2}((p)_4)_{(\kappa)_4}$ in terms of constituents $\mathfrak{B}(p)$ and $\Upsilon(p)$ is illustrated in appendix 8.8. $Q_{2,2}((p)_4)_{(\kappa)_4}$ provides boson scattering, fermion scattering, fermion pair production from bosons, fermion pair annihilation to bosons, and fermion-boson scattering amplitudes. Like $M(p)$, the $\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}$ determined in (51) and (52) do not connect superselection sectors: the $\mathfrak{h}(p_a,p_b)_{\kappa_a\kappa_b}$ are zero if κ_a and κ_b are distinct index types, a fermion with a boson index.

The scalar product (13) follows from a factorization of the constructed $Q_{k,n-k}((p)_n)_{(\kappa)_n}$. Three factors follow from consideration of the number of factors of D contributed from $*$-dual functions (11) in (51). From (50),

$$\begin{aligned} (D\cdot)_k \exp\Big(\sum_{a,b\in\mathbb{J}_{k,n}} \rho_a\rho_b \mathfrak{h}_{\kappa_a\kappa_b}\Big) &= \exp\Big(\sum_{b_1>a_1=1}^{k} \rho_{a_1}\rho_{b_1} (D\mathfrak{B}(p_{a_1}+\alpha_o p_{b_1})D^T)_{\kappa_{a_1}\kappa_{b_1}}\Big) \\ &\times \exp\Big(\sum_{b_2>a_2=k+1}^{n} \rho_{a_2}\rho_{b_2} \mathfrak{B}(\alpha_o p_{a_1}+p_{b_1})_{\kappa_{a_2}\kappa_{b_2}}\Big) \\ &\times \exp\Big(\sum_{\ell=1}^{\mathfrak{I}_{k,n}} \rho_{k+1-\ell}\rho_{k+\ell} (D\Upsilon(-p_{k+1-\ell}+p_{k+\ell})_{\kappa_{k+1-\ell}\kappa_{k+\ell}}\Big). \end{aligned} \tag{53}$$

$\sum_{b>a=1}^{k}$ designates $\sum_{a=1}^{k-1}\sum_{b=a+1}^{k}$. The limit on the third summation is

$$\mathfrak{I}_{k,n} = \min(k, n+1-k).$$

(53) describes one summation $a, b \in \mathbb{J}_{k,n}$ in (50). Additional forms for $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ also provide realizations. The description of interaction includes the summation over $\mathbb{J}_{k,n}$, the c_n, α_o and the symmetrization (45), appendix 8.8.

The upper limit in the summation to $\mathfrak{I}_{k,n}$ in (53) extends arbitrarily. Additional terms are set to zero after differentiation with respect to the $(\rho)_n$ indicated in (50). Only terms generated by differentiation with $\rho_j \in \{\rho_1, \rho_2, \ldots \rho_n\}$ contribute. This extension simplifies demonstration of the nonnegativity of scalar products in section 4.2.3.

3.3.4 Cluster expansion of VEV

Cluster expansions compose the connected functions (42) into VEV (14). The cluster expansion with connectedness, section 3.3.1, satisfy a consequence of causality, cluster decomposition A.6 in section 4.1.

The connected function sequences (27) are

$$\begin{aligned} {}^{C}\underline{\mathcal{F}} &= (0,0,0,0,{}^{C}\mathcal{F}_{1,1},0,\ldots) \\ {}^{C}\underline{\mathcal{U}} &= (0,\ldots 0,{}^{C}\mathcal{U}_{2,2},\ldots,{}^{C}\mathcal{U}_{k,n-k},\ldots). \end{aligned}$$

The two-point functions ${}^{C}\mathcal{F}_{1,1} = W_2$ are from section 3.2 and the $n \geq 4$, n-point connected functions ${}^{C}\mathcal{U}_{k.n-k}$ are constructed in section 3.3.3. VEV function sequences $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$ are the cluster expansions of ${}^{C}\underline{\mathcal{F}}$ and ${}^{C}\underline{\mathcal{U}}$.

$$\underline{\mathcal{F}} = \exp\circ({}^{C}\underline{\mathcal{F}}). \tag{54}$$

and

$$\underline{\mathcal{U}} = \exp\circ({}^{C}\underline{\mathcal{U}}). \tag{55}$$

These expansions are based on a $\circ$-product developed below in (60). Given the $\circ$-product, the exponential of a generalized function sequence $\underline{\mathcal{T}}$ provides its cluster expansion.

$$\exp\circ(\underline{\mathcal{T}}) = \Omega + \sum_{j=1}^{\infty} \frac{1}{j!}\, \underline{\mathcal{T}} \circ \underline{\mathcal{T}} \circ \ldots \underline{\mathcal{T}}. \tag{56}$$

The series (56) has j factors of $\underline{\mathcal{T}}$ in the jth term. Finally, the constructed VEV functions $\mathcal{W}_{k.n-k}$ are the elements of the $\circ$-product of the sequences of free field VEV $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, the cluster expansion of the $n \geq 4$, n-argument connected VEV functions ${}^{C}\underline{\mathcal{F}}$. The sequence (27) of constructed VEV functions is the composition

$$\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}. \tag{57}$$

The free field contributions $\underline{\mathcal{F}}$ associate the constructions with elementary particles, section 4.3, and the $n \geq 4$, n-argument connected functions $\underline{\mathcal{U}}$ exhibit interaction, sections 4.5 and 5. Equality of the cluster expansion (54) for the free field VEV $\underline{\mathcal{F}}$ with the operator expansion for VEV evaluated from the commutation relations of creation and annihilation field components is verified in appendix 8.7. The equality of (57) with

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^{C}\underline{\mathcal{F}} + {}^{C}\underline{\mathcal{U}}) \\ &= \exp\circ({}^{C}\underline{\mathcal{W}}) \end{aligned} \tag{58}$$

follows from (42) and is demonstrated below in this section.

The $\circ$-product $\underline{\mathcal{W}}$ of two generalized function sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$,

$$\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}, \tag{59}$$

is the function sequence (27) that consists of elements that are the split signed symmetrization (45) of a function sequence $\underline{w}$ constructed from arithmetic products of the component functions from $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$.

$$\begin{aligned} \mathcal{W}_{k,n-k}(\{1,n\}) &= \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_n)\; w_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) \right) \\ w_{k,n-k}(\{1,n\}) &= \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell!j!} \; \frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!}. \end{aligned} \tag{60}$$

The abbreviated notation (43) applies. A reference argument order A_o, A'_o defines the split signed symmetrization (45).

$$\begin{aligned} A_o &= \{1,\ell\} \cup \{k+1, k+j\} \\ A'_o &= \{1,n\} \setminus A_o = \{\ell+1, k\} \cup \{k+j+1, n\}. \end{aligned} \tag{61}$$

Signs $\sigma(\pi_\nu,(\kappa)_n)$ are from section 3.3.2 and the signs determine that the fields exhibit normal statistics. If, like $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are split signed symmetric (44), then the $\circ$-product simplifies to

$$\mathcal{W}_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \left(\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2} \sigma_1\sigma_2 \; \mathcal{T}_{\ell,j}(A)\; \mathcal{V}_{k-\ell,n-k-j}(A') \right) \tag{62}$$

with the abbreviated notation $\sigma_\nu = \sigma(\pi_\nu,(\kappa)_n)$ for $\nu = 1,2$. The A' are the set complements within $\{1,n\}$ of the arguments in A. The summations $\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}$ include only a subset of the permutations π_1, π_2. The summations over $\mathfrak{s}_1$ and $\mathfrak{s}_2$ include only permutations π_1, π_2 that result in a natural order of indices within each of the two factors. The

$$\{\mathfrak{s}_\nu\} \subset \{\pi_\nu\}$$

result in only the naturally ordered sets of indices A and A'. That is, if

$$A = \pi_1(\pi_2(A_o)) = i_1, i_2, \ldots i_{\ell+j},$$

the term is included in the summation $\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}$ only if $i_1 < i_2 < \ldots i_{\ell+j}$ and the $n-\ell-j$ argument indices within A' are similarly ordered. There are $\ell!j!(k-\ell)!(n-k-j)!$ of each

distinct term considering the split signed symmetry (44) of $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$. The factors in each term $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ are products of elements from $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ that include no arguments in common, and all arguments are present in each term of the expansion (60). As a consequence, the sequence $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is a sequence of generalized functions if $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are sequences of generalized functions.

In the composition (57), only the elements $\mathcal{F}_{\ell,\ell}$ are nonzero. Consequently, only terms with $j = \ell$ contribute in the construction (57).

$$\mathcal{W}_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \left(\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2} \sigma_1 \sigma_2 \, \mathcal{F}_{\ell,\ell}(A) \, \mathcal{U}_{k-\ell,n-k-\ell}(A') \right) \tag{63}$$

in the notation of (62).

The $\circ$-product (60) is commutative, associative and distributive with addition of generalized function sequences. The commutativity of $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is demonstrated by relabeling the summations in the $\circ$-product (60) $\ell' = k - \ell$ and $j' = n - k - j$. With the relabeling,

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \, \frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!}$$

becomes

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell'=0}^{k} \sum_{j'=0}^{n-k} \frac{\mathcal{V}_{\ell',j'}(A'_o)}{\ell'! j'!} \, \frac{\mathcal{T}_{k-\ell',n-k-j'}(A_o)}{(k-\ell')!(n-k-j')!}.$$

That (60) is a summation of arithmetic multiplications provides that the $\circ$-product is associative and distributive.

From (56), the commutativity, associativity and distributivity with addition of sequences of the $\circ$-product (59), and the binomial expansion provide

$$\begin{aligned} \exp\circ(\underline{\mathcal{T}} + \underline{\mathcal{V}}) &= \sum_{N=0}^{\infty} \frac{1}{N!} \left((\underline{\mathcal{T}} + \underline{\mathcal{V}}) \circ \right)^N \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\nu=0}^{N} \binom{N}{\nu} (\underline{\mathcal{T}} \circ)^{\nu} \circ (\underline{\mathcal{V}} \circ)^{N-\nu} \\ &= \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \frac{1}{N_1! N_2!} (\underline{\mathcal{T}} \circ)^{N_1} \circ (\underline{\mathcal{V}} \circ)^{N_2} \\ &= (\exp\circ(\underline{\mathcal{T}})) \circ (\exp\circ(\underline{\mathcal{V}})) \end{aligned} \tag{64}$$

from relabeling terms in summations. The expansions for $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, (54) and (55) respectively, substituted into the construction (57) provides that

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^{C}\underline{\mathcal{F}}) \circ \exp\circ({}^{C}\underline{\mathcal{U}}) \\ &= \exp\circ({}^{C}\underline{\mathcal{F}} + {}^{C}\underline{\mathcal{U}}) \end{aligned} \tag{65}$$

from the identity (64). Connected functions add in the construction (57). If both a ${}^1\underline{\mathcal{W}}$ and a ${}^2\underline{\mathcal{W}}$ are split signed symmetric and satisfy A.1-7, then ${}^1\underline{\mathcal{W}} \circ {}^2\underline{\mathcal{W}}$ also realizes A.1-7 and this leads to consideration of prime and divisible realizations [25].

Inversion of the expansion (56) defines truncated functions ${}^T\underline{\mathcal{W}}$ given a sequence $\underline{\mathcal{W}}$ [9, 50].

$$\exp \circ ({}^T\underline{\mathcal{W}}) = \underline{\mathcal{W}}. \tag{66}$$

Then,

$${}^T\underline{\mathcal{W}} = {}^C\underline{\mathcal{W}}$$

with the connected functions ${}^C\underline{\mathcal{W}}$ from (42). Similarly to RQFT, truncated functions are connected with the exception of any non-zero expectation value fields.

The constructed VEV (42) are solutions to Klein-Gordon equations,

$$(p_j^2 - m_{\kappa_j}^2)\tilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} = 0$$

for $1 \leq j \leq n$, yet exhibit interaction. Mass shell singularities of the VEV together with a lack of constraints that set momenta equal in pairs result in nontrivial scattering. Canonical formalism-compliant solutions to Klein-Gordon equations necessarily exhibit trivial physics [9, 58]. The theorems that solutions to Klein-Gordon equations necessarily exhibit trivial physics do not apply with revised axioms: interacting quantum fields are not densely defined Hermitian Hilbert space operators. The constructed scattering likelihoods that coincide with Feynman series at weak coupling demonstrate that solutions to Klein-Gordon equations are of interest. VEV that are solutions to the Klein-Gordon equation and physically nontrivial are inconsistent with Hermitian field operators. This suggests that interacting relativistic quantum fields are precluded by canonical quantization.

This completes construction of example, physically nontrivial VEV. Together with specification of the basis function space $\underline{\mathcal{P}}$, section 3.4, a construction follows from specification of the connected functions (42). The high order connected functions ${}^C\underline{\mathcal{U}}$ follow from (49) with (50), (51), (52) and (53) in section 4.2. The cluster expansions (54) and (55) with specification (32) of the two-point function in section 3.2 complete a construction. Satisfaction of the properties of RQP, prospective axioms A.1-7 in section 4.1, is verified in section 4.2.

3.3.5 Non-zero expectation value fields

Mean values of the VEV of the field $\langle\Omega|\Phi(x)_\kappa\Omega\rangle$ are set to zero in the development without loss of generality. Nonzero means are freely specified for spin zero boson fields without impact to satisfaction of the prospective axioms A.1-6 in section 4.1. The statement of A.7 presented in section 4.1 assumes zero expectation value fields. The addition of finite constants a_κ to mean zero quantum fields $\Phi(x)_\kappa \mapsto \Phi(x)_\kappa + a_\kappa$ implements fields with finite means

$$a_\kappa = \langle\Omega|\Phi(x)_\kappa\Omega\rangle.$$

The VEV functions for constant fields $\Phi(x)_\kappa = a_\kappa$ follow from the VEV (14).

$$(D\cdot)_k \, \mathcal{V}_{k,n-k}(\{1,n\}) = \prod_{\ell=1}^{k} \overline{a}_{\kappa_\ell} \prod_{j=k+1}^{n} a_{\kappa_j},$$

These VEV functions satisfy A.1-6 and an appropriately restated A.7. Applications of Dirac conjugations D use the matrix notation (12) and the abbreviated notation (43) is used for the VEV functions.

However, if κ_j is a fermion index, then addition of a finite constant to the field violates anti-commutation, and if $\Phi(x_j)_{\kappa_j}$ is a component of a non-zero spin boson field, then finite expectation values violate spacetime invariance of the laws of motion. Lorentz invariance (37) of the scalar product (13) requires

$$S(A)M(p)S(A)^T = M(\Lambda^{-1}p).$$

Lorentz covariance of the non-zero expectation values a_κ,

$$a' = S(A)a,$$

$\overline{S(A)} = S(A)$ and $[D, S(A)] = 0$ suffice to satisfy (37) except then variation of the constants a_κ with reference frame violates spacetime invariance. If $\Phi(x)_\kappa$ transforms with a scalar representation of the Lorentz group, $S(A) = \mathbb{I}$, then the constants a_{κ_j} are invariant.

$\underline{\mathcal{V}}$ is trivially split signed symmetric (44) and as a consequence of the properties of the $\circ$-product (59), and from the definition of truncated functions (66) and the composition (64) of cluster expansions,

$$\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}} \circ \underline{\mathcal{V}} = \exp \circ ({}^{C}\underline{\mathcal{F}} + {}^{C}\underline{\mathcal{U}} + {}^{T}\underline{\mathcal{V}})$$

satisfies axioms A.1-6 and an appropriately restated A.7.

3.4 The basis function spaces

3.4.1 The basis function spaces $\underline{\mathcal{P}}$

Selection of support constrained (7) basis function spaces $\underline{\mathcal{P}}$ admits nontrivial VEV realizations that are unavailable to Wightman's original development [58, 64]. Wightman selected the Schwartz tempered test functions $\underline{\mathcal{S}}$. In this development, apparently unrealizable constraints on VEV are relaxed by placing constraints on the basis function space $\underline{\mathcal{P}}$.

$\underline{\mathcal{P}}$ consists of those tempered functions with Fourier transforms with simple zeros on the appropriate negative energy mass shells. The mass shell is determined by the mass m_{κ_j} for each constituent field labeled κ_j.

$$\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$$

if

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n} (p_{j0} + \omega_j)\, \tilde{h}_n((p)_n)_{(\kappa)_n}$$

from (7) with $h_n \in \mathcal{S}(\mathbb{R}^{4n})$ and

$$\hbar c\omega_j = \hbar c\omega(\mathbf{p}_j) = \sqrt{(m_{\kappa_j}c^2)^2 + (\hbar \mathbf{p}_j)^2 c^2}$$

from (8). $\varphi_0 = h_0 \in \mathbb{C}$. Then,

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = 0 \qquad \text{when any energy} \qquad p_0 = -\omega_j$$

for $j = 1, \ldots n$. The infinitely differentiable and polynomially bounded in magnitude factors $(p_{k0} + \omega_k)$ are multipliers of tempered test functions [20]. The spaces $\mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$ are closed subspaces of the Schwartz tempered spaces and are consequently nuclear and perfect [20, 21]. $\underline{\mathcal{P}}$ is stable under Poincaré transformations (72). The Lorentz invariance of p^2 and that $(a, \Lambda)\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ for every $\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ provides that $\underline{\mathcal{P}}$ is stable under Lorentz transformations. Indeed, the Lorentz invariance of p^2 provides that a zero of the appropriate form (7) is stable with proper, orthochronous Lorentz transformation,

$$p'_{0,j} + \omega(\mathbf{p}'_j) = (0, \Lambda)(p_{j0} + \omega_j) = (p_{j0} + \omega_j)\frac{(p_{j0} - \omega_j)}{(0, \Lambda)(p_{j0} - \omega_j)}$$

for the Poincaré transformation (a, Λ). The final factor is regular and $p'_{0,j} + \omega(\mathbf{p}'_j) = 0$ if $p_{j0} + \omega_j = 0$. Then $(0, \Lambda)\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$ if $\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$.

The labels of the elements of $\mathbf{H}_{\mathcal{P}}$ include Cauchy sequences of functions in $\underline{\mathcal{P}}$ convergent in the Hilbert space norm (25). For the constructions, these include functions

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n} (p_{j0} + \omega_j)\, \tilde{g}_n((\mathbf{p})_n)_{(\kappa)_n})$$

with $g_n \in \mathcal{S}(\mathbb{R}^{3n})$ and, as a consequence, $\mathbf{H}_{\mathcal{P}}$ includes states characterized by state describing functions with point support over time [34]. The functions used by Lehmann, Symanzik and Zimmermann to isolate the creation component of a field operator in their developments of scattering for RQFT [9] are included in $\mathbf{H}_{\mathcal{P}}$. As a consequence, LSZ expressions for scattering amplitudes readily adapt to the constructions. Explicit scattering likelihoods are presented in section 4.5 and [32, 34, 36, 38].

The basis function spaces $\underline{\mathcal{P}}$ include no strictly localized states [34] but there are essentially localized states, appendix 8.17. Functions of the form (7) are anti-local [46, 55] and not of bounded spatial support. Anti-local functions do not equal zero within any finite volume. Tails of essentially localized state describing functions are physically negligible since minor support corresponds to unlikely and effectively unrepeatable observations.

Selection of the basis function spaces $\underline{\mathcal{P}}$ overcomes the challenge of discovering VEV within $\underline{\mathcal{S}}'$ that satisfy the physical conditions of RQP for any function from $\underline{\mathcal{S}}$. This long-standing problem remains without a resolution [2, 4, 9, 30, 40, 42, 58, **?**]. The realizations of relativistic quantum physics discussed here are admitted by elimination of the assertion that quantum fields (29) canonically quantize corresponding classical fields and, as a consequence, must be Hermitian. The lack of real functions within the completion of $\underline{\mathcal{P}}$ precludes Hermitian field operators but does not preclude an appropriate correspondence of classical and quantum dynamics.

3.4.2 An alternative formulation based on $\underline{\mathcal{S}}$

An equivalent formulation modifies the constructed VEV from section 3.3 and selects the Schwartz tempered functions $\underline{\mathcal{S}}$ as the basis function space. In this equivalent formulation, the VEV are necessarily not formally Hermitian if interaction is exhibited, if the $n \geq 4$, n-argument connected VEV are included.

The equivalence derives from that multiplier functions can be applied to either the generalized functions or the test functions. Multiplier functions

$$\frac{\omega \pm p_0}{2\omega} = \theta(\pm p_0)$$

if evaluated on mass shells $\hbar^2 p^2 = m^2 c^2$ with $\theta(x)$ the Heaviside step function. The zeros in (7) that distinguish the subspace $\underline{\mathcal{P}}$ from the tempered functions $\underline{\mathcal{S}}$ are provided by these multiplier functions that may alternatively be applied to the VEV as positive energy constraints. In this equivalent, alternative formulation, involutivity W.b is satisfied but formal Hermiticity W.a is abandoned to achieve interaction. With the multipliers applied to the VEV rather than to the test functions, the generalized functions become

$$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \mapsto \prod_{j=1}^{k} \prod_{\ell=k+1}^{n} \left(\frac{-p_{j0} + \omega_j}{2\omega_j} \right) \left(\frac{p_{\ell 0} + \omega_\ell}{2\omega_\ell} \right) \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}. \tag{67}$$

From section 3.3, the VEV have point support on energies on mass shells, $\delta(p_j^2 - m_{\kappa_j}^2 c^2/\hbar^2)$ for each $j \in \{1, n\}$. The resulting equivalence of formulations is:

$$\begin{array}{cc} \underline{\text{VEV}} & \underline{\text{Basis space}} \\ \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{P}} \\ \prod_{j=1}^{k} \theta(-p_{j0}) \prod_{\ell=k+1}^{n} \theta(p_{\ell 0})\, \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{S}}. \end{array} \tag{68}$$

The equivalent formulations are interesting for an analogy with a Reeh and Schlieder result [46, 68]; states described by (2) with Hermitian field operators, states described by operating on

the vacuum with powers of field operators and functions $\varphi_n((x)_n)_{(\kappa)_n}$ supported solely within an open and bounded region $\mathcal{O}$ are dense in the entire Hilbert space and not just dense for states supported within the same bounded region $\mathcal{O}$. Although the theorem of Reeh and Schlieder is demonstrated for RQFT and not axioms A.1-7 from section 4.1, the equivalent formulations in (68) display the same puzzlement. The formulation with states labeled by the anti-local functions in $\underline{\mathcal{P}}$ is equivalent to a formulation that includes states labeled by functions of bounded support from $\underline{\mathcal{S}}$. Anti-local functions have global support: anti-local functions do not vanish over any finite volume. The equality (68) demonstrates that the more evident anti-locality of the formulation based on $\underline{\mathcal{P}}$ is also exhibited in the formulations based on $\underline{\mathcal{S}}$.

4 Characteristics of the relativistic fields

From Born's rule, scalar products determine the likelihoods of observed events and these likelihoods must satisfy our observations of nature. The constructed state descriptions must satisfy established properties of quantum mechanics and relativity [9, 13, 58, 62, 63]. These properties are stated in prospective axioms A.1-7 in section 4.1. Realization of the axioms is demonstrated in section 4.2. Additional properties including correspondences with classical particles, the Hamiltonian and scattering amplitudes are demonstrated in sections 4.3-4.5.

4.1 Axioms for the VEV of relativistic fields

Early study of mathematical structures that realize quantum mechanics did not identify nontrivial realizations of relativistic quantum physics [7, 8, 9]. Methods include the proposal of axioms to characterize general properties of relativistic quantum physics. Axioms are used to define concepts, derive general results, and assess the consistency of the axioms with additional assumptions. Significantly, the only realizations of the established prospective axioms for relativistic quantum physics are physically trivial. Established axioms include the Wightman functional analytic axioms [9, 10, 58, 64], the Gårding-Wightman axioms for field operators [9, 58, 65], and the Haag-Kastler (Araki-Haag-Kastler) algebraic axioms for bounded, local Hermitian operators [9, 68, 69]. These axioms all consider Hilbert space realizations in addition to Fock space but the realizations that have been constructed are physically trivial. The constructed realizations of relativistic quantum physics are enabled by a technical weakening of the original axioms [9].

Seven prospective axioms adapted from the Wightman functional analytic axioms [9, 10, 58, 64] and RQFT [63] characterize the constructed realizations of relativistic quantum physics.

A.1) *Regularity:* the vacuum expectation values (VEV) of quantum fields are generalized functions dual to the Schwartz tempered functions $\underline{\mathcal{S}}$.

A.2) *Nonnegativity:* the state describing functions are elements of a Hilbert space. VEV define

(13) a degenerate scalar product $\mathcal{W}(\underline{g}^*, \underline{f})$ for sequences of state describing functions $\underline{g}, \underline{f}$ with $*$-dual sequences $\underline{g}^*$ from (11).

A.3) *Relativistic invariance:* inertial observers agree on likelihoods. The degenerate scalar product (13) is invariant to proper orthochronous Poincaré transformations of the state describing function sequences.

A.4) *Spectral support:* energy-momenta lie within the closed forward (nonnegative energy) cone.

A.5) *Local commutativity:* field operators commute if not causally related and similarly described particles are indistinguishable. The magnitudes of VEV are invariant with interchange of argument values if the argument values are space-like separated.

A.6) *Cluster decomposition:* likelihoods for non-entangled, distantly space-like separated state descriptions are independent. The degenerate scalar product for functions supported on distantly space-like separated volumes factors,

$$\mathcal{W}(\underline{g_1}^* \times \underline{g_2}^*, \underline{f_1} \times \underline{f_2}) \longrightarrow \mathcal{W}(\underline{g_1}^*, \underline{f_1})\mathcal{W}(\underline{g_2}^*, \underline{f_2}), \tag{69}$$

as the supports of $\underline{g_1}, \underline{f_1}$ become arbitrarily distantly space-like separated from the supports of $\underline{g_2}, \underline{f_2}$.

A.7) *Elemental stability:* Setting fields to zero expectations, the vacuum and one particle states are orthogonal to multiple argument states. If $g_0 = g_1(x)_\kappa = 0$, $\kappa \in \{1, N_c\}$ and $f_n((x)_n)_{(\kappa)_n} = 0$ for $n \neq 1$, then $\langle \underline{g}|\Omega\rangle = 0$ and $\langle \underline{g}|\Phi(\underline{f})\Omega\rangle = 0$.

The notation is developed in section 3.1 and [10]. Axioms A.2-6 are designated here as the physical conditions [32]. Regularity and nonnegativity imply that states are realized as elements of a rigged Hilbert space and satisfaction of the physical conditions applies in the Hilbert space. In the emphasized development, the Hilbert space, based on support constrained (7) function sequences $\underline{\mathcal{P}}$, section 3.4.1, is designated $\mathbf{H}_\mathcal{P}$. In the alternative development, section 3.4.2 , the basis space of function sequences is $\underline{\mathcal{S}}$.

Given generalized functions $\mathcal{U}_{n,m}$ that satisfy A.1-6, generalized functions $\mathcal{W}_{n,m}$ that also satisfy A.7 follow from

$$\mathcal{W}(\underline{f}^*, \underline{f}) = |f_0|^2 + \mathcal{F}_{1,1}(f_1^*, f_1) + \mathcal{U}((\underline{f} - {}^A\underline{f})^*, \underline{f} - {}^A\underline{f}) \tag{70}$$

with the abbreviated sequence ${}^A\underline{f}$ derived from $\underline{f}$,

$${}^A\underline{f} = (f_0, f_1(x)_1, \ldots f_1(x)_{N_c}, 0, 0, \ldots).$$

$\mathcal{F}_{1,1}$ is a free field two-point function, (32) in section 3.2. In (70), zero expectation value fields,

$$\langle \Omega|\Phi(x)_\kappa\Omega\rangle = 0$$

is asserted without loss of generality: the addition of non-zero expectation value fields in section 3.3.5 preserves satisfaction of the axioms. Satisfaction of A.1-6 is preserved in the transformation (70). The decoupling of the contributions from the $n \geq 4$, n-argument VEV in $\underline{\mathcal{U}}$ from the two-point function illustrates that with revised axioms, a two-point function equal to the free field two-point function no longer implies that the quantum field is a free field. Without densely defined Hermitian field operators, that is, without VEV that satisfy the additional constraints of formal Hermiticity W.a and involutivity W.b introduced below, the RQFT lemma [9] that provides that the Pauli-Jordan function can be a two-point VEV only for a free field does not apply to the constructions.

Using Born's rule, all inertial observers perceive the same likelihood of events if the scalar products are invariant to Poincaré transformations.

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle \underline{g}|\underline{f}\rangle, \tag{71}$$

for Poincaré transformations

$$\begin{aligned} (a,\Lambda)f_n((x)_n)_{(\kappa)_n} &= (S(A)^T\cdot)_n f_n((\Lambda^{-1}(x-a))_n)_{(\kappa)_n} \\ (a,\Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} &= \prod_{k=1}^{n} e^{-ip_k a}\,(S(A)^T\cdot)_n \tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n} \end{aligned} \tag{72}$$

from the Fourier transform (21) with properties (23), with the matrix notation (12) and with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz group. Λ is the proper orthochronous Lorentz transformation determined by $A \in \mathrm{SL}(2,\mathbb{C})$, and $a \in \mathbb{R}^4$ is a translation in spacetime. The 4×4 Lorentz transformation Λ determined from A has components

$$\Lambda_{jk} = \frac{1}{2}\mathrm{Trace}(\sigma_j A \sigma_k A^*) \tag{73}$$

with σ_j designating one of the four Pauli spin matrices [9]. $\Lambda^T g \Lambda = g$ with g the Minkowski signature matrix (24).

From the reconstruction theorem [9, 58], the invariance (71) provides that the spacetime transformation (72) is unitarily implemented: the Hilbert space operator adjoint of (a,Λ) is the inverse of (a,Λ),

$$(a,\Lambda)^* = (a,\Lambda)^{-1} = (-\Lambda^{-1}a, \Lambda^{-1}).$$

The transform (72) identifies that

$$\begin{aligned} (a_2,\Lambda_2)(a_1,\Lambda_1)\underline{f} &= S(A_2)^T S(A_1)^T \underline{f}(\Lambda_2^{-1}\Lambda_1^{-1}(x-a_1) - \Lambda_2^{-1}a_2) \\ &= (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)\underline{f} \\ &= S(A)^T \underline{f}(\Lambda_2^{-1}\Lambda_1^{-1}(x - a_1 - \Lambda_1 a_2)) \end{aligned}$$

from the group property and composition of Poincaré transformations (72). Then

$$S(A) = S(A_1)S(A_2) \tag{74}$$

results in the appropriate composition of $S(A)$ for a representation of the Poincaré group.

In a non-relativistic development, time is a universal parameter and many Hamiltonians are compatible with the non-relativistic scalar product [54]. In relativistic physics, time translations of states, and hence, the Hamiltonian, must comply with Poincaré invariance (71) of likelihoods. With the Fourier transform (21) and Poincaré transformations (72), time translations by λ are

$$\tilde{f}_n((p)_n)_{(\kappa)_n} \to \prod_{j=1}^{n} e^{-ip_{j0}\lambda}\, \tilde{f}_n((p)_n)_{(\kappa)_n}$$

in the n-argument subspace of the momentum domain. A state describing function $f(x_0 - \lambda, \mathbf{x})$ evaluates the field at an advanced time.

$$\int dx\ \Phi(x) f(x_0 - \lambda, \mathbf{x}) = \int dx\ \Phi(x_0 + \lambda, \mathbf{x}) f(x).$$

A subgroup of the Poincaré group, temporal translation is unitarily implemented,

$$U(\lambda) f((x)_n)_{(\kappa)_n} = f((x_0 - \lambda, \mathbf{x})_n)_{(\kappa)_n}. \tag{75}$$

The Fourier transform (21) has properties (23), and as a consequence,

$$e^{-ipy} \tilde{f}(p - q) \tag{76}$$

and

$$e^{iq(x-y)} f(x - y) \tag{77}$$

are a Fourier transform pair. The support for the function in (76) is translated from $f(x)$ in spacetime by $y \in \mathbb{R}^4$ and in energy-momentum by $q \in \mathbb{R}^4$. A change in sign of spatial and momentum translations follows from the Minkowski signature in Lorentz vector products $px = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}$.

A.4 follows from the observation that energy-momenta Lorentz vectors lie within the closed forward cone

$$\overline{V}^+ = \{p :\ p^2 \geq 0 \text{ and } p_0 \geq 0\}.$$

If $|p\rangle$ is a generalized eigenfunction of momentum with eigenvalue p, then for any state describing function g_k with k arguments,

$$\langle g_k | p \rangle = 0$$

if $p \notin \overline{V}^+$. As a consequence and illustrated below, the supports of the Fourier transforms of the generalized functions

$$\tilde{T}_n((p)_n)_{(\kappa)_n} = \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \widetilde{g_k^*}((p)_k)_{(\kappa)_k} \tilde{f}_{n-k}((p)_{k+1,n})_{(\kappa)_{k+1,n}} \tag{78}$$

are limited to $\mathcal{E}_n^+$ [9, 10, 58, 64]. $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are the VEV functions (14), $g_k \in \mathcal{P}_{(\kappa)_k}(\mathbb{R}^{4k})$, $f_{n-k} \in \mathcal{P}_{(\kappa)_{k+1,n}}(\mathbb{R}^{4n-4k})$ and

$$\mathcal{E}_n^+ = \{(p)_n : \ p_n \in \overline{V}^+, \ p_{n-1} + p_n \in \overline{V}^+, \ldots, \ p_2 + \ldots p_n \in \overline{V}^+, \ p_1 + p_2 + \ldots p_n = 0\}. \quad (79)$$

The support limitation follows from spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [41], chapters 7-10 [24]). The unitary spacetime translation operator in $\mathbf{H}_{\mathcal{P}}$ is

$$U(a) = \int dE(p) \ e^{-ipa}$$

with

$$dE(p) \sim dp \ |p\rangle\langle p|$$

a resolution of the identity in $\mathbf{H}_{\mathcal{P}}$. Every $p \in \overline{V}^+$. Unitarity of $U(a)$ follows from Poincaré invariance A.3 of the scalar product and Hermiticity of the densely defined generators of translations, the energy-momentum, appendix 8.3 and Stone's theorem. Translations of the fields are

$$U(a)\Phi(x_j)_{\kappa_j}U(a)^{-1} = \Phi(x_j + a)_{\kappa_j},$$

and the vacuum is translation invariant.

$$U(a)\Omega = \Omega.$$

Then, translation of arguments ℓ though n is expressed

$$\begin{aligned} &\langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell U(a)^{-1} \ldots U(a) \Phi_n U(a)^{-1} \Omega \rangle \\ &\quad = \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell \ldots \Phi_n \Omega \rangle \end{aligned}$$

with an abbreviated notation for fields $\Phi_j = \Phi(x_j)_{\kappa_j}$. The scalar product (6) with functions g_{n-k} of translated support for arguments ℓ through n provides

$$\begin{aligned} F(q) &= \int da \ e^{iqa} \int d(x)_n \ \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} g_{n-k} \\ &= \int da \int d(x)_n \int e^{iqa - ipa} \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} dE(p) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} f_{n-k} \\ &= (2\pi)^4 \int d(x)_n \int \delta(q-p) \ \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} dE(p) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} g_{n-k}. \end{aligned}$$

$F(q) = 0$ if $q \notin \overline{V}^+$ for each choice of n, k, ℓ and $(\kappa)_n$ from A.4 and the properties of the translation operator $U(a)$. From the equivalent expression (13) for scalar product, Parseval's equality (22) and the properties of Fourier transforms (23),

$$\begin{aligned} F(q) &= \int da \ e^{iqa} \int d(x)_n \ \mathcal{W}_{k,n-k}((x)_{\ell-1}, (x+a)_{\ell,n})_{(\kappa)_n} \ f_k^* g_{n-k} \\ &= \int da \int d(p)_n \ e^{iqa - i\sum_{j=\ell}^n p_j a} \ \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \ \widetilde{f_k^*} \tilde{g}_{n-k} \\ &= (2\pi)^4 \int d(p)_n \ \delta(q - \textstyle\sum_{j=\ell}^n p_j) \ \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \ \widetilde{f_k^*} \tilde{g}_{n-k}. \end{aligned}$$

Then, $F(q) = 0$ unless

$$\sum_{j=\ell}^{n} p_j \in \overline{V}^+$$

implies that the support of (78) includes only p_j from $\mathcal{E}_n^+$ (79). Development follows similarly for $\ell \leq k$ from the sesquilinearity of the scalar product (6). As a consequence, to satisfy the observation that all energy-momentum lie in the closed forward cone, the support of (78) includes only p_j from $\mathcal{E}_n^+$.

In Wightman's original axioms, the support of the state describing functions $\tilde{f}_k, \tilde{g}_{n-k}$ are not constrained and as a consequence, to satisfy the support limitation (78), the support of the VEV functions $\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}$ is limited to $\mathcal{E}_n^+$. In the emphasized constructions, the supports of the constructed VEV are limited to mass shells and the supports (18) of elements of $\underline{\mathcal{P}}$ limit the energy-momentum support to positive energies. Together, the joint support of VEV functions and functions in $\underline{\mathcal{P}}$ satisfies A.4 [34]. Support constrained functions enable joint satisfaction of local commutativity, positive energy support and Poincaré covariance, an unmet and apparently unattainable task within RQFT developments when interaction is manifest [4, 9, 32, 42]. Alternatively, the supports of the state describing functions are based on $\underline{\mathcal{S}}$ and unconstrained but the VEV functions do not satisfy the formal Hermiticity condition in the Wightman axioms, section 2.

Local commutativity is that linear combinations of VEV with transpositions of arguments conditionally vanish.

$$\langle \underline{f} | \Phi_1 \dots (\Phi_k \Phi_{k+1} \pm \Phi_{k+1} \Phi_k) \dots \Phi_n \Omega \rangle = 0$$

if the points x_k and x_{k+1} are space-like, $(x_k - x_{k+1})^2 < 0$, $1 \leq k < n$, and the notation is abbreviated, $\Phi_k = \Phi(x_k)_{\kappa_k}$. Space-like separations specify that the points in the supports of the quantum fields are not causally related. The sign is determined by particle statistics to satisfy normal commutation relations [9].[i] Verification of local commutativity uses functions of bounded support with space-like separated support, for example, tempered test functions $\underline{\mathcal{S}}$. In the constructions based on $\underline{\mathcal{P}}$, local commutativity reduces to split signed symmetry (44) of the VEV with exchanges of arguments or $*$-dual function arguments. The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ lack functions of bounded support and the constructed VEV exhibit unconditional split signed symmetry if based upon $\underline{\mathcal{P}}$, section 3.3 and [32, 34]. For the realizations emphasized in these notes, A.5 could be tightened to unconditional split signed symmetry of VEV but to include free field developments with the basis function spaces extended to $\underline{\mathcal{S}}$, only conditional local commutativity is required in A.5 [32].[j] Free field VEV are locally commutative but not split signed symmetric if the basis function space is $\underline{\mathcal{S}}$.

[i] For brevity, the possibility of parastatistics [9] is not discussed here.

[j] For a free field, the field $\Phi_o(f) = \Phi_o^+(f)$ with $\Phi_o^+(f)$ the creation operator if $f \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$, section 3.2 and appendix 8.7. The creation operators commute.

Condition A.6 is another manifestation of causality: great spatial separation implies independence of nonentangled local observables. A.6 is stronger than the condition used in [32, 34, 38] that was also designated as cluster decomposition. Earlier constructions in [32, 34, 38] satisfy formal Hermiticity described below. However, in section 3.3 it is illustrated that formal Hermiticity conflicts with cluster decomposition A.6 in constructions with nonzero spin and in some scalar field examples. Both conditions, A.6 or the cluster decomposition condition used in [32, 34, 38], imply that the vacuum is the sole translation-invariant state. In common with Wightman's development, formal Hermiticity was maintained in [38] and resulted in truncated functions that are not connected. In the earlier development that satisfied formal Hermiticity, the strong cluster decomposition condition A.6 is not satisfied. The constructions in section 3.3 abandon formal Hermiticity to establish the causal cluster decomposition property A.6. Cluster decomposition is a nonlinear condition and relates the description of interaction across orders of the VEV.

Condition A.6 implies that: the vacuum $|\Omega\rangle$ is the only translationally invariant state [34, 58]; that states with sufficiently isolated and space-like separated support are described by free particles, section 4.3; that the quantum field (29) elevates to Hilbert space operators, section 4.4; and the essential independence of the local observables of non-entangled, spatially distant bodies, appendices 8.2.8 and 8.2.9.

A.7 provides satisfaction of nonnegativity A.2 together with regularity A.1 for the constructions. However, satisfaction of A.7 is not necessary for regularity. Regularity and nonnegativity generally require that $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ for $n \geq 2$ in the constructions, section 4.2.6. Satisfaction of formal Hermiticity precludes setting $\mathcal{W}_{0,n} = \mathcal{W}_{1,n} = 0$ for $n \geq 2$ but kinematic constraints can result in $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ for nonzero $\mathcal{W}_{0,n}$ and $\mathcal{W}_{1,n}$. For realizations with only a single mass, $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ if $n \geq 2$ due to conservation of energy-momentum: neither the vacuum nor a single particle can create a cascade of particles with each particle of the same finite mass as the decaying particle.

A.7 applies for free field theory constructed with the basis function space $\underline{\mathcal{P}}$ [32] or with the modified free field VEV (68) and the basis function space $\underline{\mathcal{S}}$. A.7 does not hold for free field theory constructed with the basis function space $\underline{\mathcal{S}}$.

Confined particles, those that do not appear with their support isolated from the support of other arguments, have $\mathcal{F}_{1,1} = 0$ for the κ_j included in their description. A $|f_0|^2$ term ($\mathcal{W}_{0,0} = 1$) is required in any construction that includes a vacuum state. Elemental stability A.7 expresses that if a state is prepared with only one elementary particle, or together with A.6 that if one elementary particle is greatly isolated from the support of every other body, then the elementary particle is stable until it encounters another body. Whether or which elementary particles are confined is not determined by the axiom and a discussion of confinement awaits better understanding of bound states within the constructions. Confined species extends this development.

Wightman's original axioms [9, 10, 58, 64] are distinguished here by two additional as-

sumptions.[k] Wightman's axioms support the canonical formalism's conjectured elevation of classical fields to Hermitian quantum field operators, Wightman's prospective axioms include assumptions that imply the fields in the VEV (5) are Hermitian Hilbert space field operators:

W.a) *Formal Hermiticity:* the generalized functions that define the degenerate scalar product (13) satisfy

$$\mathcal{W}_{k,n-k}((x)_n))_{(\kappa)_n} = W_n((x)_n))_{(\kappa)_n}$$

independently of k. In this case, the degenerate scalar product is

$$\mathcal{W}(\underline{g}^*, \underline{f}) = \underline{W}(\underline{g}^* \times \underline{f})$$

for a Wightman functional $\underline{W}$ and the product (4).

W.b) *Involutivity:* $\underline{\mathcal{P}} = \underline{\mathcal{S}}$. Satisfaction of the axioms applies for all sequences of Schwartz tempered functions.

Together with A.1-2, satisfaction of W.a and W.b implies that the field is realized as densely defined Hermitian Hilbert space operators. Condition W.a results in the simplified form (15) for the scalar product. Sequences of tempered functions $\underline{\mathcal{S}}$ are a $*$-involutive algebra for the product (4) and involution (11). However, satisfaction of both conditions W.a-b preclude the constructed nontrivial realizations of relativistic quantum physics. Physically trivial examples, free fields, Wick polynomials and generalized free fields, satisfy both W.a and W.b. Either, but not both W.a and W.b can be satisfied by the physically nontrivial constructions in section 3.3. In section 3.4.2, it is demonstrated that involutivity W.b can be satisfied if formal Hermiticity W.a is abandoned: with a non-Hermitian embellishment to the VEV presented in section 3.3, the constructions can be based on $\underline{\mathcal{S}}$. In [32], it is demonstrated that W.a can be satisfied for a physically nontrivial neutral scalar field if involutivity W.b is abandoned.

The technically revised axioms A.1-7 result in substantial departures from established RQFT properties. Without the assumption that field operators are densely defined Hermitian operators, axioms A.1-7 are demonstrably realizable, several "no go" results [9, 48] for RQFT do not apply, and the RQFT demonstrations of PCT and spin-statistics theorems [9, 58] require review to include the physically nontrivial constructions. Without Hermitian field operators, analytic extensions of VEV functions to imaginary time (Wick rotations) are not necessarily symmetric with transpositions of arguments. Consequently, the analytic extensions of VEV functions to imaginary times (the Schwinger functions) are not necessarily the moment functions of a generalized random process [2, 40, 59]. The Osterwalder-Schrader axioms [59] for reconstruction of a RQFT from a Euclidean field theory do not apply to the constructed VEV.

[k] The assumptions are not called out as separate axioms but formal Hermiticity and involutivity are assumed within a statement of the axioms [9, 10, 58, 64].

4.2 Satisfaction of the axioms

The constructions in section 3 satisfy axioms A.1-A.7 in section 4.1. The established physical properties of quantum mechanics and relativity [9, 13, 62, 63] are satisfied.

4.2.1 Summary of constructions

The construction (58) of VEV functions $\underline{\mathcal{W}}$ with connected functions (42) suffices to satisfy axioms A.1-7. The constructions are explicit example nontrivial realizations of relativistic quantum physics. While sufficient to satisfy A.1-7, the selected forms are not necessary to satisfaction of A.1-7. The constructed VEV satisfy A.1-7 for states described by function sequences from $\mathbf{H}_{\mathcal{P}}$, the completion in the Hilbert space norm (25) of the basis function spaces $\underline{\mathcal{P}}$ that satisfy (7). Or equivalently, the construction is for function sequences from $\underline{\mathcal{S}}$ with modified VEV, section 3.4.2. For the constructed VEV:

A.1: Demonstrations of regularity apply if the factors $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ in the connected functions (42) after evaluation of the energy-momentum support delta functions are polynomially bounded growth, locally Lebesgue-summable functions of the momenta within the domains (79), masses are finite, and the number of spacetime dimensions equals or exceeds three (2+1). Massless particles require four (3+1) or more spacetime dimensions and additional constraints on the support of functions in $\underline{\mathcal{P}}$, [37] and section 4.2.6. Regularity requires that the singularities of nonconstant $\mathfrak{h}(p_1, p_2)$ in the VEV (55) with (49) and (50) are excluded from the support of the scalar product (13), and that the divergent extrapolation of the connected functions (49) to two-point function is eliminated (70). A demonstration of regularity is in section 4.2.6.

A.2: The nonnegativity of the scalar product (13) follows from the construction (57), and the nonnegativity of $\circ$-products of nonnegative constituent sequences of generalized functions $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, section 4.2.2. The nonnegativity of $\underline{\mathcal{F}}$ is well-established [9, 58] and demonstrated in the current context in appendix 8.7. The nonnegativity of $\underline{\mathcal{U}}$ is demonstrated in section 4.2.3 and follows from the nonnegativity of ${}^{C}\underline{\mathcal{U}}$, the cluster expansion (55), and that the $\circ$-product preserves the nonnegativity and split signed symmetry of function sequences.

A.3: Section 4.2.4 includes a demonstration of relativistic invariance (71). Relativistic invariance follows from the relativistic covariance of $\underline{\mathcal{F}}$ and the expansions (49), (50) and (55) for $\underline{\mathcal{U}}$. The array triple $S(A)$, $M(p)$ and D from an underlying free field $\underline{\mathcal{F}}$ establishes the realization of the Lorentz group for a construction. Translation invariance follows from energy-momentum conservation.

A.4: The limitation of the support of VEV to mass shells together with the zeros (18) on negative energy mass shells in the support of the functions from the basis spaces $\underline{\mathcal{P}}$

implements the spectral support condition. In the evaluation of scalar products (13), either

$$p_j + p_{j+1} \ldots + p_n \in \overline{V}^+$$

explicitly since each $p_\ell \in \overline{V}^+$ and convex cones are closed under addition, or energy-momentum conservation provides that this sum of energy-momenta is the negative of $p_1 + p_2 \ldots + p_{j-1}$ with each of $p_1, \ldots p_{j-1}$ explicitly in the closed backward cone. Demonstrations of spectral support are included in [34, 38].

A.5: For constructions based on function sequences $\underline{\mathcal{P}}$ from section 3.4, local commutativity follows from the split signed symmetry (44) of the constructed VEV.

A.6: Cluster decomposition follows from the connectedness of the functions ${}^C\underline{\mathcal{W}}$ in (42) and the cluster expansion (58) for the VEV functions $\underline{\mathcal{W}}$. The demonstration of cluster decomposition is in section 4.2.5.

A.7: VEV function sequences that satisfy A.1-6 determine sequences $\underline{\mathcal{W}}$ that also satisfy elemental stability A.7, (70). Elemental stability follows for mean zero fields if $\mathcal{W}_{k,0} = \mathcal{W}_{0,k} = \mathcal{W}_{k,1} = \mathcal{W}_{1,k} = 0$ for $k \geq 2$. For mean zero fields, $\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0$. If kinematic constraints do not suffice to satisfy elemental stability, then the constructions must violate formal Hermiticity W.a. The introduction of non-zero expectation value fields preserves satisfaction of A.1-7 with the appropriately restated A.7, section 3.3.5.

Free fields, Wick polynomials of free fields and generalized free fields satisfy formal Hermiticity W.a with a basis function space $\underline{\mathcal{S}}$. Satisfaction of A.1-6 with both formal Hermiticity W.a and involutivity W.b appears to be peculiar to physically trivial VEV.

There is a realization of relativistic quantum physics for: every realization of free fields (selection of N_c and a triple of $N_c \times N_c$ arrays $M(p), D, S(A)$ with the $S(A)$ a representation of the Lorentz group (74)); nonnegative measure $d\sigma(\lambda)$ (that determines the c_n, the relative contributions of connected functions); Lorentz invariant measure $d\mu_{\mathfrak{B}}(p)$ (that characterize interactions); and Lorentz invariant and nonnegative measure $d\mu_{\Upsilon}(p)$, and constant α_o (that characterize interactions). In constructions with multiple species, distinct α_o, $d\mu_{\mathfrak{B}}(p)$, $d\mu_{\Upsilon}(p)$ and coupling constants c_n can be introduced for each indecomposable array triple $M(p), D, S(A)$. Decomposable triples $M(p), D, S(A)$ are discussed in appendix 8.12.

Generalizations to the constructions presented include odd order scalar field VEV [34], additional forms in the expansion (53) for connected functions in terms of the matrices $\mathfrak{B}(p)$ and $\Upsilon(p)$, extension of $d\mu_{\mathfrak{B}}(s)$ to complex-valued measures [34, 36], spaces of generalized functions in addition to $\underline{\mathcal{S}}$ [29], and the inclusion of massless particles, [37] and section 4.2.6.

4.2.2 Nonnegative $\circ$-products

The $\circ$-product of two VEV function sequences (27) that each provide a degenerate scalar product (13) provides a degenerate scalar product. The demonstration follows in this section.

By definition (60), the $\circ$-product $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is the split signed symmetrization (45) of a sequence $\underline{w}$. From (46), if $\underline{w}$ provides a degenerate scalar product, then $\underline{\mathcal{W}}$ provides a degenerate scalar product. With $\underline{w}$ from (60), it is demonstrated that $w(\underline{f}^*, \underline{g})$ is the degenerate scalar product of elements $\underline{f}, \underline{g}$ in the tensor product of two linear vector spaces. These linear vector spaces have degenerate scalar products specified by the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$, respectively. The preservation of nonnegativity for scaled sequences (47) provides that $\underline{\mathcal{T}_a}(\underline{f}^*, \underline{g}) \geq 0$ and $\underline{\mathcal{V}_a}(\underline{f}^*, \underline{g}) \geq 0$ with $\underline{\mathcal{T}}_a$ and $\underline{\mathcal{V}}_a$ scaled versions of the sequences $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$. Linear vector spaces with degenerate scalar products are also designated pre-Hilbert spaces.

If both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ provide degenerate scalar products, then the tensor product of the pre-Hilbert spaces that result from scaled $\underline{\mathcal{V}}$ and $\underline{\mathcal{T}}$ has elements labeled $\underline{f} \otimes \underline{g}$ and a degenerate scalar product

$$w((\underline{f}_1 \otimes \underline{f}_2)^*, \underline{g}_1 \otimes \underline{g}_2) = \mathcal{T}_a(\underline{f}_1^*, \underline{g}_1)\, \mathcal{V}_a(\underline{f}_2^*, \underline{g}_2). \tag{80}$$

The scales (47) on the degenerate scalar products are $a_\ell = 1/\ell!$ in both instances. In the pre-Hilbert spaces of function sequences based on $\underline{\mathcal{P}}$ for generalized functions $\underline{\mathcal{T}_a}$ and $\underline{\mathcal{V}_a}$, fields $\Phi_\mathcal{T}$ and $\Phi_\mathcal{V}$ are defined (29). These operations extend to the tensor product space. The elevations $\Phi_\mathcal{T} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_\mathcal{V}$ to the tensor product space are

$$(\Phi_\mathcal{T} \otimes \mathbb{I})\underline{f} \otimes \underline{g} = (\Phi_\mathcal{T} \underline{f}) \otimes \underline{g}$$

and

$$(\mathbb{I} \otimes \Phi_\mathcal{V})\underline{f} \otimes \underline{g} = \underline{f} \otimes (\Phi_\mathcal{V} \underline{g}).$$

The composite vacuum is $\Omega = \Omega_\mathcal{T} \otimes \Omega_\mathcal{V}$. To condense notation, the arguments of the fields are understood. The demonstration that the $\circ$-product preserves nonnegativity is a variation of a construction due to Borchers and Uhlmann [25, 48]. $|O(f_n)\Omega\rangle$ with

$$O(f_n) = \sum_{(\kappa)_n} \int d(x)_n \left(\sum_{\ell=0}^{n} \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!} \right) f_n((x)_n)_{(\kappa)_n}$$

are elements of the tensor product space with

$$(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell} = \left(\prod_{j=1}^{\ell} (\Phi_\mathcal{T}(x_j)_{\kappa_j} \otimes \mathbb{I}) \right) \left(\prod_{j=\ell+1}^{n} (\mathbb{I} \otimes \Phi_\mathcal{V}(x_j)_{\kappa_j}) \right).$$

In the tensor product space, $\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle$ is the degenerate scalar product for function sequences $\underline{f}, \underline{g}$ from $\underline{\mathcal{P}}$.

$$\begin{aligned} \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \\ \times \langle \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!}\, \Omega | \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^j (\mathbb{I} \otimes \Phi_\mathcal{V})^{m-j}}{j!(m-j)!}\, \Omega \rangle \\ \times \overline{f_n((x)_n)}_{(\kappa)_n}\, g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned}$$

The definition of scalar product, the operators $\Phi_{\mathcal{T}}(x_j)_{\kappa_j} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_{\mathcal{V}}(x_j)_{\kappa_j}$, the $*$-dual (11) and the relation between VEV and VEV functions (14) result in

$$\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \frac{\mathcal{V}_{n-\ell,m-j}(A'_o)}{(n-\kappa)!(m-j)!} \times f_n((x)_n)^*_{(\kappa)_n} g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

for A_o, A'_o from (61). Then substitution of the expression (60) for functions $w_{k,n-k}$ identifies that

$$w(\underline{f}^*, \underline{g}) = \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle. \tag{81}$$

As a consequence. $w(\underline{f}^*, \underline{f}) \geq 0$ since it is the squared norm of a vector in the tensor product space.

Satisfaction of nonnegativity A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ follows from (46), (47) and (60) if both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ satisfy A.1-2. For $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$,

$$\mathcal{W}(\underline{f}^*, \underline{f}) \geq 0$$

if $\underline{\mathcal{U}}$ satisfies nonnegativity since it is established that $\underline{\mathcal{F}}$ is positive, [9, 58] and appendix 8.7. Satisfaction of nonnegativity A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ reduces to demonstration that the $\underline{\mathcal{U}}$ in the cluster expansion (55) of the connected functions (49) satisfies nonnegativity.

4.2.3 Nonnegativity of $\underline{\mathcal{U}}$

In this section it is demonstrated that the constructed split signed symmetric sequences (27) of generalized functions $\underline{\mathcal{U}}$ from the cluster expansion (55) of the connected functions ${}^C\underline{\mathcal{U}}$ in (49) provides a degenerate scalar product (13) for state describing function sequences from $\underline{\mathcal{P}}$. With the result of section 4.2.2 that the $\circ$-products of sequences that each provide a degenerate scalar product (13) provides a degenerate scalar product, the nonnegativity of $\underline{\mathcal{U}}$ provides that $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ satisfies nonnegativity A.2.

A demonstration of the nonnegativity of ${}^C\underline{\mathcal{U}}(\underline{f}^*, \underline{f})$ suffices to demonstrate the nonnegativity of $\underline{\mathcal{U}}(\underline{f}^*, \underline{f})$. The construction (55) of $\underline{\mathcal{U}}$ as $\exp \circ ({}^C\underline{\mathcal{U}})$, a positively weighted summation of the $\circ$-products of generalized function sequences that provide degenerate scalar products, provides that $\underline{\mathcal{U}}$ satisfies nonnegativity if ${}^C\underline{\mathcal{U}}$ satisfies nonnegativity, section 4.2.2. The sequences $\underline{f} \in \underline{\mathcal{P}}$ are terminating and the number of contributing terms in scalar products (13) and the $\circ$-exponentials are therefore finite.

From substitution of the description (50) of $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ into the construction (49) of the connected VEV functions, the connected VEV functions ${}^C\mathcal{U}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}}$ are the split

signed symmetrization (45) of

$$\tilde{\mathfrak{u}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} = \iint \frac{d\sigma(\lambda)du}{(2\pi)^4}\, \lambda^{n+m} \prod_{k=1}^{n+m} \left(e^{-ip_k u}\, \delta(p_k^2 - \lambda_{ck}^{-2})\right)\, \mathfrak{q}_{n,n+m}((p)_{n+m})_{(\kappa)_{n+m}} \tag{82}$$

with

$$\mathfrak{q}_{n,n+m}((p)_{n+m})_{(\kappa)_{n+m}} = \prod_{j=1}^{n+m} \left(\frac{d}{d\rho_j}\right) \exp\Big(\sum_{a,b\in\mathbb{J}_{n,n+m}} \rho_a \rho_b\, \mathfrak{h}(p_a, p_b)_{\kappa_a\kappa_b}\Big)$$

from (50) evaluated at $(\rho)_{n+m} = 0$ after differentiation. The split signed symmetrization (45) of a sequence that provides a degenerate scalar product also provides a degenerate scalar product (46). Then, ${}^C\mathcal{U}(\underline{f}^*, \underline{f}) \geq 0$ if $\mathfrak{u}(\underline{f}^*, \underline{f}) \geq 0$.

$$\begin{aligned}\mathfrak{u}(\underline{f}^*, \underline{f}) \;&= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(p)_{n+m}\; (D\cdot)_n \tilde{\mathfrak{u}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times \overline{\tilde{f}_n}(-p_n, \ldots - p_1)_{\kappa_n \ldots \kappa_1}\, \tilde{f}_m(p_{n+1}, \ldots p_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned}$$

follows from (13). This scalar product is expressed using Fourier transforms (21), the Parseval's equality (22) definition for the Fourier transform of generalized functions, and the definition (11) for $*$-dual functions. The Fourier transform (21) of a $*$-dual function is

$$\int \frac{d(x)_n}{(2\pi)^{2n}}\, e^{-ip_1x_1} \ldots e^{-ip_nx_n} f^*((x)_n)_{(\kappa)_n} = (D^T\cdot)_n \overline{\tilde{f}_n}((-p)_{n,1})_{(\kappa)_{n,1}}.$$

Relabeling summation variables $p_j \mapsto -p_{n-j+1}$ and $\kappa_j \mapsto \kappa_{n-j+1}$ for $j = 1, \ldots n$,

$$\begin{aligned}\mathfrak{u}(\underline{f}^*, \underline{f}) \;&= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(p)_{n+m}\; \mathcal{M}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times \overline{\tilde{f}_n}((p)_n)_{(\kappa)_n}\, \tilde{f}_m(p_{n+1}, \ldots p_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned} \tag{83}$$

with

$$\mathcal{M}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} = (D\cdot)_n \tilde{\mathfrak{u}}_{n,m}(-p_n, \ldots - p_1, p_{n+1}, \ldots p_{n+m})_{\kappa_n, \ldots \kappa_1, \kappa_{n+1}, \ldots \kappa_{n+m}}. \tag{84}$$

Then, if the array of functions $\mathcal{M}_{n,m}$ in (83) factors nonnegatively,

$$\mathcal{M}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} = \int d\mu_{\mathcal{M}}(\mathfrak{s})\; \overline{F_{\mathcal{M},n}(\mathfrak{s}, (p,\kappa)_n)}\, F_{\mathcal{M},m}(\mathfrak{s}, (p,\kappa)_{n+1,n+m}) \tag{85}$$

with $d\mu_{\mathcal{M}}(\mathfrak{s})$ a nonnegative measure on a set of variables $\mathfrak{s}$, then $\mathfrak{u}(\underline{f}^*, \underline{f}) \geq 0$. In this instance,

$$\mathfrak{u}(\underline{f}^*, \underline{f}) = \int d\mu_m(\mathfrak{s})\; |G_{\mathcal{M}}(\mathfrak{s})|^2$$

with

$$G_{\mathcal{M}}(\mathfrak{s}) \;\; = \sum_{n} \sum_{(\kappa)_n} \int d(p)_n \; F_{\mathcal{M},n}(\mathfrak{s}, (p,\kappa)_n) \, \tilde{f}_n((p)_n)_{(\kappa)_n}.$$

The $m, (p,\kappa)_{n+1,n+m}$ are summation variables independent of the $n, (p,\kappa)_n$.

The array of functions $\mathcal{M}_{n,m}$ factors nonnegatively. The factorization follows from: the form (50) of $\mathfrak{q}_{n,n+m}$; the relation (84) of $\mathcal{M}_{n,m}$ and $\mathfrak{q}_{n,n+m}$; the definition (51) of $\mathfrak{h}(p_a, p_b)_{\kappa_a\kappa_b}$; the factorization (53) in section 3.3.3,

$$\begin{aligned}(D\cdot)_n \exp\Big(\sum_{a,b\in\mathbb{J}_{n,n+m}} \rho_a\rho_b\mathfrak{h}_{\kappa_a\kappa_b}\Big) &= \exp\Big(\sum_{b_1>a_1=1}^{n} \rho_{a_1}\rho_{b_1}(D\mathfrak{B}(p_{a_1}+\alpha_o p_{b_1})D^T)_{\kappa_{a_1}\kappa_{b_1}}\Big) \\ \times \exp\Big(\sum_{b_2>a_2=n+1}^{n+m} \rho_{a_2}\rho_{b_2}\mathfrak{B}(\alpha_o p_{a_2}+p_{b_2})_{\kappa_{a_2}\kappa_{b_2}}\Big) \; &\exp\Big(\sum_{\jmath=1}^{\infty}\rho_{n+1-\jmath}\rho_{n+\jmath}(D\Upsilon)_{\kappa_{n+1-\jmath}\kappa_{n+\jmath}}\Big);\end{aligned} \tag{86}$$

the definitions (52) of $\mathfrak{B}(p)_{\kappa_a\kappa_b}$ and $\Upsilon(p)_{\kappa_a\kappa_b}$; and properties (33) and (34) of the array $M(p)_{\kappa_1\kappa_2}$ from the two-point function (32).

Substitution of the identity (34) for transformations of $M(p)$ into the definition (52) of $\mathfrak{B}(p)$ provides that

$$D\mathfrak{B}(p)D^T = \mathfrak{B}(-p)^* \tag{87}$$

with the Hermitian transpose

$$\mathfrak{B}^*(-p) = \overline{\mathfrak{B}(-p)}^T.$$

Then, substitution of (87) into the first factor of (86) results in

$$\exp\Big(\sum_{b_1>a_1=1}^{n} \rho_{a_1}\rho_{b_1}(D\mathfrak{B}D^T)_{\kappa_{a_1}\kappa_{b_1}}\Big) = \exp\Big(\sum_{b_1>a_1=1}^{n} \rho_{a_1}\rho_{b_1}\overline{\mathfrak{B}(-p_{a_1}-\alpha_o p_{b_1})}_{\kappa_{b_1}\kappa_{a_1}}\Big).$$

With the relabeling of the $*$-dual summation variables $p_j \mapsto -p_{n-j+!}$ and $\kappa_j \mapsto \kappa_{n-j+!}$ for $j = 1, \ldots n$, if the summation indices $a_1, b_1 \in \{1, n\}$ are relabeled $a_1 \mapsto b_1$ and $b_1 \mapsto a_1$, then the first factor in the factorization (86) becomes

$$\exp\Big(\sum_{b>a=1}^{n} \rho_a\rho_b(D\mathfrak{B}(p_a+\alpha_o p_b))D^T)_{\kappa_a\kappa_b}\Big) \mapsto \exp\Big(\sum_{b>a=1}^{n} \rho_a\rho_b\overline{\mathfrak{B}(\alpha_o p_a+p_b))}_{\kappa_a\kappa_b}. \tag{88}$$

This is the same form as the second factor of (86) except with independent variables $n, (p,\kappa)_n$ replacing $m, (p,\kappa)_{n+1,n+m}$, and complex conjugation.

In the third, last factor of (86), the nonnegative matrix $DM(s) = C^*(s)C(s)$ from (33) substituted into (52) provides that

$$D\Upsilon(p) = \int d\mu_\Upsilon(s) \; e^{-sp} C^*(s)C(s). \tag{89}$$

The exponential function expands.

$$\exp(\sum_{\jmath=1}^{\infty}\rho_{n+1-\jmath}\rho_{n+\jmath}(D\Upsilon)_{\kappa_{n+1-\jmath}\kappa_{n+\jmath}}) = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\sum_{\jmath=1}^{\infty}\rho_{n+1-\jmath}\rho_{n+\jmath}(D\Upsilon(-p_{n+1-\jmath}+p_{n+\jmath}))_{\kappa_{n+1-\jmath}\kappa_{n+\jmath}}\right)^N .$$

Then, the identity for summation,

$$\sum_a\sum_b x_a y_b = \left(\sum_a x_a\right)\left(\sum_b y_b\right), \tag{90}$$

and linearity enables reordering of the finite summations with the integration in (89). Only a finite number of the ρ_j are non-zero, those with $j \in \{1, n+m\}$. Substitution of (89) results in

$$\sum_{\jmath=1}^{\infty}\rho_{n+1-\jmath}\rho_{n+\jmath}(D\Upsilon(-p_{n+1-\jmath}+p_{n+\jmath})))_{\kappa_{n+1-\jmath}\kappa_{n+\jmath}} = \sum_{\jmath_1=1}^{\infty}\sum_{\ell_1=1}^{N_c}\int d\mu_\Upsilon(s_1)\left(\rho_{n+1-\jmath_1}e^{s_1 p_{n+1-\jmath_1}}\overline{C(s_1)}_{\ell_1\kappa_{n+1-\jmath_1}}\right)\left(\rho_{n+\jmath_1}e^{-s_1 p_{n+1-\jmath_1}}C(s_1)_{\ell_1\kappa_{n+\jmath_1}}\right)$$

using the expression for Hermitian transpose $C(s)^*$. Similarly,

$$\left(\sum_{\jmath=1}^{\infty}\rho_{n+1-\jmath}\rho_{n+\jmath}(D\Upsilon(-p_{n+1-\jmath}+p_{n+\jmath}))_{\kappa_{n+1-\jmath}\kappa_{n+\jmath}}\right)^N = \sum_{\jmath_1=1}^{\infty}\sum_{\ell_1=1}^{N_c}\int d\mu_\Upsilon(s_1)\ldots \times\sum_{\jmath_N=1}^{\infty}\sum_{\ell_N=1}^{N_c}\int d\mu_\Upsilon(s_N)\prod_{\nu=1}^{N}\left(\rho_{n+1-\jmath_\nu}e^{s_\nu p_{n+1-\jmath_\nu}}\overline{C(s_\nu)}_{\ell_\nu\kappa_{n+1-\jmath_\nu}}\right)\left(\rho_{n+\jmath_\nu}e^{-s_\nu p_{n+\jmath_\nu}}C(s_\nu)_{\ell_\nu\kappa_{n+\jmath_\nu}}\right).$$

The relabeling of $*$-dual summation variables $p_j \mapsto -p_{n-j+1}$ and $\kappa_j \mapsto \kappa_{n-j+1}$ for $j = 1,\ldots n$, provides that the third factor of (86) results from terms

$$\left(\sum_{\jmath=1}^{\infty}\rho_\jmath\rho_{n+\jmath}(D\Upsilon(p_\jmath+p_{n+\jmath}))_{\kappa_\jmath\kappa_{n+\jmath}}\right)^N = \sum_{\jmath_1=1}^{\infty}\sum_{\ell_1=1}^{N_c}\int d\mu_\Upsilon(s_1)\ldots \times\sum_{\jmath_N=1}^{\infty}\sum_{\ell_N=1}^{N_c}\int d\mu_\Upsilon(s_N)\prod_{\nu=1}^{N}\left(\rho_{\jmath_\nu}e^{-s_\nu p_{\jmath_\nu}}\overline{C(s_\nu)}_{\ell_\nu\kappa_{\jmath_\nu}}\right)\left(\rho_{n+\jmath_\nu}e^{-s_\nu p_{n+\jmath_\nu}}C(s_\nu)_{\ell_\nu\kappa_{n+\jmath_\nu}}\right). \tag{91}$$

substituted into the series expansion of the exponential function.

Substitution of (88) and (91) into (84) provides the nonnegative factorization of the array of functions $\mathcal{M}_{n,m}$. From linearity, the expression (84) for $\mathcal{M}_{n,m}$, (50) for $\mathfrak{q}_{n,n+m}$, the factorization (53) in section 3.3.3 and the properties of the array $M(p)_{\kappa_1\kappa_2}$ in the two-point function (32), substitution of (88) and (91) results in

$$\begin{aligned}\mathcal{M}_{n,m} = \iint \frac{d\sigma(\lambda)du}{(2\pi)^4} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\jmath_1=1}^{\infty} \sum_{\ell_1=1}^{N_c} \int d\mu_\Upsilon(s_1) \dots \sum_{\jmath_N=1}^{\infty} \sum_{\ell_N=1}^{N_c} \int d\mu_\Upsilon(s_N) \\ \times \overline{F_{\mathcal{M},n}(\lambda, u, N, (s,\ell,\jmath)_N, (p,\kappa)_n)}\; F_{\mathcal{M},m}(\lambda, u, N, (s,\ell,\jmath)_N, (p,\kappa)_{n+1,n+m})\end{aligned} \tag{92}$$

with

$$\begin{aligned}F_{\mathcal{M},n}(\lambda, u, N, (s,\ell,\jmath)_N, (p,\kappa)_n) = \lambda^n \prod_{j=1}^{n} \left(e^{-ip_j u}\, \delta(p_j^2 - \lambda_{cj}^{-2})\, \frac{d}{d\rho_j} \right) \\ \times \exp\Big(\sum_{b>a=1}^{n} \rho_a \rho_b \mathfrak{B}(\alpha_o p_a + p_b)_{\kappa_a\kappa_b} \Big) \prod_{\nu=1}^{N} \left(\rho_{\jmath_\nu} e^{-s_\nu p_{\jmath_\nu}} C(s_\nu)_{\ell_\nu \kappa_{\jmath_\nu}} \right).\end{aligned}$$

evaluated at $(\rho)_n = 0$ after differentiation. The $p_{\jmath_\nu}$ in $F_{\mathcal{M},n}$ include $p_1, p_2 \dots p_n$. The $n, (p,\kappa)_n$ and $m, (p,\kappa)_{n+1,n+m}$ are independent variables, and in $F_{\mathcal{M},m}$, the parameters $\rho_{\jmath_\nu}$ for the differentiations are independent of the $\rho_{\jmath_\nu}$ in $F_{\mathcal{M},n}$.

Inspection of (92) provides the variables

$$\mathfrak{s} = \lambda, u, N, (s,\ell,\jmath)_N$$

and the measure on $\mathfrak{s}$ with nonnegative weight is

$$d\mu_{\mathcal{M}}(\mathfrak{s}) = \iint \frac{d\sigma(\lambda)du}{(2\pi)^4} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\jmath_1=1}^{\infty} \sum_{\ell_1=1}^{N_c} \int d\mu_\Upsilon(s_1) \dots \sum_{\jmath_N=1}^{\infty} \sum_{\ell_N=1}^{N_c} \int d\mu_\Upsilon(s_N)$$

with $d\sigma(\lambda)$ from (48) and $d\mu_\Upsilon(s)$ from (52). This demonstrates the nonnegative factorization (85) of the array of functions $\mathcal{M}_{n,m}$.

Finally, $\mathcal{U}(\underline{f}^*, \underline{f})$ is nonnegative as a consequence of the definition (55) of

$$\underline{\mathcal{U}} = \exp \circ ({}^C\underline{\mathcal{U}}),$$

the expansion (56) of the $\circ$-exponential, the nonnegativity and split signed symmetry preserving properties of the $\circ$-product, section 4.2.2, and the nonnegativity preservation (46) of split signed symmetrization (45).

4.2.4 Poincaré invariance

The Poincaré covariance of the constructed VEV is established in [38]. Invariance of the degenerate scalar product (13) follows from the conservation of energy-momentum and properties (37) of the representations of the Lorentz group. Compliant example representations of the Lorentz group are included in section 3.2.2.

Poincaré invariance of likelihoods is that

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle \underline{g}|\underline{f}\rangle,$$

for Poincaré transformations (72) of the state describing functions,

$$(a,\Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} = \prod_{k=1}^{n} e^{-ipa}\,(S(A)^T\cdot)_n\tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n}.$$

The scalar product (13), the Fourier transform (21), the Parseval's equality definition for the Fourier transform of a generalized function (22), and the dual of functions (11) result in

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\,(D\cdot)_n\widetilde{\mathcal{W}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}}\prod_{\ell=1}^{n+m}e^{-ip_\ell a}$$
$$\times\overline{(S(A)^T\cdot)_n\tilde{g}_n((-\Lambda^{-1}p)_{n,1})}_{\kappa_{n,1}}(S(A)^T\cdot)_{n+1,n+m}\tilde{f}_m((\Lambda^{-1}p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

in the matrix notation (12). Translation invariance is verified by noting that the support of $\widetilde{\mathcal{W}}_{k,n-k}$ includes only the surface with energy-momentum conserved; $p_1+p_2+\ldots p_n = 0$ from the connected VEV functions (42) and the cluster expansion (65). Energy-momentum conservation, $p_1 + p_2 + \ldots p_n = 0$, implements translation invariance.

Reordering summations, substitution of $\overline{S(A)}D = DS(A)$ from (37), the indicated substitutions $\Lambda^{-1}p_j \mapsto p_j$ for summation variables, and that the determinant of the Lorentz transformation Λ is unity results in

$$\begin{aligned}\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\,(D\cdot)_n(S(A)\cdot)_{n+m}\widetilde{\mathcal{W}}_{n,m}((\Lambda p)_{n+m})_{(\kappa)_{n+m}}\\ &\qquad\times\overline{\tilde{g}_n}((-p)_{n,1})_{\kappa_{n,1}}\tilde{f}_m((p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}\\ &= \langle \underline{g}|\underline{f}\rangle\end{aligned}$$

if

$$(S(A)\cdot)_n\widetilde{\mathcal{W}}_{k,n-k}((p)_n) = \widetilde{\mathcal{W}}_{k,n-k}((\Lambda^{-1}p)_n).$$

There is exactly one factor of $S(A)$ for each field component argument κ_j and consequently two factors of $S(A)$ for every factor of $M(p_\ell)_{\kappa_j\kappa_\ell}$ or $\mathfrak{h}(p_j,p_\ell)_{\kappa_j\kappa_\ell}$ in each term in $\mathcal{W}_{k,n-k}$.

By construction, the $N_c \times N_c$ matrices $M(p)$ in (32) and the $\mathfrak{h}(p_1, p_2)$ from (51) and (52) transform with the same representation of the Lorentz group.

$$((S(A)\cdot)_2 M(p)) = S(A)M(p)S(A)^T$$

from (12), and from condition (37),

$$S(A)M(p)S(A)^T = M(\Lambda^{-1}p).$$

Then if $\mathfrak{h}(p_1, p_2) = \mathfrak{B}(\alpha_o p_1 + p_2)$,

$$\begin{aligned} S(A)\mathfrak{h}(p_1, p_2)S(A)^T &= \int d\mu_{\mathfrak{B}}(s)\, e^{is(\alpha_o p_1 + p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_{\mathfrak{B}}(s)\, e^{is(\alpha_o p_1 + p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_{\mathfrak{B}}(s')\, e^{is'\Lambda^{-1}(\alpha_o p_1 + p_2)} M(s') \\ &= \mathfrak{h}(\Lambda^{-1}p_1, \Lambda^{-1}p_2) \end{aligned}$$

with the change of variables $s' = \Lambda^{-1}s$ and similarly if $\mathfrak{h}(p_1, p_2) = \mathfrak{B}(p_1 + \alpha_o p_2)$. If $\mathfrak{h}(p_1, p_2) = \Upsilon(-p_1 + p_2)$,

$$\begin{aligned} S(A)\mathfrak{h}(p_1, p_2)S(A)^T &= \int d\mu_{\Upsilon}(s)\, e^{s(p_1 - p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_{\Upsilon}(s)\, e^{s(p_1 - p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_{\Upsilon}(s')\, e^{s'\Lambda^{-1}(p_1 - p_2)} M(s') \\ &= \mathfrak{h}(\Lambda^{-1}p_1, \Lambda^{-1}p_2). \end{aligned}$$

These results follow from Lorentz invariance of the measures $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$, the change of variables $s' = \Lambda^{-1}s$, the Lorentz invariance of the Minkowski signature $\Lambda^T g \Lambda = g$, and that $ps = p^T gs$ is a Lorentz scalar. The generalized functions $\delta(p_{i_1} + \ldots p_{i_n})$ and $\delta(p_k^2 - \lambda_{ck}^{-2})$ are Lorentz scalars. In all three cases, each κ_j uniquely associates with a factor of M, $\mathfrak{B}$ or Υ, apply in every term in the cluster expansion (65) with (49), (50), (55) and (54). Each factor of M or $\mathfrak{h}$ introduces paired indices κ_a, κ_b with $a \neq b$, and as a consequence, with the exception of Lorentz scalar fields, only even order VEV appear in the constructions.

The constructed VEV provide a Poincaré invariant scalar product (13) that satisfies relativistic invariance A.3.

4.2.5 Cluster decomposition

Satisfaction of cluster decomposition A.6 for the constructions is demonstrated in this section. Satisfaction of the strong form of cluster decomposition A.6 follows from the connectivity of the

truncated functions (42) and the cluster expansion (65). This is an established result in RQFT [9, 50] but the constructed $\underline{\mathcal{U}}$ do not generally satisfy the Wightman axioms. A.6 provides that

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \longrightarrow \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f})$$

as the supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the supports of $\underline{g}, \underline{f}$.

Labeling summation variables in scalar products (13) with $\mathcal{W}(\underline{\psi}^*, \underline{\varphi})$ labeled by n, m and $(x, \kappa)_{n+m}$, and $\mathcal{W}(\underline{g}^*, \underline{f})$ labeled by ℓ, k and $(y, \mu)_{\ell+k}$ results in

$$\begin{aligned}\mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f}) = \sum_{n.m} \sum_{\ell,k} \sum_{(\kappa)_{n+m}} \sum_{(\mu)_{\ell+k}} \iint d(x)_{n+m} d(y)_{\ell+k} \\ \times (D\cdot)_n\, \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\, (D\cdot)_\ell\, \mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}} \\ \times \overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}} \varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}} \overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}} f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{93}$$

The $\times$-product of function sequences (4), the $*$-dual of function sequences (11), and labeling summation variables with the same designations for ψ, φ and g, f as (93) results in

$$\begin{aligned}\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) = \sum_{n.m} \sum_{\ell,k} \sum_{(\kappa)_{n+m}} \sum_{(\mu)_{\ell+k}} \iint d(x)_{n+m} d(y)_{\ell+k} \\ \times (D\cdot)_{n+\ell}\, \mathcal{W}_{n+\ell,m+k}((x)_n, (y)_\ell, (x)_{n+1,n+m}, (y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ \times \overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}} \varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}} \overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}} f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{94}$$

The assignment of summation variables partitions the arguments into two volumes P and P' suitable to test satisfaction of cluster decomposition (69). P consists of points $(x)_{n+m}$ within the dominant support of $\psi_n \varphi_m$ and P' consists of points $(y)_{\ell+k}$ within the dominant support of $g_\ell f_k$. To test cluster decomposition (69), the supports of ψ_n and φ_m are taken to be arbitrarily distantly space-like separated from the supports of g_ℓ and f_k: every $x_j \in P$ can be considered to be arbitrarily greatly space-like separated from any $y_j \in P'$.

The difference of (93) and (94) and distributivity provides that cluster decomposition is satisfied if

$$\begin{aligned}\mathcal{W}_{n+\ell,m+k}((x)_n, (y)_\ell, (x)_{n+1,n+m}, (y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\, \mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}},\end{aligned}$$

when every $x_j \in P$ are arbitrarily space-like separated from every $y_{j'} \in P'$ in evaluation of the scalar product (94). From the identification of truncated functions (66), $\underline{\mathcal{W}}$ has elements

$$\mathcal{W}_{n+\ell,m+k}((x)_n, (y)_\ell, (x)_{n+1,n+m}, (y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} = (\exp \circ ({}^T\mathcal{W}))_{n+\ell,m+k}$$

with the indicated association of arguments

$$(x,\kappa)_n,(y,\mu)_\ell,(x,\kappa)_{n+1,n+m},(y,\mu)_{\ell+1,\ell+k} \leftrightarrow (x,\kappa)_{1,n+m+\ell+k}$$

on the left- and right-hand sides, respectively. The sequence of truncated functions ${}^T\underline{\mathcal{W}}$ consists of the sum of the connected functions ${}^C\underline{\mathcal{U}}$ and the free field two-point function, section 3.3.4. The number of $*$-dual function arguments is $n+\ell$ and $m+k$ is the number of function arguments in the scalar products (93) and (94).

The elements of the sequence ${}^T\underline{\mathcal{W}}$ from (42) are connected, and due to the great space-like separations of the points in P from the points in P', any ${}^T\mathcal{W}_{\nu_1,\nu_2}$ that includes a point from P and a point from P' is nearly zero. The arguments in any non-zero ${}^T\mathcal{W}_{\nu_1,\nu_2}$ must be all from within P or all from within P'. Designate characteristic functions for P and P',

$$\Lambda_P((x)_{n+m},(y)_{\ell+k}) = \begin{cases} 1 & \text{if } (x)_{n+m}\in P \text{ and } \ell=k=0 \\ 0 & \text{otherwise} \end{cases} \tag{95}$$

and

$$\Lambda_{P'}((x)_{n+m},(y)_{\ell+k}) = \begin{cases} 1 & \text{if } (y)_{\ell+k}\in P' \text{ and } n=m=0 \\ 0 & \text{otherwise.} \end{cases}$$

$\Lambda_P = 1$ if $(x)_{n+m}$ is within the dominant support of $\psi_n\varphi_m$ and $\ell,k=0$, and $\Lambda_{P'}=1$ if $(y)_{\ell+k}$ is within the dominant support of $g_\ell f_k$ and $n,m=0$ in the evaluation (94) of $\mathcal{W}(\underline{\psi}^*\times\underline{g}^*,\underline{\varphi}\times\underline{f})$. Then, in the evaluation of the scalar product (94) with arbitrarily great space-like separation of P and P',

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^T\underline{\mathcal{W}}) \\ &\approx \exp\circ(\Lambda_P{}^T\underline{\mathcal{W}} + \Lambda_{P'}{}^T\underline{\mathcal{W}}) \\ &= (\exp\circ(\Lambda_P{}^T\underline{\mathcal{W}}))\circ(\exp\circ(\Lambda_{P'}{}^T\underline{\mathcal{W}})) \\ &= (\Lambda_P\underline{\mathcal{W}})\circ(\Lambda_{P'}\underline{\mathcal{W}}). \end{aligned} \tag{96}$$

from the identity (64) and that $\Lambda_P+\Lambda_{P'}=1$ within the dominant support of the contributing ${}^T\underline{\mathcal{W}}$. $\Lambda_P\underline{\mathcal{V}}$ designates the sequence (27)

$$(\mathcal{V}_{0.0},\Lambda_P(x_1)\mathcal{V}_{1,0}(x_1)_{\kappa_1},\ldots,\Lambda_P((x)_{n+m})\mathcal{V}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}},\ldots).$$

The approximation follows from the neglected volume in evaluation of (93) and (94) due to the small difference between $\exp\circ({}^T\underline{\mathcal{W}})$ and $\exp\circ((\Lambda_P+\Lambda_{P'})\,{}^T\underline{\mathcal{W}})$. The approximation can be made arbitrarily accurate with increases in support volumes and space-like separation. From (95), permutations in the $\circ$-product (60) that remains in the last line of (96) that transpose elements of P with elements of P' do not contribute. From $\mathcal{W}_{0,0}=1$ and the identity (62) for the $\circ$-product, the contributing elements of the sequence $\underline{\mathcal{W}}=(\Lambda_P\underline{\mathcal{W}})\circ(\Lambda_{P'}\underline{\mathcal{W}})$ in (96) are

$$\begin{aligned} &\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ &\qquad\qquad \approx \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\,\mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}}. \end{aligned}$$

This demonstrates that for the construction (65) of $\underline{\mathcal{W}}$ from connected functions ${}^T\underline{\mathcal{W}}$,

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \longrightarrow \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f})$$

as the dominant supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the dominant supports of $\underline{g}, \underline{f}$.

If one of the function sequences $\underline{\psi}, \underline{\varphi}, \underline{g}, \underline{f}$ are in the equivalence class of zero, then that function sequence is translation invariant and the question arises whether a translation of support implements cluster decomposition A.6. Designate the function sequence in the equivalence class of zero as $\underline{h}$. While the translation shifts any function features of $\underline{h}$, these features are Lebesgue measure zero and could be uniformly distributed over space, and the unshifted sequence equals the shifted sequence in the Hilbert space norm. For a translation T implemented in the Poincaré group,

$$\|\underline{h} - T\underline{h}\| = 0,$$

a consequence of the Cauchy-Schwarz-Bunyakovsky inequality, the unitarity of translation $\|T\underline{h}\| = \|\underline{h}\|$, and $\|\underline{h}\| = 0$.

Nevertheless, if any one of the $\underline{\psi}, \underline{\varphi}, \underline{g}, \underline{f}$ are in the equivalence class of zero, then cluster decomposition A.6 is satisfied trivially, 0=0. From the Cauchy-Schwarz-Bunyakovsky inequality,

$$|\mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f})|^2 \leq \mathcal{W}(\underline{\psi}^*, \underline{\psi})\, \mathcal{W}(\underline{\varphi}^*, \underline{\varphi}) \mathcal{W}(\underline{g}^*, \underline{g})\, \mathcal{W}(\underline{f}^*, \underline{f}) = 0$$

if one of $\underline{\psi}, \underline{\varphi}, \underline{g}, \underline{f}$ are in the equivalence class of zero. If either of $\underline{\varphi}, \underline{f}$ are in the equivalence class of zero, then

$$\underline{\varphi} \times \underline{f} = (\varphi_0 f_0, \ldots, \sum_{\ell=0}^{n} \varphi_\ell(x_1, \ldots x_\ell)_{(\kappa)_{1,\ell}}\, f_{n-\ell}(x_{\ell+1}, \ldots x_n)_{(\kappa)_{\ell+1,n}}, \ldots)$$

from (4) and every element of $\underline{\varphi} \times \underline{f}$ is in the equivalence class of zero. Either φ_0 or f_0 is zero, and the Lebesgue integration over the variables included in the function factor that is equivalent to zero is zero. In the constructions, section 3, every scalar product (13) reduces to a Lebesgue integration over momenta after evaluation of the generalized functions. Then, from the Cauchy-Schwarz-Bunyakovsky inequality,

$$|\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f})|^2 \leq \mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\psi} \times \underline{g}) \mathcal{W}(\underline{\varphi}^* \times \underline{f}^*, \underline{\varphi} \times \underline{f}) = 0.$$

Similarly for $\underline{\psi}^* \times \underline{g}^*$. This demonstrates cluster decomposition A.6 if one or more of the function sequences $\underline{\psi}, \underline{\varphi}, \underline{g}, \underline{f}$ are in the equivalence class of zero.

4.2.6 Regularity

If the constructed VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are generalized functions from $\mathcal{S}'(\mathbb{R}^{4n})$ then regularity A.1 is satisfied. The construction, (57) with (63), (32), (54), (49) and (55), displays

the VEV as finite sums of products of connected functions (42) with factors that have no arguments in common. Then, demonstration of regularity reduces to demonstration that the $n \geq 4$, n-argument connected functions are elements of $\mathcal{S}'(\mathbb{R}^{4n})$ since the two-point function (32) is an element of $\mathcal{S}'(\mathbb{R}^8)$. The $n \geq 4$, n-argument connected functions from (42), ${}^C\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$, are products of elementary generalized functions but since products are not generally defined, these particular products require justification.

To satisfy regularity A.1 for the physically nontrivial realizations, elemental stability A.7 is adopted. Elemental stability is implemented by a scalar product of the form (70). Avoidance of the extrapolation of ${}^C\mathcal{U}_{k,2n-k}$ to the singular $2n = 2$ [34, 38] follows from (70) and the characterization of $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ from (50) in (42). For functions in $\underline{\mathcal{P}}$, constructions with a single finite mass m and zero expectation fields, conservation of energy sets all contributions from $\mathcal{W}_{0,n}$ and $\mathcal{W}_{n,0}$ to zero for $n \geq 2$. If conservation of energy precludes decay of isolated elementary particles, then satisfaction of formal Hermiticity W.a is consistent with $\mathcal{W}_{1,n-1}(f_1^* f_{n-1}) = 0$ and $\mathcal{W}_{n-1,1}(f_{n-1}^* f_1) = 0$. More generally, $\mathcal{W}_{1,n-1}(f_1^* f_{n-1}) = 0$ and $\mathcal{W}_{n-1,1}(f_{n-1}^* f_1) = 0$ is required but is inconsistent with formal Hermiticity. In the constructions, regularity A.1 requires elemental stability A.7.

Whether the connected functions ${}^C\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$ satisfy regularity A.1 rests on whether the scalar products (13) are finite. The scalar products are determined by summations over submanifolds of $(p)_n \in \mathbb{R}^{4n}$ determined by the energy-momentum conservation and mass shell delta functions contributed by the connected VEV functions (42) and the zeros on the negative energy mass shells from functions in $\underline{\mathcal{P}}$. The functions $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ in the connected VEV functions are polynomially bounded growth, locally Lebesgue-summable functions within the domains included by the summations in the scalar product (13).

The singularities of the $N_c \times N_c$ array of functions $\mathfrak{h}(p_1, p_2)$ are removed from consideration. Regular $\mathfrak{h}(p_1, p_2)$ are selected. From (51), $\mathfrak{h}(p_a, p_b)_{\kappa_a\kappa_b}$ equals $\mathfrak{B}(\alpha_o p_a + p_b)_{\kappa_a\kappa_b}$, $\mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a\kappa_b}$ or $\Upsilon(-p_a + p_b)_{\kappa_a\kappa_b}$ depending on whether the arguments are both $*$-dual arguments, both function arguments, or one $*$-dual and one function argument in the evaluation of the scalar product (13). $\alpha_o \geq 0$ is a dimensionless, nonnegative real parameter. From (52),

$$\begin{aligned} \mathfrak{B}(p) &= \int d\mu_{\mathfrak{B}}(s)\, e^{isp} M(s) \\ \Upsilon(p) &= \int d\mu_{\Upsilon}(s)\, e^{-sp} M(s). \end{aligned}$$

There are no nontrivial finite Lorentz invariant measures [9] and as a consequence, $\mathfrak{B}(p)$ and $\Upsilon(p)$ are either constant or diverge at $p = 0$. The trivial Lorentz invariant measures are $d\mu_{\mathfrak{B}}(s), d\mu_{\Upsilon}(s) = c\delta(s)ds$. Non-constant $\mathfrak{B}(p)$ and $\Upsilon(p)$ are singular at $p = 0$ due to the divergence of the summations $\int d\mu_{\mathfrak{B}}(s)$ and $\int d\mu_{\Upsilon}(s)$. From the definition (52) of $\mathfrak{B}(p)$ and $\Upsilon(p)$, the Fourier transform property (23) and that the $M(p)$ are multinomials of the energy-momentum components, it follows that nonconstant $\mathfrak{B}(p)$ and $\Upsilon(p)$ are derivatives of Källén-Lehmann forms. Källén-Lehmann forms are at least as singular at $p^2 = 0$ as the Pauli-Jordan

function [56]. From the definition (52), $\mathfrak{B}(p)$ and $\Upsilon(p)$ are as singular at energy-momenta $p^2 = 0$ as Källén-Lehmann forms are singular at spacetime points $x^2 = 0$. The energy-constrained support of VEV functions and functions from $\underline{\mathcal{P}}$, and the selection (51) of $\mathfrak{h}(p_1, p_2)$ suffice to exclude singularities at $p^2 = 0$ from consideration in satisfaction of regularity A.1. For energy-momentum p_j, p_k with either both $p_{j0}, p_{k0} > 0$ or both $p_{j0}, p_{k0} < 0$, $(\alpha_o p_j + p_k)^2 > \alpha_o^2 \lambda_j^{-2} + \lambda_k^{-2}$. The singularity of $\mathfrak{B}(\alpha_o p_j + p_k)$ would be encountered only if p_j is a $*$-dual argument ($p_{j0} j = -\omega_j < 0$) and p_k is a function argument ($p_{k0} = \omega_k > 0$), or p_k is a $*$-dual argument and p_j is a function argument. These singularites are excluded by the selection (51) of $\mathfrak{h}(p_1, p_2)$. Similarly for $\mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a \kappa_b}$. For $\Upsilon(-p_j + p_k)$, the singularities are encountered only if both arguments are associated with a function or both are associated with a $*$-dual function. The form (51) excludes the singularities of $\mathfrak{h}(p_1, p_2)$ from the intervals included in the summations that determine scalar products (13). The measures $d\mu_{\mathfrak{B}}(s)$ and $d\mu_{\Upsilon}(s)$ are selected to ensure convergence of the summations (52) for $p^2 > 0$ [38], and then the constructed $\mathfrak{h}(p_1, p_2)$ are regular .

In this section, the number of spacetime dimensions is considered and is designated d.

Each connected function (42) in the constructions is supported solely on submanifolds with energy and momentum conserved. The contributions of the free field VEV are regular and need not be considered in this section. After evaluation of the mass shell delta functions and with consideration of the zeros in the energy support of functions in $\underline{\mathcal{P}}$ and $\underline{\mathcal{P}}^*$, the Fourier transforms of the $n \geq 4$, n-argument connected functions include only momenta on the manifold defined by

$$\prod_{j=1}^{n} \frac{1}{2\omega_j} \, \delta(\omega_1 \ldots + \omega_k - \omega_{k+1} \ldots - \omega_n) \, \delta(\mathbf{p}_1 + \mathbf{p}_2 \ldots + \mathbf{p}_n) \tag{97}$$

with

$$\omega_j = \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}$$

from (8).

Factors of $1/(2\omega_j)$ are multipliers of tempered functions for finite masses and are not considered further. Within the submanifold of $\mathbb{R}^{3n}$ with momentum conserved,

$$\mathbf{p}_n = -\mathbf{p}_1 \ldots - \mathbf{p}_{n-1}. \tag{98}$$

Energy conservation is $\delta(E_k((\mathbf{p})_n))$ with

$$\begin{aligned} E_k((\mathbf{p})_n) &= \sum_{j=1}^{k} \omega_j - \sum_{j=k+1}^{n} \omega_j \\ &= \sum_{j=1}^{n} s_j \omega_j. \end{aligned} \tag{99}$$

$s_j = -1$ if $j \in \{k+1, n\}$ and equals 1 otherwise. $\delta(E_k((\mathbf{p})_n))$ defines a generalized function except possibly for points on the surface $E_k((\mathbf{p})_n) = 0$ with a vanishing gradient, $\nabla E_k((\mathbf{p})_n) = 0$ [19].

The components of the gradient $\nabla E_k((\mathbf{p})_n)$ within the submanifold with momentum conserved are

$$\begin{aligned} \frac{dE_k((\mathbf{p})_n)}{d\mathbf{p}_j} &= s_j \frac{d\omega_j}{d\mathbf{p}_j} - \frac{d\omega_n}{d\mathbf{p}_n}\,\frac{d\mathbf{p}_n}{d\mathbf{p}_j} \\ &= s_j \frac{\mathbf{p}_j}{\omega_j} + \frac{\mathbf{p}_n}{\omega_n} \end{aligned} \tag{100}$$

from (99) with $d-1$ dimensional momentum vectors $\mathbf{p}_j$, the constrained $\mathbf{p}_n$ from (98), and $j \in \{1, n-1\}$. Summing squares provides that when the gradient vanishes, $\mathbf{p}_j^2/\omega_j^2 = \mathbf{p}_n^2/\omega_n^2$. Then, substitution of ω_j provides that the gradient vanishes if and only if

$$s_j \frac{\mathbf{p}_j}{m_{\kappa_j}} = -\frac{\mathbf{p}_n}{m_{\kappa_n}}$$

for each $j \in \{1, n-1\}$ and with the constrained $\mathbf{p}_n$ from (98). Then both the function and its gradient vanish,

$$E_k((\mathbf{p})_n) = \nabla E_k((\mathbf{p})_n) = 0,$$

only if

$$E_k((\mathbf{p})_n) = \left(\sqrt{\frac{c^2}{\hbar^2} + \frac{\mathbf{p}_n^2}{m_{\kappa_n}^2}}\right) \sum_{j=1}^{n} s_j m_{\kappa_j} = 0 \tag{101}$$

from the assignments

$$\frac{\mathbf{p}_j}{m_{\kappa_j}} = -s_j \frac{\mathbf{p}_n}{m_{\kappa_n}}$$

that set the gradient to zero, and the resultant

$$\omega_j = m_{\kappa_j} \sqrt{\frac{c^2}{\hbar^2} + \frac{\mathbf{p}_n^2}{m_{\kappa_n}^2}}.$$

A neighborhood of those points with both a zero energy and vanishing gradient is

$$\frac{\mathbf{p}_j}{m_{\kappa_j}} = s_j \frac{\mathbf{p}_1}{m_{\kappa_1}} + \mathbf{e}_j$$

with $\|\mathbf{e}_j\| < \epsilon \ll 1$, $0 < \|\mathbf{p}_1\|$ and $j \in \{2, n-1\}$. In this neighborhood,

$$\mathbf{p}_n = -\sum_{j=1}^{n-1} \mathbf{p}_j = -\frac{m_{\kappa_n}}{m_{\kappa_1}}\,\mathbf{p}_1 - \sum_{j=2}^{n-1} \mathbf{e}_j$$

using (101) and

$$\omega_j \approx \frac{m_{\kappa_j}}{m_{\kappa_1}}\omega_1 + s_j m_{\kappa_j}\frac{\mathbf{p}_1\cdot\mathbf{e}_j}{\omega_1} + m_{\kappa_1}m_{\kappa_j}\frac{\mathbf{e}_j^2}{2\omega_1} - m_{\kappa_1}m_{\kappa_j}\frac{(\mathbf{p}_1\cdot\mathbf{e}_j)^2}{2\omega_1^3}$$

to second order in small quantities and with $\mathbf{e}_n = -\sum_{\ell=2}^{n-1}\mathbf{e}_\ell$ and for $j \in \{2, n\}$.

On the submanifold with momentum and energy conserved, and within a neighborhood of the points where the gradient $\nabla E_k((\mathbf{p})_n)$ equals zero,

$$\begin{aligned} E_k((\mathbf{p})_n) &\approx \frac{m_{\kappa_1}}{2\omega_1^3}\sum_{j=2}^{n} s_j m_{\kappa_j}\left(\omega_1^2\,\mathbf{e}_j^2 - (\mathbf{p}_1\cdot\mathbf{e}_j)^2\right) \\ &= \frac{R^2}{2\omega_1^3}\,(\alpha\mathbf{p}_1^2 + \beta m_{\kappa_1}^2) \end{aligned}$$

from (101), polar coordinates for the $(n-2)$ spatial vectors $\mathbf{e}_j$, and with the definitions

$$R^2 = \sum_{j=2}^{n-1}\mathbf{e}_j^2,$$

$\mathbf{u}_1 = \mathbf{p}_1/\|\mathbf{p}_1\|$ and

$$\alpha R^2 = m_{\kappa_1}\sum_{j=2}^{n} s_j m_{\kappa_j}\left(\mathbf{e}_j^2 - (\mathbf{u}_1\cdot\mathbf{e}_j)^2\right), \qquad \beta R^2 = m_{\kappa_1}\sum_{j=2}^{n} s_j m_{\kappa_j}\,\mathbf{e}_j^2.$$

The result is that energy conservation (97) defines a generalized function except possibly for the points with the gradient $\nabla E_k((\mathbf{p})_n)$ vanishing when $E_k((\mathbf{p})_n) = 0$, the points with $R = 0$. For $R \approx 0$,

$$\begin{aligned} \delta(E_k((\mathbf{p})_n)) &= \delta(\frac{R^2}{2\omega_1^3}\,(\alpha\mathbf{p}_1^2 + \beta m_{\kappa_1}^2)) \\ &= \frac{2\omega_1^3}{R^2}\,\delta(\alpha\mathbf{p}_1^2 + \beta m_{\kappa_1}^2) + \frac{2\omega_1^3}{\alpha\mathbf{p}_1^2 + \beta m_{\kappa_1}^2}\,\delta(R^2). \end{aligned}$$

Since (97) is a generalized function except when both the energy (99) and gradient (100) vanish, that is, when $R > 0$, $\delta(\alpha\mathbf{p}_1^2 + \beta m_{\kappa_1}^2)$ is regular.

For $n \geq 4$ and a sufficient number of dimensions d, the singularities of the energy-momentum conserving delta functions (97) are locally summable for the regular selection (51) of $\mathfrak{h}(p_1, p_2)$. The determinant of the Jacobian matrix to polar coordinates for $(\mathbf{e})_{2,n-1}$ contributes

$$R^{(d-1)(n-2)-1}$$

in d spacetime dimensions for the nth order connected functions. Then, the summations in evaluation of the degenerate scalar product (13) include

$$\frac{R^{(d-1)(n-2)-1}}{R^2}\,dR = R^{(d-1)(n-2)-3}\,dR$$

in the neighborhood of the singularities and $d \geq 3$ suffices for the summations to converge and the constructed $\underline{\mathcal{W}}$ to be continuous linear functionals dual to tempered functions. Terms

$$R^{(d-1)(n-2)-1}\delta(R^2)$$

do not contribute for $n \geq 4, d \geq 3$ [19]. $d \geq 3$ suffices for finite masses m_{κ_j} and $d \geq 4$ includes massless particles [37].

To include massless particles, $m_\kappa = 0$, the development follows the finite mass case except that new singularities are encountered. The positive and negative mass shells intersect at $\mathbf{p}^2 = 0$, and ω^{-1} and the derivatives of $\omega = \sqrt{\mathbf{p}^2}$ diverge at $\mathbf{p}^2 = 0$. Selection of basis function spaces $\underline{\mathcal{P}}$ with Fourier transforms with infinite order zeroes at each $\mathbf{p}_j^2 = 0$ regularizes the development [37]. With massless particles, a neighborhood of the points with a vanishing gradient (100) is defined by

$$\frac{\mathbf{p}_j}{\sqrt{\mathbf{p}_j^2}} = s_j \frac{\mathbf{p}_1}{\sqrt{\mathbf{p}_1^2}} + \mathbf{e}_j$$

with the constraint that

$$\mathbf{e}_j^2 + 2s_j \frac{\mathbf{p}_1 \cdot \mathbf{e}_j}{\sqrt{\mathbf{p}_1^2}} = 0.$$

Unlike the finite mass case, the perturbations $\mathbf{e}_j$ must preserve the unit lengths of $\mathbf{p}_j/\sqrt{\mathbf{p}_j^2}$. The constraint to unit length reduces the number of degrees of freedom in the summation over $(\mathbf{e})_{2,n-1}$. In polar coordinates, the summation contributes $R^{(d-2)(n-2)-1}$ and as a consequence, $d = 4$ is required to satisfy regularity with the inclusion of massless particles.

Finite masses, elemental stability A.7, the regular selection (51) for the arrays $\mathfrak{h}(p_1, p_2)$ and three or more spacetime dimensions d suffice for the constructed $\underline{\mathcal{W}}$ to satisfy regularity A.1. Inclusion of massless elementary particles requires four or more dimensions.

4.3 Relation to free fields

The cluster expansion (63), the selection of two-point VEV (42) with (32), and connectedness provide that the constructed realizations of RQP are described by free fields when the supports of each argument of the state describing functions are sufficiently localized to be widely space-like separated. The connectedness and that at least two $*$-dual arguments and at least two function arguments appear in each contributing factor of the $n \geq 4$, n-arguments connected functions ${}^{C}\mathcal{U}_{k,n-k}$ provides that the contributions of the constructed VEV functions $\mathcal{W}_{k,n-k}$

reduce to contributions from the free field VEVs $\mathcal{F}_{k,n-k}$. Each two-point function factor contributing to $\mathcal{F}_{k,n-k}$ has one $*$-dual and one function argument. In these instances with the supports of each argument of the state describing functions $\underline{f}$ sufficiently isolated and widely space-like separated, the states are readily interpretable as consisting entirely of free particles. The scalar products of states with the support of each argument sufficiently widely space-like separated and localized equal the scalar products for free fields. Free fields are readily interpreted in the Wigner classification [66] of the elementary particles. Free fields are composed from irreducible representations of the Poincaré group. Mass m_κ and spin label the irreducible representations in the finite mass cases emphasized here.

4.4 The Hamiltonian and Hilbert space field operators

In this section, the Hamiltonians for the constructions are discussed and it is demonstrated that the fields $\Phi(\underline{f})$ constructed in section 3.3 are unbounded Hilbert space operators. The Hamiltonian is the generator of time translation for the realization of the Poincaré group.

The expectations (28) of p^2,

$$\langle g_k|p_j^2 f_n\rangle = m_{\kappa_j}^2\langle g_k|f_n\rangle$$

for $1 \leq j \leq n$, are Poincaré invariant, the Hermitian momentum observables are $\mathbf{p}_j$, and consequently the Hamiltonian, p_{j0}, the Hermitian generator of time translation, is determined from $\mathbf{p}_j^2$ and that energies are nonnegative. As a consequence of the limitation of the momentum support of the VEV to mass shells and that the intersection of the supports of the VEV with the support (18) of functions from $\underline{\mathcal{P}}$ includes only positive energies, the Hamiltonian, the generator p_0 of time translations of a state describing function, derives from the single-argument subspace operator (75).

$$e^{-ip_{j0}\lambda} = e^{-i\omega_j\lambda} \tag{102}$$

using (8) and the translation is by $\lambda = ct$. This follows from (75) with the Fourier transform (21). In an n-argument subspace of $\mathbf{H}_{\mathcal{P}}$,

$$U(\lambda) = \prod_{j=1}^{n} e^{-i\omega_j\lambda} \tag{103}$$

implements temporal translation. The $\sum_{j=1}^{n} p_j$ are densely defined Hermitian operators in the n-argument subspaces and correspond to energy and momentum. Hermiticity follows from Poincaré invariance of the scalar product (13) and Stone's theorem [24]. In the subspace of the vacuum, $n = 0$, the Poincaré transformations (72) are $(a, \Lambda) = \mathbb{I}$. In multiple argument subspaces, an association of single argument subspace operators with the classical dynamical variables of corresponding particles is generally not determined. An association of the arguments of state describing functions with the properties of classical particles necessarily applies only for

appropriate state describing functions, sections 4.3 and 5.2, or for VEV that lack interaction. The exhibition of interaction is described by the VEV (5): VEV determine the Hilbert space scalar product (6) and consequently the likelihoods of observations. The n and k-argument state describing functions are not orthogonal for $n \neq k$ when scalar products exhibit interaction. The evolution of state describing functions and the exhibition of interaction are discussed further in sections 4.5 and 5.

Demonstrated below, satisfaction of the prospective axioms A.1-7 in section 4.1 suffices to define the constructed quantum fields $\Phi(\underline{f})$ as Hilbert space operators. The quantum fields (29) are the multiplication (4) of function sequences and this product preserves Hilbert space norm-equivalence classes. The expansion (3) for states as products of the field applies for function sequences from $\underline{\mathcal{P}}$: the domain of the field (29) includes this dense set of elements within $\mathbf{H}_{\mathcal{P}}$. Demonstrated below, the field is not a bounded operator and as a consequence, the field is at best only densely defined, [24], section 7 and appendix 8.2.5.

For a space-like translation T and an element $|\underline{h}\rangle$ of the null space of $\mathbf{H}_{\mathcal{P}}$, satisfaction of cluster decomposition (69) in section 4.1 (axiom A.6) provides that

$$\begin{aligned}
\|\underline{f} \times (\underline{g} + T\underline{h})\|^2 &= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \times T\underline{h} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | T\underline{h} \rangle + \langle T\underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle T\underline{h} | T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | \underline{h} \rangle + \langle \underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle \underline{h} | \underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle \\
&= \|\underline{f} \times \underline{g}\|^2
\end{aligned}$$

from linearity of the scalar product, evaluation of the norm (25), translation invariance of the scalar product ($\|T\underline{g}\| = \|\underline{g}\|$), the Cauchy-Schwarz-Bunyakovsky inequality ($|\langle \underline{h} | \Omega \rangle| \leq \|\Omega\| \|\underline{h}\|$), and $\|\underline{h}\| = 0$. Satisfaction of cluster decomposition for elements $\underline{h}$ in the equivalence class of zero is demonstrated in section 4.2.5. Sequences $\underline{h}$ in the equivalence class of zero are translation invariant, $T\underline{h} = \underline{h}$ in the Hilbert space norm, section 4.2.5. As a consequence, the field operators $\Phi(\underline{f})$ in (29) preserve equivalence classes of the Hilbert space norm (25).

$$\|\Phi(\underline{f})\,(\underline{g} + \underline{h})\| = \|\Phi(\underline{f})\,\underline{g}\|$$

for any $\|\underline{h}\| = 0$. Repetition of this argument with $(\underline{f} + \underline{h}) \times \underline{g}$ demonstrates that the quantum field $\Phi(\underline{f})$ is also independent of the representative used for $\underline{f}$. The constructed fields are Hilbert space operators.

A quantum field $\Phi(\underline{f})$ (29) would be bounded if there is a real constant C such that $\|\Phi(\underline{f})\underline{v}\| \leq C\,\|\underline{v}\|$ for all $|\underline{v}\rangle \in \mathbf{H}_{\mathcal{P}}$, [47] and appendix 8.2.5. However, the fields $\Phi(\underline{f})$ (29) are unbounded and consequently are not continuous anywhere within $\mathbf{H}_{\mathcal{P}}$, [47] and section 7.

A demonstration of unboundedness for a free scalar boson field suffices for the constructions of physical interest. Any construction that includes unconfined scalar bosons includes the free boson field VEV. Then, the cluster decomposition axiom A.6 in section 4.1 provides that the

field $\Phi(\underline{f})$ is unbounded if the free boson field is unbounded. For free bosons, the VEV functions result from

$$\mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_S \prod_{j=1}^{k} W_2(x_j, x_{i_j})_{\kappa_j \kappa_{i_j}}$$

and are otherwise zero from (54) in section 3.2. For a product function,

$$\underline{f^n} = (0, 0 \ldots, \prod_{\ell=1}^{n} f(x_\ell)_{\kappa_\ell}, \ldots),$$

application of (4) and (29) results in

$$\Phi(\underline{f})\underline{f^n} = (0, 0 \ldots, f(x_1)_{\kappa_1} \prod_{\ell=2}^{n+1} f(x_\ell)_{\kappa_\ell}, \ldots).$$

Then the norm (25) for the free boson field provides

$$\|\underline{f^n}\|^2 = n!\, W_2(\underline{f}^*, \underline{f})^n$$

and

$$\|\Phi(\underline{f})\underline{f^n}\|^2 = (n+1)!\, W_2(\underline{f}^*, \underline{f})^{n+1}$$

with

$$W_2(\underline{f}^*, \underline{f}) = \sum_{\kappa_1=1}^{N_c} \sum_{\kappa_2=1}^{N_c} W_2(f^*_{\kappa_1}\, f_{\kappa_2})_{\kappa_1 \kappa_2}$$

from (13). Then

$$\frac{\|\Phi(\underline{f})\underline{f^n}\|^2}{\|\underline{f^n}\|^2} = (n+1)\, W_2(\underline{f}^*, \underline{f}) > C^2$$

for any finite real constant C, some $n \in \mathbb{N}$ and some $|\underline{f^n}\rangle \in \mathbf{H}_{\mathcal{P}}$. The constructed quantum fields $\Phi(\underline{f})$ are unbounded.

4.5 Scattering likelihoods

Interaction is exhibited in changes to momenta and particle numbers. A lack of orthogonality of the descriptions of states with differing particle numbers and momenta implements interaction. From Born's rule, scalar products (13) provide the state transition likelihoods and in relativistic physics, these likelihoods are Poincaré invariant. The state descriptions are covariant. In the constructions, the VEV rather than a Hamiltonian describe interaction. The scalar products of classical particle-like states, those with the support of each argument isolated, unentangled, and well described by a single momentum, approximate the scalar products of free fields, section

4.3. Free fields have established interpretations as classical particles [7, 9, 23, 54, 63, 66]. If two classical particle-like state describing functions are separated by a great time difference,

$$\lim_{\lambda\to\infty}\langle \underline{f}|e^{-iH\lambda}\underline{g}\rangle,$$

the separations of the supports of $\underline{f}$ and $\underline{g}$ become time-like and the transition likelihood is no longer dominated by the free field VEV contribution. In this case, transition likelihoods follow from the physically nontrivial VEV and exhibition of interaction manifests in the state transition likelihoods. Observations of distinct momenta and particle numbers become likely.

Scattering likelihoods are proportional to large time difference transition likelihoods evaluated with incoming and outgoing states described by their momenta [7, 23, 54, 63]. For the example of a single neutral scalar field and states described by product functions

$$f_n((x)_n) = \prod_j \ell(x_j;\lambda_j,\widehat{\mathbf{p}}_j),$$

scattering amplitudes are

$$S_{n,m} = \lim_{\lambda\to\infty}\langle U(\lambda)\ell(\lambda,\widehat{\mathbf{p}}_{n+1})\dots\ell(\lambda,\widehat{\mathbf{p}}_{n+m})|U(-\lambda)\ell(-\lambda,\widehat{\mathbf{p}}_1)\dots\ell(-\lambda,\widehat{\mathbf{p}}_n)\rangle$$

with $U(\lambda)$ the unitary operator (103) that translates the states in time. The parameters $\lambda_j,\widehat{\mathbf{p}}_j$ in the functions control a phase and the center of a localized momentum support, respectively. The designation $\ell(\lambda_j,\widehat{\mathbf{p}}_j)$ indicates the values of the parameters of the state describing functions $\ell(x_j;\lambda_j,\widehat{\mathbf{p}}_j)$. The LSZ (Lehmann-Symanzik-Zimmermann) expressions [9] for scattering amplitudes use functions with Fourier transforms

$$\tilde{\ell}(p_j;\lambda_j,\widehat{\mathbf{p}}_j) = e^{ip_{j0}\lambda_j}(\omega_j + p_{j0})\tilde{f}(\mathbf{p}_j - \widehat{\mathbf{p}}_j) \tag{104}$$

with λ_j a real parameter, $\widehat{\mathbf{p}}_j$ a momentum vector and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ is a Schwartz tempered test function. From section 3.4, $\ell(x_j;\lambda,\widehat{\mathbf{p}}_j)$ is a function that labels an element in the constructed Hilbert space $\mathbf{H}_{\mathcal{P}}$. A convenient choice of test function are the most classical-like state describing functions, Gaussian functions,

$$\tilde{f}(\mathbf{p}) = \left(\frac{L^2}{\pi}\right)^{3/2} e^{-L^2\mathbf{p}^2} > 0. \tag{105}$$

These $\tilde{f}(\mathbf{p})$ are point-wise nonnegative delta sequences heavily weighted near zero momentum when nearing the plane wave limit $L\to\infty$ and

$$\int d\mathbf{p}\ \tilde{f}(\mathbf{p}-\widehat{\mathbf{p}}) = 1.$$

The LSZ scattering amplitudes are VEV of products of fields

$$\Phi(\ell(\lambda,\widehat{\mathbf{p}})) = \int dp\ (\omega + p_0) e^{ip_0\lambda} \tilde{f}(\mathbf{p}-\widehat{\mathbf{p}})\ \tilde{\Phi}(p)$$

in this scalar field example. The VEV functions $\tilde{\mathcal{W}}_{k,n-k}((p)_n)$ are (17) with (14). Temporal translations of the field evaluated with the state describing functions (104) are independent of time if the parameter λ_j is set to λ.

$$\begin{aligned} U(\lambda)\Phi(\ell(\lambda,\widehat{\mathbf{p}}))U(\lambda)^{-1} &= \int dp\ (\omega+p_0) e^{-i(\omega-p_0)\lambda} \tilde{f}(\mathbf{p}-\widehat{\mathbf{p}})\ \tilde{\Phi}(p) \\ &= \Phi(\ell(0,\widehat{\mathbf{p}})) \end{aligned}$$

due to the limitation of the spectral support of the constructed VEV to mass shells.

In a more familiar notation,

$$U(\lambda)\Phi(\ell(\lambda,\widehat{\mathbf{p}}))U(\lambda)^{-1} = i\int d\mathbf{x}\ \hat{u}(\lambda,\mathbf{x})\ \overset{\leftrightarrow}{\partial}_o\ \Phi(\lambda,\mathbf{x})$$

with

$$f(x)\ \overset{\leftrightarrow}{\partial}_o\ g(x) = f(x)\dot{g}(x) - \dot{f}(x)g(x),$$

$\dot{f}(x)$ the first time derivative of $f(x)$ and

$$\hat{u}(\lambda,\mathbf{x}) = \frac{1}{2\pi}\int d\mathbf{p}\ e^{i\omega\lambda} e^{-i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}-\widehat{\mathbf{p}})$$

is a smooth solution of the Klein-Gordon equation.

For the Gaussian functions (105), the plane wave scattering amplitudes are the limits described by momenta $(\mathbf{p})_{n+m}$.

$$\begin{aligned} S_{n,m} &= \lim_{\substack{L\to\infty \\ \lambda\to\infty}} \langle U(\lambda)\ell(\lambda,\mathbf{p}_{n+1})\dots\ell(\lambda,\mathbf{p}_{n+m})|U(-\lambda)\ell(-\lambda,\mathbf{p}_1)\dots\ell(-\lambda,\mathbf{p}_n)\rangle \\ &= \lim_{L\to\infty} \langle \ell(0,\mathbf{p}_{n+1})\dots\ell(0,\mathbf{p}_{n+m})|\ell(0,\mathbf{p}_1)\dots\ell(0,\mathbf{p}_n)\rangle. \end{aligned}$$

Evaluation of the mass shell deltas in the VEV from section 3 simplify the expression and the resulting quadrature is readily evaluated in the plane wave limit [34]. From the scalar product (13) for scalar field VEV,

$$\begin{aligned} S_{n,m} &= \lim_{L\to\infty} \left(\frac{L}{\sqrt{\pi}}\right)^{3(n+m)} \int d(\mathbf{p})_{n+m} \prod_{j=1}^{n} e^{-L^2(-\mathbf{p}_j-\mathbf{p}_j)^2} \prod_{\ell=n+1}^{n+m} e^{-L^2(\mathbf{p}_\ell-\mathbf{p}_\ell)^2} \\ &\qquad \times \widetilde{\mathcal{W}}_{n,m}((-\omega(\mathbf{p}),\mathbf{p})_n, (\omega(\mathbf{p}),\mathbf{p})_{n+1,n+m}) \\ &= \widetilde{\mathcal{W}}_{n,m}((-\omega(\mathbf{p}),-\mathbf{p})_n, (\omega(\mathbf{p}),\mathbf{p})_{n+1,n+m}) \end{aligned} \tag{106}$$

with $\omega(\mathbf{p}_j) = \omega(-\mathbf{p}_j)$ from (8). Each

$$(\omega_j \pm p_{k0})\,\delta(p_k^2 - \lambda_c^2) = \delta(p_{k0} \pm \omega_k)$$

and the sign is determined by whether p_k is the argument of a function or a $*$-dual function (11). Each energy is evaluated on the appropriate mass shell. Non-forward amplitudes result if only the connected contribution ${}^{C}\widetilde{\mathcal{W}}_{n,m}$ of the VEV is considered. Then for the example connected function (17), the forward amplitudes are

$$S_{n,m} = c_{n+m}\,\delta(\omega(\mathbf{p}_1)\ldots+\omega(\mathbf{p}_n)-\omega(\mathbf{p}_{n+1})\ldots-\omega(\mathbf{p}_{n+m}))\delta(\mathbf{p}_1\ldots+\mathbf{p}_n-\mathbf{p}_{n+1}\ldots-\mathbf{p}_{n+m}).$$

Up to a factor of i, this amplitude equals the first order expansion from a Feynman-Dyson series with an interaction Hamiltonian density $\mathcal{H}_{int}(x) = \sum_\ell a_\ell : \Phi(x)^\ell :$ with $\ell \geq 4$ and $a_\ell = c_\ell\,(2\pi)^{2\ell-4}/\ell!$, appendix 8.9. c_ℓ is from (48). For $n = 4$, the non-forward, elastic scattering amplitude is

$$S_{2,2} = c_4\delta(\omega(\mathbf{p}_1) + \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3) - \omega(\mathbf{p}_4))\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \tag{107}$$

that corresponds to a Yukawa-like equivalent potential in first Born approximation [36]. For this scalar field example, the scattering cross sections of the realizations of relativistic quantum mechanics equal first order contributions from Feynman series. The phase difference of forward and non-forward amplitudes is unobservable in the scattering limit applicable to Feynman-Dyson series. In the example of Compton scattering, the cross sections deviate from first order Feynman series results for extremely relativistic exchange momenta (small distances) [36].

The scattering likelihoods are independent of time in the plane wave limit [34]. In this development, the propagation of the support of localized states can be followed through intermediate times. The lack of time dependence in finite time, plane wave limits is understood from a Hamiltonian that contributes only a phase in the plane wave limit and that plane waves uniformly cover all space.

5 Non-relativistic quantum-classical correspondences

> We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. At least I do, because I'm an old enough man that I haven't got to the point that this stuff is obvious to me. Okay, I still get nervous with it... You know how it always is, every new idea, it takes a generation or two until it becomes obvious that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm not sure there's no real problem. – Richard Feynman, 1982, p. 471 in [17].

Understanding quantum mechanics as the description of nature has been a persistently controversial topic [5, 8, 16, 45, 53, 60, 67]. Quantum mechanics is often regarded as "strange," perhaps referring to the incompatibility of the quantum description of nature with well-established classical understandings. Emphasized in the Einstein-Podolosky-Rosen (EPR) paradox, [16] and appendix 8.5, the quantum description of nature contradicts classical concepts. In the classical concept, every particle is distinguishable and described by determined classical dynamical variables. In the quantum description, particles are indistinguishable, their locations and momenta are never both described with arbitrary precision, and state descriptions are elements of Hilbert spaces that linearly expand in terms of other state descriptions. Although quantum effects like the Heisenberg uncertainty bounds are difficult to observe with massive bodies, Schrödinger's cat paradox [53] illustrates that quantum mechanics can not be relegated to a "strange microscopic world."

A quantum-classical correspondence provides an approximate description of the quantum dynamics by classical dynamical variables. Quantum-classical correspondences apply when the classical description is a sufficiently accurate approximation. When sufficient, classical dynamical variables represent the support of appropriate state describing functions, those well characterized by their location and momenta. Classical description applies to perception of the quantum state.

One realization of relativistic quantum dynamical is generally perceived with multiple scenario-dependent classical descriptions. In addition to the correspondences from scattering, $-g/r$ potential classical interactions provide quantum-classical correspondences for state describing functions with non-relativistic and isolated support, and bound states provide correspondences with particles other than the elementary particles.

Brief interval state transition amplitudes for state describing functions with non-relativistic and isolated support are the primary focus of this section. For states that are well-represented by finite mass, point-like particles with non-relativistic relative velocities over brief intervals, evolution of the support of appropriate state describing functions is approximated by the evolution of classical dynamical variables in Newtonian mechanics with $-g/r$ pair potentials. Quantum-classical correspondences are derived from analysis of the support of the state describing functions. The quantum-classical correspondences are conditional, conditioned on properties of the state describing functions, and particular state describing functions correspond with each classical description. Significant spatial overlap of supports, entanglements, and widely distributed spatial supports preclude detailed quantum-classical correspondences.

Quantum-classical correspondences include scattering likelihoods. In scattering, initial states and scattered products are perceived as classical particles. Cluster decomposition A.6 provides that isolated initial and final scattered states are readily interpreted as freely propagating classical particles [31]. Scattering likelihoods enable comparison of the constructions with Feynman-Dyson series. Scattering likelihoods of the constructions include approximations of Feynman series likelihoods. Example scattering amplitudes are provided in [32, 34, 36] and section 4.5. Scattering likelihoods include relativistic correspondences.

Quantum-classical correspondences based on associating the support of state describing functions with classical dynamical variables is the primary topic of this section, This correspondence appears in Schrödinger's 1926 study of non-relativistic linear harmonic oscillators [51] and Ehrenfest's extensions [14, 43]. The supports of the most classical-like solutions to the Schrödinger equation are well-represented by corresponding classical dynamical variables. In 1+1 spacetime, particular Gaussian functions $\psi(x, \lambda)$ satisfy the Schrödinger equation.

$$\psi(x, ct) = \exp\left(-\frac{(x - A\cos wt)^2}{4\sigma^2} - i\frac{\beta x \sin wt}{\hbar} - i\phi(t)\right)$$

satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\psi = i\hbar\frac{d\psi}{dt}$$

and the solution is characterized by a mass m, spring constant k, and oscillation amplitude A,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}}, \qquad w = \sqrt{\frac{k}{m}}$$
$$\beta = \sqrt{mk}\,A, \qquad \phi(t) = \frac{w}{2}t - \frac{kA^2}{4\hbar w}\sin 2wt.$$

The breadth of support of $\psi(x, ct)$ over x is described by σ and w is the oscillation frequency. Selected as the peak likelihood, the representative for the support of this function is the trajectory

$$x(\lambda) = A\cos w\lambda/c$$

of classical linear harmonic motion. The quantum-classical correspondence is most evident when the spread of support is small with respect to the amplitude of the motion, $\sigma^2 \ll A^2$. As $m \to \infty$,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}} \ll A^2 = \frac{2E}{k} \tag{108}$$

with the total classical energy $E = \frac{1}{2}kA^2$ a constant of the motion. The support of the function $\psi(x, \lambda)$:

1. one state describing function exhibits the correspondence with classical dynamical variables m, k and A. Given classical dynamical variables $m.k.A$, σ is determined and the particular state describing function $\psi(x, t)$ follows. The Gaussian (most classical-like) state describing function that exhibits the quantum-classical correspondence has a particular spread σ;

2. the $\psi(x, \lambda)$ with quantum-classical correspondences are among the most-classical particle-like descriptions: the state describing functions meet the Heisenberg lower bound on the breadths of support over location and momentum, appendix 8.6;

3. and any classical energy E can be matched by a quantum mechanical description, a state describing function $\psi(x, ct)$.

Analogously to the association of energy eigenfunctions and energies, particular state describing functions associate with each selection of values for the classical dynamical variables. This quantum-classical correspondence does not apply for all state describing functions. Energy eigenfunctions of the linear harmonic oscillator,

$$e_n(x, ct) = H_n(\alpha x)\, \exp\left(-i(n + \frac{1}{2})wt - \frac{1}{2}\alpha^2 x^2\right)$$

with H_n the nth Hermite polynomial [1] and

$$\alpha^2 = \frac{\sqrt{mk}}{\hbar},$$

do not exhibit a quantum-classical correspondence. The energy eigenfunctions do not follow classical trajectories of linear harmonic motion, with the exception of $n = 0$ and $A = 0$. The energy eigenfunctions exhibit quantized energies

$$E = (n + \frac{1}{2})\hbar w$$

and the temporal evolution includes only a phase variation, and not an evolution of the support of the state describing function.

To simplify development and notation, discussions within this section is generally limited to a single, neutral scalar field, $N_c = 1$.

5.1 The support of state describing functions

Likelihoods of observation determine the physically relevant support of states. Likelihoods are determined from Born's rule: the likelihoods of perception equal the squared magnitudes of normalized scalar products (13). From A.3, Born's rule likelihoods are invariant to Poincaré transformations and state descriptions are covariant. The spatial support of a state describing function $\varphi_n((x)_n)$ at a selected time follows from a real-valued function over $(\mathbf{y})_k \in \mathbb{R}^{3k}$,

$$|\langle (\mathbf{y})_k | \varphi_n \rangle|^2 \tag{109}$$

with $|(\mathbf{y})_k\rangle \in \mathbf{H}_{\mathcal{P}}$ described by a k-argument state describing function with spatial support concentrated very near the points $(\mathbf{y})_k \in \mathbb{R}^{3k}$ and the temporal support is concentrated on the time of observation. As the support of $|(\mathbf{y})_k\rangle$ becomes more localized, the momentum domain support includes more relativistic momenta. For the scalar product (13) from section 3, significant likelihoods occur for state describing functions $\varphi_n((x)_n)$ with $k \neq n$. n-argument

state describing functions include multiple particle numbers in their expansions over elements of the Hilbert space. Similarly, the momentum support of $\varphi_n((x)_n)$ at a selected time is

$$|\langle(\mathbf{w})_k|\tilde{\varphi}_n\rangle|^2 \tag{110}$$

with $|(\mathbf{w})_k\rangle$ described by a k-argument function within the Fourier transform domain of $\mathbf{H}_{\mathcal{P}}$ with momentum support concentrated very near $(\mathbf{w})_k \in \mathbb{R}^{3k}$ and temporal point support on the time of observation.

Due to factors $\delta(p_j^2 - \lambda_{cj}^{-2})$ in the constructed VEV (32) and (42), any state describing function in $\mathbf{H}_{\mathcal{P}}$ is equivalent to

$$\tilde{\varphi}_n((p)_n) = \prod_{j=1}^{n} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\widehat{\mathbf{p}})_n). \tag{111}$$

$\tilde{f}_n((\mathbf{p})_n) \in \mathcal{S}(\mathbb{R}^{3n})$ is a function with dominant support centered on $(\mathbf{p})_n = 0$ and with an inverse Fourier transform with dominant support centered on $(\mathbf{x})_n = 0$. Additional p_{j0} dependence from $\tilde{\varphi}_n((p)_n)$ is eliminated by the substitutions $p_{j0} = \omega(\mathbf{p}_j)$. As a consequence, definition of the time and spatial support of functions $\varphi_n((x)_n)_{(\kappa)_n}$ derived from the inverse Fourier transform of a $\tilde{\varphi}_n((p)_n)_{(\kappa)_n}$ are ambiguous. Scalar products (109) and (110) are determined. The representation (111) of $\tilde{\varphi}_n((p)_n)$ is of point support at time zero with spatial supports centered on the $\mathbf{y}_j$ and momentum supports centered on the $\widehat{\mathbf{p}}_j$. The inverse Fourier transform of $\tilde{\varphi}_n$ is

$$\begin{aligned}
\varphi_n((x)_n) &= \int \frac{d(p)_n}{(2\pi)^{2n}} \prod_{j=1}^{n} e^{ip_jx_j} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\widehat{\mathbf{p}})_n) \\
&= \int \frac{d(p_0)_n}{(2\pi)^{\frac{n}{2}}} \int \frac{d(\mathbf{p})_n}{(2\pi)^{\frac{3n}{2}}} \prod_{j=1}^{n} \left(-i\frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2}-\Delta_j}\right) e^{ip_{j0}x_{j0}} e^{-i\mathbf{p}_j\cdot(\mathbf{x}_j-\mathbf{y}_j)} \tilde{f}_n((\mathbf{p}-\widehat{\mathbf{p}})_n) \\
&= (2\pi)^{\frac{n}{2}} \prod_{j=1}^{n} \left(-i\frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2}-\Delta_j}\right) e^{-i\widehat{\mathbf{p}}_j\cdot(\mathbf{x}_j-\mathbf{y}_j)} \delta(x_{j0}) f_n((\mathbf{x}-\mathbf{y})_n)
\end{aligned} \tag{112}$$

with Δ_j the Laplacian for argument $\mathbf{x}_j \in \mathbf{R}^3$, and the derivative of $\delta(t)$ is a generalized function [19]. These are functions that describe a state at a specified time in the one reference frame. Time is generally kept externally to the observed and perceptions are assigned a time in a particular observer's coordinate frame, the coordinate frame of the representation (111).

For illustration, from (111) and (112), delta sequences include normalization of

$$\tilde{g}_k((p)_k) = \prod_{j=1}^{k} e^{i\mathbf{p}_j\cdot\mathbf{y}_j}(p_{j0}+\omega_j)\; e^{-L^2(\mathbf{p}_j-\mathbf{w}_j)^2}. \tag{113}$$

The $\tilde{g}_k((p)_k)$ in (113) describe $|(\mathbf{y})_k\rangle$ as $L^2 \to 0$. $\tilde{g}_k((p)_k)$ describes an $|(\mathbf{w})_k\rangle$ for $L^2 \to \infty$.

From (7), function sequences $\underline{\varphi}, \underline{\psi} \in \underline{\mathcal{P}}$ are functions

$$\tilde{\varphi}_n((p)_n) = \prod_{j=1}^{n} (\omega_j + p_{j0}) \tilde{f}_n((\mathbf{p})_n)_{\kappa_1 \ldots \kappa_n}$$
$$\tilde{\psi}_n((p)_n) = \prod_{j=1}^{n} (\omega_j + p_{j0}) \tilde{g}_n((\mathbf{p})_n)_{\kappa_1 \ldots \kappa_n}$$

with $f_n, g_n \in \mathcal{S}(\mathbb{R}^{3n})$. From the VEV constructed in section 3, the scalar products (13) expressed in the momentum domain are

$$\begin{aligned} \langle \underline{\varphi} | \underline{\psi} \rangle &= \sum \int d(p)_{n+m} \, \langle \tilde{\Phi}(p_1)_{\kappa_1} \ldots \tilde{\Phi}(p_n)_{\kappa_n} \Omega | \tilde{\Phi}(p_{n+1})_{\kappa_{n+1}} \ldots \tilde{\Phi}(p_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ &\qquad \times \overline{\tilde{\varphi}_n}(-p_1, \ldots - p_n)_{\kappa_1 \ldots \kappa_n} \, \tilde{\psi}_m(p_{n+1}, \ldots p_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \\ &= \sum \int d(p)_{n+m} \, \overline{\tilde{f}_n}(p_1, \ldots p_n)_{\kappa_1 \ldots \kappa_n} \tilde{g}_m(p_{n+1}, \ldots p_{n+m})_{\kappa_{n+1} \ldots \kappa_{n+m}} \prod_{j=1}^{n+m} \delta(p_{j0} - \omega_j) \\ &\qquad \times T_{n,m}((-\mathbf{p})_n, (\mathbf{p})_{n+1,n+m})_{(\kappa)_{n+m}} \delta(p_1 \ldots + p_n - p_{n+1} \ldots - p_{n+m}) \end{aligned} \tag{114}$$

from (6), (11), (21), (14), relabeling summation variables and the Fourier transform of generalized functions [19]. The summations are over all $n, m \in \mathbb{N}_0$, $1 \leq \kappa_j \leq N_c$ with $1 \leq j \leq n+m$, and $(p)_{n+m} \in \mathbb{R}^{4n+4m}$. The functions $T_{n,m}((\mathbf{p})_{n+m})_{(\kappa)_{n+m}}$ follow from the VEV, from the cluster expansion (58) of (32) and (42). For the single scalar field and two-argument subspace, $T_{2,2}((\mathbf{p})_4)$ follows from (148) and (149) presented below. The $T_{n,m}((\mathbf{p})_{n+m})_{(\kappa)_{n+m}}$ include factors of $1/(2\omega_j)$ from evaluation of the mass shell delta functions $\delta(p_j^2 - \lambda_{cj}^{-2})$. Limited to positive energies by the properties of functions from $\mathbf{H}_{\mathcal{P}}$, from (8),

$$2\omega_j \delta(p_2^2 - \lambda_{cj}^{-2}) = \delta(p_{j0} - \omega_j).$$

In (114), $(p)_n \mapsto (-p)_n$ is the change of variables for the first n four-dimensional summation variables associated with the $*$-dual functions in the scalar product (114).

The physically relevant spatial support (109) of a state describing function is Poincaré covariant (71).

5.2 Properties of states with classical descriptions

Verified daily, classical dynamical variables provide accurate representatives for the perceptions of quantum state descriptions when the support of states is "macroscopic," that is, classical particle-like. A quantum state description is classical particle-like if the spatial support of each argument is isolated and well-represented by a single location, the support of the Fourier transform of each argument is well-represented by a single momentum, and supports are not

entangled. The Michelson interferometer illustrates that localization is necessary, and a relativistic collision illustrates the necessity of isolation to a quantum-classical correspondence, appendix 8.4. The EPR paradox [16] illustrates that descriptions must not be entangled for a quantum-classical correspondence to apply. If the dominant support of states is classical particle-like, then the support of the arguments of state describing functions is well represented by classical dynamical variables.

In this section, the properties that determine whether the support of states is "macroscopic" or non-relativistic are discussed. Dominant support, localization, representative locations and momenta, isolation and non-relativistic support are described below in the instance of one neutral scalar field and more generally, the definitions apply for each constituent function labeled by field component κ_j.

Dominant support and representative location: The physically relevant dominant support of a state describing function $\varphi_n((x)_n)$ is evaluated by comparison of norms within distinct, common, finite volumes within $\mathbb{R}^3$. A convenient characterization for the dominant support of the jth argument of $\varphi_n((x)_n)$ is the spherical volume $V(\mathbf{u}_j)$ centered on a location $\mathbf{u}_j$ with

$$\int_{V(\mathbf{u}_j)} d\mathbf{y}_j \; |\langle (\mathbf{y})_k | \varphi_n \rangle|^2 = (1-\epsilon) \int_{\mathbb{R}^3} d\mathbf{y}_j \; |\langle (\mathbf{y})_k | \varphi_n \rangle|^2 \tag{115}$$

with $(\mathbf{y})_k$ the delta function-like element (113) in (109). $1-\epsilon$ is the relative likelihood that a location measurement for the jth argument will lie within the volume $V(\mathbf{u}_j)$ of dominant support. Designate the radius of this sphere $V(\mathbf{u}_j)$ by R_j. Then, the representative location $\mathbf{u}_j$ achieves a specified likelihood $1-\epsilon$ with the minimal R_j, and then the dominant volume of support is $V(\mathbf{u}_j)$.

Localization: The jth argument of a state describing function $\varphi_n((x)_n)$ is localized near $\mathbf{y}_j$ if the radius R_j of the dominant support is within a tolerance of interest L_j.

$$L_j > R_j.$$

If the jth argument of $\varphi_n((x)_n)$ is sufficiently localized near $\mathbf{u}_j$, then

$$\langle \psi_n | g(\mathbf{x}_j) \varphi_n \rangle \approx g(\mathbf{u}_j) \langle \psi_n | \varphi_n \rangle \tag{116}$$

for multiplier functions $g(\mathbf{x})$ of slow variation within the dominant support of $\varphi_n((x)_n)$. $\mathbf{u}_j$ is the representative value of the argument $\mathbf{x}_j$, the jth argument of $\varphi_n((x)_n)$. $g(\mathbf{x}_j)\varphi_n$ designates the n-argument function

$$g(\mathbf{x}_j)\varphi_n(x_1, x_2, \ldots x_j, \ldots x_n).$$

One example tolerance L_j is an R_j small with respect to the separation of slits in Young's double slit: the observation is propagation through the slits followed by localized detection far behind the slits. If the radius of dominant support R_j is small with respect to the slit separation, then no interference is observed. Conversely, if R_j is much larger than the slit separation and

$\mathbf{u}_j$ illuminates both slits, then the quantum state description behaves as a wave rather than a classical particle.

From the connectedness of constructed VEV (41), if the spatial support of a state describing function φ_n is localized,

$$\int_{V(\mathbf{u}_j)} d\mathbf{y}_j \, |\varphi_n((0,\mathbf{y})_k)|^2 = (1-\epsilon) \int_{\mathbb{R}^3} d\mathbf{y}_j \, |\varphi_n(0,(\mathbf{y})_k)|^2,$$

then this localized support $V(\mathbf{u}_j)$ is indicative of the physically relevant spatial support (109). Although the dominant support of the state describing function $\varphi_n((x)_n)$ does not equal the physically relevant support, it does approximately characterize the physically relevant dominant support. The connectedness of VEV provides that scalar products (13) are dominated by points within a limited distance of the dominant support.

Representative momenta: The physically relevant dominant momentum support is characterized by a spherical volume $U(\mathbf{w}_j)$ centered on a momentum $\mathbf{w}_j$ in the Fourier transform domain such that

$$\int_{U(\mathbf{w}_j)} d\mathbf{q}_j \, |\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2 = (1-\epsilon) \int_{\mathbb{R}^3} d\mathbf{q}_j \, |\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2 \tag{117}$$

with $(\mathbf{q})_k$ the delta function-like element (113) in (110). Designate the radius of this sphere $U(\mathbf{w}_j)$ by Q_j. The representative momentum $\mathbf{w}_j$ achieves a specified likelihood $1-\epsilon$ with the minimal Q_j, and then the dominant volume of momentum support is $U(\mathbf{w}_j)$.

Similarly to location, the representative momentum support of the Fourier transform $\tilde{\varphi}_n((p)_n)$ of a state describing function may be "localized." That is, the description of the momenta may be well-characterized by the representative momentum $\mathbf{w}_j$. If the dominant support of the jth argument of a state description $\tilde{\varphi}_n((p)_n)$ is a sufficiently localized near a representative momentum $\mathbf{w}_j$, then

$$\langle \tilde{\psi}_n | \tilde{g}(\mathbf{p}_j) \tilde{\varphi}_n \rangle \approx \tilde{g}(\mathbf{w}_j) \langle \tilde{\psi}_n | \tilde{\varphi}_n \rangle \tag{118}$$

for multiplier functions $\tilde{g}(\mathbf{p})$ of slow variation within the volume of dominant support over $\mathbf{p}_j$ of $\tilde{\varphi}_n((p)_n)$.

Isolated: A localized argument $\mathbf{x}_j$ is isolated if there is little likelihood that any neighborhood that lies within the dominant support is included within a dominant support of any other argument of the state describing function. The support of a localized argument $\mathbf{x}_j$ is isolated if

$$(\mathbf{u}_j - \mathbf{u}_\ell)^2 > (R_j + R_\ell)^2 \tag{119}$$

for all $\ell \in \{1, n\}$ with $\ell \neq j$. Two arguments are isolated from each other if the spatial separation of the two representative locations $\mathbf{u}_\ell$ exceeds the sum of the two radii of dominant support R_ℓ. If isolation is satisfied, only the jth argument associates to great likelihood with the volume of space $V(\mathbf{u}_j)$.

Cluster decomposition A.6 provides that if the supports of every argument of state describing functions φ_n and ψ_m are sufficiently isolated, then only two-point VEV contribute to $\langle\varphi_n|\psi_m\rangle$. In this case, scalar products (13) approximately vanish unless particle number is conserved.

$$\langle\varphi_n|\psi_m\rangle \approx \langle\varphi_n|\psi_m\rangle\ \delta_{n,m}.$$

Non-relativistic: For finite mass particles, non-relativistic physics applies if the momentum domain supports of the state describing functions φ_n are sufficiently near zero. The support of an argument $\mathbf{p}_j$ is non-relativistic if $\lambda_c^2\mathbf{p}_j^2 \ll 1$ for every momenta $\mathbf{p}_j$ within the dominant support. This is equivalent to $\hbar^2\mathbf{p}_j^2 \ll (mc)^2$. For non-relativistic momenta,

$$\begin{aligned} \mathbf{p}_j^2 &\ll \lambda_c^{-2} \\ \langle\varphi_n|\omega_j\varphi_n\rangle &\approx \langle\tilde{\varphi}_n|(\lambda_c^{-1} + \tfrac{1}{2}\lambda_c\mathbf{p}_j^2)\tilde{\varphi}_n\rangle \\ \mathbf{w}_j(\lambda) &\approx \lambda_c^{-1}\dot{\mathbf{u}}_j(\lambda). \end{aligned} \tag{120}$$

This description applies in the reference frame determined by the coordinates p_j.

Non-relativistic classical trajectories $\mathbf{u}_j(\lambda)$ are spatial vectors,

$$\mathbf{u}_j(\lambda) = (u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)).$$

Associated spacetime vectors $u_j(\lambda)$ are designated

$$u_j(\lambda) = (0, u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)). \tag{121}$$

Derivatives with respect to the temporal parameter $\lambda = ct$ are designated

$$\dot{\mathbf{u}}_j(\lambda) = \frac{d\mathbf{u}_j(\lambda)}{d\lambda}. \tag{122}$$

Trajectories are twice differentiable curves in $\mathbb{R}^3$ that specify one classical body's history of locations $\mathbf{u}_j(\lambda)$ and velocities $\dot{\mathbf{u}}_j(\lambda)$. Momenta $\hbar\mathbf{w}_j(\lambda)$ are Euclidean three-vectors associated with the particle trajectories $\mathbf{u}_j(\lambda)$.

$$\begin{aligned} \lambda_c\mathbf{w}_j(\lambda) &= \gamma_j\,\dot{\mathbf{u}}_j(\lambda) \\ &\approx \dot{\mathbf{u}}_j(\lambda) \end{aligned} \tag{123}$$

with

$$\gamma_j = \frac{1}{\sqrt{1 - \dot{\mathbf{u}}_j^2}}$$

and the approximation applies in non-relativistic ($\dot{\mathbf{u}}_j^2 \ll 1$) instances. The energy-momentum Lorentz vector of a finite mass particle is $p_j = (\omega(\mathbf{w}_j), \mathbf{w}_j)$ with $\omega(\mathbf{p})$ from (8) and $p_j^2 = \lambda_c^{-2}$.

The most classical particle-like descriptions meet the Heisenberg uncertainty bound for the breadth of location and momentum support. The association of spatial volumes with non-relativistic classical dynamical variables occurs on relatively large spatial scales, for spatial supports with extents large with respect to Compton wavelengths. Classical states have defined locations and momenta; quantum state descriptions include uncertainties in the perception of momentum and location. These uncertainties have a lower bounded product, [13] and appendix 8.6.

The quantum-classical correspondence discussed in this section applies only as long as the support of a state describing function remains isolated and well-represented by a single location and momentum, and particle number is conserved. Non-relativistic quantum-classical correspondences are limited to finite mass elementary particles, $m > 0$. This discussion of non-relativistic quantum-classical correspondences does not apply to the inherently relativistic massless particles.

Appropriate functions: If $\tilde{f}_n((\mathbf{p})_n, \lambda) \in \mathcal{S}(\mathbb{R}^{3n})$ is a function dominantly supported near $(\mathbf{p})_n = 0$ with inverse Fourier transforms localized near $(\mathbf{x})_n = 0$, then state describing functions

$$\tilde{\varphi}_n((p)_n; \lambda) = \prod_{j=1}^{n} e^{i\mathbf{p}_j \cdot \mathbf{u}_j(\lambda)} (p_{j0} + \omega_j) \tilde{f}_n((\mathbf{p} - \mathbf{w}(\lambda))_n; \lambda) \tag{124}$$

have spatial supports centered on $\mathbf{u}_j(\lambda)$ and momentum supports centered on $\mathbf{w}_j(\lambda)$. The more closely the breadth of the support for each argument of f_n satisfies the Heisenberg uncertainty bound, the more classical particle-like the state describing functions (124). Additional properties include that the description lacks entanglement and the $\mathbf{u}_j(\lambda)$ are sufficiently space-like separated to satisfy isolation. With appropriate selections for $\mathbf{u}_j(\lambda)$, $\mathbf{w}_j(\lambda)$ and $\tilde{f}_n((\mathbf{p})_n, \lambda)$, the support of (124) is non-relativistic (120), isolated (119), localized (116) and has representative momenta (118). From the Fourier transform pair (76) with (77), (124) has an inverse Fourier transform (21),

$$\varphi_n((x)_n; \lambda) = \prod_{j=1}^{n} \left(-i \frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2} - \Delta_j} \right) e^{-i\mathbf{w}_j(\lambda) \cdot (\mathbf{x}_j - \mathbf{u}_j(\lambda))} f_n((x - u(\lambda))_n; \lambda) \tag{125}$$

with Δ_j the Laplacian in $\mathbb{R}^3$ for the argument $\mathbf{x}_j$. In (125), classical trajectories $u_j(\lambda) = (0, \mathbf{u}_j(\lambda))$ are representatives for the volumes of dominant spatial support, and the $\hbar \mathbf{w}_j(\lambda)$ are representatives for the dominant momentum supports. The temporal parameter λ is included in f_n in (125) to implement deformation of $f_n((x)_n; \lambda)$ with time evolution (75). Unitary time evolution of $\varphi_n((x)_n; 0)$ results in a growth in the breadth of support of $f_n((x)_n; \lambda)$: classically, a spreading of the spatial support over time follows from uncertainty in the initial momenta. Example classical particle-like state describing functions include Gaussians with spreads determined by a complex λ-dependent parameter. The functions $\varphi_n((\mathbf{x})_n; \lambda)$ are infinitely differentiable with respect to $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$.

Position and momentum eigenfunctions are not suitable for quantum-classical correspondences. These eigenfunctions are among the least classical particle-like by the consideration that location and momentum are both determined in a classical description. Position eigenfunctions provide no indication of momentum and particle number is not necessarily conserved since momentum supports are relativistic when sufficiently localized. Similarly, there is no indication of location from momentum eigenfunctions. For scattering events, before plane wave limits, likelihoods $|\langle(\mathbf{q})_k|\tilde{\varphi}_n\rangle|^2$ are described as classical particle-like since asymptotically the supports of the state describing functions are sufficiently spatially separated that cluster decomposition A.6 provides sufficient localization (116) and isolation (119) to describe φ_n with corresponding particles.

5.3 Correspondence from the evolution of support along trajectories

Evident in scattering likelihoods, the constructed VEV from section 3.3 exhibit interaction. Interaction is also manifest in the brief interval transition likelihoods. In this section, a non-relativistic correspondence of classical particle trajectories with the time evolution of the support of state describing functions is developed. This quantum-classical correspondence associates classical dynamical variables with the evolution of particle-like support of state describing functions. The interactions that follow are generally distinct from the interactions characterizing the scattering amplitudes. This quantum-classical correspondence applies for particle-like state describing functions, those with isolated (119), non-relativistic (120) support well represented by one location (116) and one momentum (118). Plane wave scattering amplitudes apply for state describing functions with overlapping supports.

Our perceptions of "macroscopic" quantum state descriptions appear classical. The quantum-classical correspondence discussed in sections 1, 2 and 5.2 represents isolated volumes of support of the state describing functions $\varphi_n(\lambda)$ with classical dynamical variables. This representation requires isolation (119) to unambiguously associate an identifiable classical particle with the support of one state describing function argument. Accurate representations follow for localized support. The classical dynamical variables follow Newtonian dynamics in non-relativistic instances. More generally, if we calculate that the bodies move guided by classical fields $A(x)_\kappa$ with sources on the bodies (e.g., the Lorentz force and Maxwell's equations in electrodynamics, or geometrodynamical gravity) and this dynamics of classical particles produces an accurate approximation to the evolution of particle-like supports of state describing functions, then the quantum dynamics is well represented by the classical field theory. This correspondence is valid regardless of whether there is a canonical quantization. A correspondence of classical dynamics with quantum dynamics is established by the representation of the motions of the particle-like supports of state describing functions by the classical dynamical variables.

A quantum-classical correspondence is established if the evolution of a particle-like state describing function is well represented by the evolution of classical dynamical variables. This correspondence of classical particle trajectories $\mathbf{u}_j(\lambda)$ with the evolution of a state describing

function is implemented with a family of state describing functions $\varphi_n(\lambda)$.

$$|U(-\lambda)\hat{\varphi}_n(0)\rangle \approx e^{i\phi_I(\lambda)}|\hat{\varphi}_n(\lambda)\rangle \tag{126}$$

for times $\lambda > 0$. In (126), the state describing functions are both normalized in the Hilbert space norm (25),

$$\hat{\varphi}_n(\lambda) = \frac{\varphi_n(\lambda)}{\|\varphi_n(\lambda)\|}. \tag{127}$$

The unitary mapping $U(-\lambda)$ (75) of the normalized state describing functions $\hat{\varphi}_n(0)$ is the quantum mechanical evolution. For a state describing function $\varphi_n(\lambda)$ of the form (124), the Fourier transform of the quantum evolution is

$$U(-\lambda)\tilde{\varphi}_n(0) = \prod_{j=1}^{n} e^{i\omega_j\lambda} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(0)}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{w}(0))_n;0)$$

with the substitution (103) for the time translation operator. The nearly equal state describing function is a copy of the initial state described by $\varphi_n(0)$ translated along the trajectories $\mathbf{u}_j(\lambda)$, centered of the momenta $\mathbf{w}(\lambda)$ and with a broadening of support that follows from the Heisenberg uncertainty principle. The second state describing function is

$$\tilde{\varphi}_n(\lambda) = \prod_{j=1}^{n} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{w}(\lambda))_n;\lambda)$$

with $\dot{\mathbf{u}}(\lambda) \approx \lambda_c\mathbf{w}(\lambda)$ in non-relativistic approximation. Time λ is advanced in the state describing function $U(-\lambda)\varphi_n(0)$ to match the advancement of time in the description of trajectories. The condition (126) states that $|\varphi_n(\lambda)\rangle$ represents the same state as $|U(-\lambda)\varphi_n(0)\rangle$ and since states are described by rays, changes to norms are not of interest. The phase $\phi_I(\lambda)$ in the correspondence (126) is a purely quantum mechanical consideration following from the descriptions of states as rays in a complex Hilbert space. The correspondence (126) asserts that the evolution of the support of $\varphi_n(0)$ is perceived to follow trajectories $\mathbf{u}_j(\lambda)$ that satisfy classical, Newtonian dynamics. A quantum-classical correspondence is justified for particle-like state describing functions, those with spatial and momentum support that is well represented by one location (116) and one momentum (118), and with the spatial isolation (119) of supports necessary to associate the supports of arguments of state describing functions with identifiable particles. Newtonian mechanics applies if the supports are also non-relativistic (120).

From Born's rule, the squared magnitude of the scalar product

$$|\langle U(-\lambda)\hat{\varphi}_n(0)\rangle|\hat{\varphi}_n(\lambda)\rangle|^2$$

is the likelihood that the state described by $U(-\lambda)\hat{\varphi}_n(0)$ is perceived as the state described by $\hat{\varphi}_n(\lambda)$. Justified by Poincaré invariance of the scalar product (71), the reference time is taken as zero without loss of generality.

The most likely to be observed trajectories $\mathbf{u}_j(\lambda)$ optimize the correspondence (126). Conversely, given a trajectory $\mathbf{u}_j(\lambda)$, (126) defines a matching distorted function,

$$\varphi_n(\lambda) = e^{-i\phi_I(\lambda)} U(-\lambda)\hat{\varphi}_n(0)\rangle, \tag{128}$$

whether there is a quantum-classical correspondence or not. These distorted functions (128) are not of interest to identify corresponding trajectories $\mathbf{u}_j(\lambda)$. The correspondence (126) provides physical insight only when the classical dynamical variables represent the state describing functions well. The most classical particle-like state describing functions are Gaussians and the evolution of Gaussians remain Gaussian with complex spreading parameters as long as non-relativistic approximations apply. Gaussian functions initially exhibit the strongest quantum-classical correspondence: for real spreading parameters, the Heisenberg uncertainty lower bound on location and momentum support spreads is met. Support spreading from the unitary evolution of the most classical particle-like state describing function $\varphi_n(0)$ exceeds the Heisenberg uncertainty lower bound when $\lambda > 0$.

The approximate equality of two state describing functions (126) is established in the Hilbert space norm (25). The phase $\phi_I(\lambda)$ is determined to minimize separation.

$$\begin{aligned} 0 &\approx \left\| U(-\lambda)\hat{\varphi}_n(0) - e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda) \right\|^2 \\ &= 2 - 2\Re e\left(e^{i\phi_I(\lambda)} \langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle \right). \end{aligned}$$

The maximization of $\Re e\left(e^{i\phi_I(\lambda)} \langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle \right)$ determines $\phi_I(\lambda)$. Polar decomposition [49] of $\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle$ is

$$\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle = |\langle U(-\lambda)\varphi_n(0) | \varphi_n(\lambda)\rangle|\, e^{-i\phi_I(\lambda)}. \tag{129}$$

Then, with this selection for phase $\phi_I(\lambda)$,

$$\begin{aligned} 0 &\approx \left\| U(-\lambda)\hat{\varphi}_n(0) - e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda) \right\|^2 \\ &= 2 - 2\,|\langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle|\,. \end{aligned}$$

Below it is demonstrated that the phase is additive and homogeneous over time.

$$\begin{aligned} \phi_I(a+b) &= \phi_I(a) + \phi_I(b) \\ \phi_I(0) &= 0. \end{aligned}$$

The Cauchy-Schwarz-Bunyakovsky inequality provides that

$$|\langle U(-\lambda)\hat{\varphi}_n(0) | \hat{\varphi}_n(\lambda)\rangle| \leq 1 \tag{130}$$

in the normalization (127). The upper bound is achieved for $\lambda = 0$.

It is convenient to designate the scalar product of normalized state describing functions

$$I(\lambda) = \frac{\langle U(-\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle}{\|\varphi_n(0)\|\ \|\varphi_n(\lambda)\|} \tag{131}$$

and from Poincaré invariance of the scalar product,

$$\|U(-\lambda)\varphi_n(0)\| = \|\varphi_n(0)\|.$$

From the selection of phase (129),

$$|I(\lambda)| = e^{i\phi_I(\lambda)} I(\lambda).$$

This scalar product (131) varies with the unknown trajectories $\mathbf{u}_j(\lambda)$ and momenta $\mathbf{w}_j(\lambda)$. $\mathbf{u}_j(0), \dot{\mathbf{u}}_j(0)$ are considered initial conditions for the corresponding classical trajectories. Likelihood is $|I(\lambda)|^2$.

A quantum-classical correspondence (126) follows from two considerations:

C.1) the maximal likelihood $|I(\lambda)|^2$ is near unity

C.2) the dominant supports of the evolved functions (125) remain non-relativistic (120) and classical particle-like.

The corresponding classical dynamics follows from the trajectories $\mathbf{u}_j(\lambda)$.

Interaction is exhibited in the relative motions of the corresponding classical particles. The motion of the center of momentum is free, and irrelevant to determination of relative trajectories. In non-relativistic approximation, particle numbers are conserved. The state describing functions $\tilde{\varphi}_n(\lambda)$ (125) are composite functions over $6n$ functions $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda) = \lambda_c \mathbf{w}_j(\lambda)$. The center of momentum of the n interacting classical particles representative of $\varphi_n(\lambda)$ evolves as a single free body independently of the relative motion of the bodies. Independence applies in non-relativistic approximation (120) when length contraction and time dilation are negligible. Poincaré invariance (71) of the scalar product (13) and covariance (72) of state descriptions $\varphi_n(\lambda)$ provide that a Poincaré transformation equates any scalar product to the scalar product in a center of momentum reference frame determined by the n trajectories $\mathbf{u}_j(\lambda)$ with momenta $\mathbf{w}_j(\lambda)$. In this center of momentum frame and for non-relativistic momenta,

$$\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda) = 0, \quad \text{and} \qquad \sum_{j=1}^{n} m_{\kappa_j} \mathbf{w}_j(\lambda) = 0.$$

Transformation to this frame is a boost described in appendix 8.11, plus a translation to collocate the center of mass

$$\frac{\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda)}{\sum_{j=1}^{n} m_{\kappa_j}}$$

with the origin of coordinates. As a consequence, only $n-1$ of the pairs $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$ are independent variables determining $I(\lambda)$.

For the most likely $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$, satisfaction of the quantum-classical correspondence (126) is not improved by any modification to the $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$. For the optimal trajectory,

$$\begin{aligned} \frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} &= 0 \\ \frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} &= 0 \end{aligned} \tag{132}$$

individually for each component of the $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda) = \lambda_c \mathbf{w}_j(\lambda)$. Here, partial derivatives with respect to a spatial vector designates the gradient vector

$$\frac{\partial F}{\partial \mathbf{u}_j} = \frac{\partial F}{\partial u_{jx}}, \frac{\partial F}{\partial u_{jy}}, \frac{\partial F}{\partial u_{jz}}.$$

Chain rule expansion of the derivative of $|I(\lambda)|^2$ includes partial derivatives with respect to the $n-1$ independent $\mathbf{u}_j(\lambda), \dot{\mathbf{u}}_j(\lambda)$.

$$\frac{d|I(\lambda)|^2}{d\lambda} = \sum_{j=1}^{n-1} \frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} \cdot \dot{\mathbf{u}}_j(\lambda) + \sum_{j=1}^{n-1} \frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} \cdot \ddot{\mathbf{u}}_j(\lambda) + \frac{\partial |I(\lambda)|^2}{\partial \lambda} \tag{133}$$

with $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$ held constant in the evaluation of

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda}.$$

If the optimality conditions (132) are satisfied, then the trajectories that minimize the loss of likelihood (131) with increasing λ minimize

$$\frac{d|I(\lambda)|^2}{d\lambda} = \frac{\partial |I(\lambda)|^2}{\partial \lambda} \approx 0. \tag{134}$$

The product rule and $|I(\lambda)| \approx 1$ for brief λ provide that (134) is satisfied if and only if

$$\frac{\partial |I(\lambda)|}{\partial \lambda} \approx 0.$$

Sesquilinearity of the scalar product (13) and $U(-\lambda) = e^{iH\lambda}$ (103) provide that the partial

derivative of $|I(\lambda)|$ (131) evaluated at $\lambda = 0$ is

$$\begin{aligned} 0 &= \frac{\partial |I(\lambda)|}{\partial \lambda}\Big|_{\lambda=0} \\ &= \frac{\partial}{\partial \lambda} \frac{e^{i\phi_I(\lambda)} \langle e^{iH\lambda} \hat{\varphi}_n(0) | \varphi_n(\lambda)\rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|}\Big|_{\lambda=0} \\ &= i\frac{\partial \phi_I(\lambda)}{\partial \lambda} - i\frac{\langle H\varphi_n(0)|\varphi_n(0)\rangle}{\|\varphi_n(0)\|^2} + \left(\frac{\partial}{\partial \lambda} \frac{\langle \varphi_n(0)|\varphi_n(\lambda)\rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|}\right)\Big|_{\lambda=0} \\ &= i\frac{\partial \phi_I(\lambda)}{\partial \lambda} - i\frac{\langle H\varphi_n(0)|\varphi_n(0)\rangle}{\|\varphi_n(0)\|^2}. \end{aligned} \tag{135}$$

The second line follows from the polar decomposition (129). H is the Hamiltonian (103) in the n-argument subspace. The third line in evaluated at $\lambda = 0$ noting that $|I(0)|^2 = 1$ from (130). The third term of the third line,

$$\left(\frac{\partial}{\partial \lambda} \frac{\langle \varphi_n(0)|\varphi_n(\lambda)\rangle}{\|\varphi_n(0)\| \, \|\varphi_n(\lambda)\|}\right)\Big|_{\lambda=0} = 0$$

from the differentiability of the scalar product and that, from the Cauchy-Schwarz-Bunyakovsky inequality, $\lambda = 0$ is a maximum of the normalized scalar product. The second term in the final line of (135) is the expectation of the energy of $\varphi_n(0)$, and (135) relates this energy to the derivatives of the phase $\phi_I(\lambda)$. In section 5.4.3, this relation (135) determines the classical dynamical variables that correspond to the quantum dynamics. The identity (135) relates classical dynamical variables to the constructed VEV and particle-like, non-relativistic state describing functions. In section 5.4.2, $\phi_I(\lambda)$ is evaluated in classical dynamical variables after brief interval, non-relativistic, limited acceleration approximations.

First order brief interval expansions do not suffice to identify the corresponding trajectories $\mathbf{u}_j(\lambda), \mathbf{w}_j(\lambda)$ using the optimality criteria (132). $\lambda = 0$ is maximum of likelihood for all trajectories $\mathbf{u}_j(\lambda)$. At $\lambda = 0$,

$$\frac{\partial |I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} = \frac{\partial |I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} = 0$$

independently of $\mathbf{u}_j(\lambda), \mathbf{w}_j(\lambda)$. With an expansion of the notation to display the dependence of the likelihood (137) on the trajectories $\mathbf{u}_j(\lambda)$ with $\mathbf{w}_j(\lambda)$, designate

$$I(\lambda) = I((\mathbf{u}(\lambda), \mathbf{w}(\lambda))_{n\text{-}1}, \lambda).$$

From the differentiability of the likelihood, a Taylor series polynomial provides an expression for $|I(\lambda)|^2$ with small changes to trajectories from their initial values.

$$|I((\mathbf{u}(\lambda), \mathbf{w}(\lambda))_{n\text{-}1}, \lambda)|^2 \approx |I(0)|^2 + (s - s(0))\frac{d|I(0)|^2}{ds} + \frac{1}{2}(s - s(0))^2 \frac{d^2|I(0)|^2}{ds^2} + \dots \tag{136}$$

with only one component of the $\mathbf{u}_j(\lambda)$ or $\mathbf{w}_j(\lambda)$ different from its value at $\lambda = 0$. $|I(0)|^2$ is $|I((\mathbf{u}(0), \mathbf{w}(0))_{n-1}, 0)|^2$. The variable s equals this one non-zero component of $\mathbf{u}_j(\lambda)$ or component of $\mathbf{w}_j(\lambda)$, and $s(0)$ is $\mathbf{u}_j(0)$ or $\mathbf{w}_j(0)$. From the Cauchy-Schwarz-Bunyakovsky inequality (130), $|I((\mathbf{u}(0), \mathbf{w}(0))_{n-1}, 0)|^2$ is recognized as a maxima for any $(\mathbf{u}(0), \mathbf{w}(0))_{n-1}$ and as a consequence

$$\frac{d|I(0)|^2}{ds} = 0$$

and

$$|I((\mathbf{u}(\lambda), \mathbf{w}(\lambda))_{n-1}, \lambda)|^2 \approx |I(0)|^2 + \frac{1}{2}(s - s(0))^2 \frac{d^2|I(0)|^2}{ds^2} + \dots$$

with $(s - s(0))^2 = \mathcal{O}(\lambda^2)$ for brief interval expansions $s(\lambda) \approx s(0) + \lambda \dot{s}(0)$. As a consequence, first order brief interval expansions such as the development in section 5.4.2 do not suffice to identify the corresponding trajectories $\mathbf{u}_j(\lambda), \mathbf{w}_j(\lambda)$ using the optimality criteria (132). The energy relation (135) relates the classical dynamical variables and quantum dynamics for the brief interval approximations. This development is in section 5.4.3.

Representations of the evolution of the state describing function by a single classical trajectory per body (126) deteriorate with increased propagation intervals. Considering the classical correspondence for each classical body, nearby initial conditions produce trajectories that diverge with time. With sufficient divergence of the corresponding trajectories, and with the consequent increased spatial spread of support, association of the support of a state describing function with a classical body, satisfaction of C.2, is lost over time. Another perspective on the limited duration of the quantum-classical correspondence (126) is suggested by the Riemann-Lebesgue lemma [49]. The likelihood

$$|\langle U(-\lambda)\varphi_n(0)|e^{i\phi_I(\lambda)}\varphi_n(\lambda)\rangle|^2$$

will asymptotically converge to zero for $\lambda \to \infty$ unless the phases $\exp(i\omega_k\lambda)$ from the Hamiltonian (102) are compensated by the state describing function $\varphi_n(\lambda)$. The phase $\phi_I(\lambda)$ does not suffice to compensate $\exp(i\omega_k\lambda)$ since $\phi_I(\lambda)$ does not vary with momenta $(\mathbf{p})_n$. An uncompensated phase proportional to λ leads to an asymptotically vanishing scalar product.

Persistent classical correspondences such as planetary motions over great periods include recurring observations. Presumably, each of these observations leave the observed in a state very near the propagation of the initial state that corresponds with a particular classical description. Effective observations result from scatter and emission of radiation, and perturbations of the motion of additional bodies. Inclusion of the effects of recurring observations on observed long term classical correspondences is an additional insight requiring further development.

5.4 Two body correspondence

This section is a substantial digression to evaluate the scalar product for two-argument state describing functions. For two-argument state describing functions, the non-relativistic corre-

spondence is with two classical bodies. Two classical bodies are described by a freely evolving center of momentum and one independent trajectory $\mathbf{u}_1(\lambda)$. Non-relativistic, brief interval, limited acceleration approximations for the scalar products (126) and derivatives are evaluated. For two corresponding classical bodies, classical trajectories are available, for example, appendix 8.13. The example evaluation uses the VEV for a single neutral scalar field from section 3 and state describing functions (125). For the most classical-like states, it is shown that the energy correspondence (135) determines the state describing functions from corresponding classical parameters and the quantum dynamical parameters. Lacking the additional quantum dynamical principle introduced in section 1, the correspondence with observed classical dynamics determines the quantum dynamics of state describing functions with non-relativistic, isolated support. The additional principle would be in addition to axioms A.1-7 of section 4.1.

For two-argument state describing functions, satisfaction of the quantum-classical correspondence (126) maximizes the likelihood

$$|I(\lambda)|^2 = \frac{|\langle e^{iH\lambda}\varphi_2(0)|\varphi_2(\lambda)\rangle|^2}{\langle\varphi_2(0)|\varphi_2(0)\rangle\langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle} \tag{137}$$

and satisfies the energy relation (135). Following the development in section 5.2, appropriate two-argument state describing functions

$$\varphi_2(x_1, x_2; \lambda)$$

have Fourier transforms

$$\tilde{\varphi}_2(p_1, p_2; \lambda) = \prod_{j=1}^{2} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0} + \omega_j)\ \tilde{f}_2(\mathbf{p}_1 - \mathbf{w}_1(\lambda), \mathbf{p}_2 - \mathbf{w}_2(\lambda); \lambda). \tag{138}$$

f_2 is dominantly localized near $(\mathbf{x})_n = 0$ with Fourier transform dominantly supported near $(\mathbf{p})_2 = 0$. Poincaré invariance is exploited to express scalar products in the center of momentum coordinate frame determined by the corresponding classical locations $\mathbf{u}_j(\lambda)$ and momenta $\mathbf{w}_j(\lambda)$. In the resulting center of momentum coordinate frame with the center of mass collocated with the origin,

$$\mathbf{u}_2(\lambda) = -\mathbf{u}_1(\lambda) \tag{139}$$

and $\mathbf{w}_2(\lambda) = -\mathbf{w}_1(\lambda)$. For non-relativistic (120) momenta, $\lambda_c\mathbf{w}_j(\lambda) \approx \dot{\mathbf{u}}_j(\lambda)$. In this reference frame, an abbreviated notation,

$$\mathbf{u}(\lambda) = \mathbf{u}_1(\lambda), \tag{140}$$

is substituted. Similarly, $\mathbf{w}(\lambda) = \mathbf{w}_1(\lambda)$.

5.4.1 Non-relativistic approximation

Non-relativistic approximations for the scalar products of state describing functions $U(-\lambda)\varphi_2(0)$ and $\varphi_2(\lambda)$ are developed for selected state describing and VEV functions in this section.

The form (138) of $\tilde{f}_2$ is selected to eliminate consideration of the motion of the center of momentum in the evaluation of the likelihoods $|I(\lambda)|^2$. With the selected form and in evaluation of the scalar products that evaluate (137), description of the motion of the center of momentum is factored from description of the relative motion of the two bodies and thus eliminated from the likelihood (137). The selected $\tilde{f}_2$ separates in Jacobi sum and difference coordinates.

$$\tilde{f}_2(\mathbf{p}_1-\mathbf{w}(\lambda),\mathbf{p}_2+\mathbf{w}(\lambda);\lambda) = \tilde{f}_M(\mathbf{p}_1+\mathbf{p}_2;\lambda)\tilde{f}_I(\frac{\mathbf{p}_1-\mathbf{p}_2}{2}-\mathbf{w}(\lambda);\lambda). \tag{141}$$

f_M describes the center of momentum, and f_I describes the relative motion of two corresponding classical bodies designated 1 and 2. Reliable association of classical bodies with arguments requires isolation (119). (141) is expressed in the center of momentum reference frame with the abbreviated notation (140). The supports of $\tilde{f}_M(\mathbf{p};\lambda)$ and $\tilde{f}_I(\mathbf{p};\lambda)$ are centered on the origins in both the spatial and momentum domains. The Jacobi coordinates are designated

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{p}_1+\mathbf{p}_2, & \mathbf{q}_2 &= \tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2) \\ \mathbf{q}_3 &= \mathbf{p}_3+\mathbf{p}_4, & \mathbf{q}_4 &= \tfrac{1}{2}(\mathbf{p}_3-\mathbf{p}_4) \end{aligned} \tag{142}$$

and then

$$\begin{aligned} \mathbf{p}_1 &= \tfrac{1}{2}\mathbf{q}_1+\mathbf{q}_2, & \mathbf{p}_2 &= \tfrac{1}{2}\mathbf{q}_1-\mathbf{q}_2 \\ \mathbf{p}_3 &= \tfrac{1}{2}\mathbf{q}_3+\mathbf{q}_4, & \mathbf{p}_4 &= \tfrac{1}{2}\mathbf{q}_3-\mathbf{q}_4. \end{aligned}$$

The determinant of the Jacobian matrix for the coordinate transformation $(\mathbf{p})_4 \mapsto (\mathbf{q})_4$ is unity.

In the Jacobi coordinates, conservation of momentum separates, $\mathbf{q}_1, \mathbf{q}_3$ center of momentum dependence from $\mathbf{q}_2, \mathbf{q}_4$ internal motion dependence.

$$\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4) = \delta(\mathbf{q}_1+\mathbf{q}_3).$$

Factors of ω_j+p_{j0} after evaluation of mass shell delta functions in the state describing functions become approximately constant in non-relativistic approximation, $\omega_j \approx \lambda_c^{-1}$ if $\lambda_c^2\mathbf{p}_j^2 \ll 1$. The remaining separation of variables considerations are factoring the momentum dependence in the unitary time translation $U(-\lambda)$ and in the energy conservation delta function.

Separation of variables in the time translation (103) follows from non-relativistic approximation (120) for the Hamiltonian (102). Factors of $\omega_j + p_{j0}$ commute with the Hamiltonian (102) and then the time evolution of the state describing function is

$$U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) = e^{i(\omega_1+\omega_2)\lambda}\ \tilde{f}_M(\mathbf{p}_1+\mathbf{p}_2;0)\tilde{f}_I(\frac{\mathbf{p}_1-\mathbf{p}_2}{2}-\mathbf{w}(0);0).$$

The Hamiltonian in the two-argument subspace is $\omega_1 + \omega_2$. Taylor theorem polynomial approximation results in

$$\begin{aligned} \omega_1 + \omega_2 &= \omega(\tfrac{1}{2}\mathbf{q}_1 + \mathbf{q}_2) + \omega(\tfrac{1}{2}\mathbf{q}_1 - \mathbf{q}_2) \\ &\approx 2\omega(\mathbf{q}_2) + \tfrac{1}{4}\lambda_c \mathbf{q}_1^2 \end{aligned} \tag{143}$$

from (8) and with a binomial expansion for $\lambda_c^2 \mathbf{q}_1^2 \ll 1$. Then, for the state describing function (138) in non-relativistic instances, time translation also factors in Jacobi coordinates,

$$U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2; 0) \approx \left(e^{i\frac{1}{4}\lambda_c \mathbf{q}_1^2 \lambda}\tilde{f}_M(\mathbf{q}_1; 0)\right)\left(e^{2i\omega(\mathbf{q}_2)\lambda}\tilde{f}_I(\mathbf{q}_2 - \mathbf{w}(0); 0)\right). \tag{144}$$

Non-relativistic approximation of the Hamiltonian limits the interval λ that a non-relativistic approximation of time translation applies. For negligible error,

$$\left(\omega_1 + \omega_2 - 2\omega(\mathbf{q}_2) - \frac{1}{4}\lambda_c \mathbf{q}_1^2\right)\lambda \ll \pi.$$

Indeed, the non-relativistic approximations to ω_j are polynomials in $\mathbf{p}_j$ and are qualitatively distinct from the Hamiltonian. $\sqrt{\lambda_c^{-2} - \Delta}$ is an anti-local operator [55] while powers of the Laplacian Δ are local.

Separation of variables in the energy conservation delta function also follows from the non-relativistic approximation, (143).

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 \approx 2\omega(\mathbf{q}_2) - 2\omega(\mathbf{q}_4) \tag{145}$$

since $\mathbf{q}_1 = \mathbf{q}_3$ due to the conservation of momentum. Only the momenta in the description of the internal motion appear in the energy conservation delta function.

With time evolution applied to define the evolved center of mass state describing function,

$$\tilde{f}_M(\mathbf{p}; \lambda) = e^{i\frac{1}{4}\lambda_c \mathbf{p}^2 \lambda}\tilde{f}_M(\mathbf{p}; 0), \tag{146}$$

non-relativistic approximation of the Hamiltonians (144) with (146) provides that the state describing functions include a common factor.

$$\begin{aligned} \tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2; \lambda) &= \tilde{f}_M(\mathbf{q}_1; \lambda)\, \tilde{f}_I(\mathbf{q}_2 - \mathbf{w}(\lambda); \lambda) \\ U(-\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2; 0) &= \tilde{f}_M(\mathbf{q}_1; \lambda)\, e^{2i\omega(\mathbf{q}_2)\lambda}\tilde{f}_I(\mathbf{q}_2 - \mathbf{w}(0); 0) \end{aligned} \tag{147}$$

The non-relativistic descriptions of the two, two-argument state describing functions in the likelihood (137) include the description of the center of momentum motion as a common factor $\tilde{f}_M(\mathbf{q}_1; \lambda)$.

Cluster expansion (58) of connected functions (42) for the VEV functions of a single neutral scalar field provides that scalar products of two-argument functions follow from VEV

$$\langle\widetilde{\Phi}(p_2)\widetilde{\Phi}(p_1)\Omega|\widetilde{\Phi}(p_3)\widetilde{\Phi}(p_4)\Omega\rangle = \widetilde{\mathcal{F}}_{2,2}((p)_4) + \widetilde{\mathcal{U}}_{2,2}((p)_4). \tag{148}$$

The VEV consist of a four-argument connected function $\mathcal{U}_{2,2}((x)_4)$ from (17) and a free field contribution $\mathcal{F}_{2,2}((x)_4)$ with the two-point VEV function (32).

$$\begin{aligned}\widetilde{\mathcal{U}}_{2,2}((p)_4) &= c_4\,\delta(p_1+p_2+p_3+p_4)\prod_{j=1}^{4}\delta(p_j^2-\lambda_c^{-2})\\ \widetilde{\mathcal{F}}_{2,2}((p)_4) &= (\delta(p_1+p_3)\delta(p_2+p_4)+\delta(p_1+p_4)\delta(p_2+p_3))\prod_{j=3}^{4}\theta(p_{0j})\delta(p_j^2-\lambda_c^{-2}).\end{aligned} \tag{149}$$

Due to the factors $\delta(p_j^2-m^2)$ that limit energy-momenta p_j to mass shells, the factor $\theta(p_{03})\theta(p_{04})$ is redundant with the mass shell zeros from state describing functions in $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$.

With

$$g_2(\mathbf{p}_1,\mathbf{p}_2;\lambda) = \prod_{j=1}^{2} e^{i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}\;\tilde{f}_2(\mathbf{p}_1-\mathbf{w}_1(\lambda),\mathbf{p}_2-\mathbf{w}_2(\lambda);\lambda), \tag{150}$$

the frequency domain representation (114) of the scalar product with state describing functions (138) and the VEV (148) becomes

$$\begin{aligned}\langle\varphi_2(\lambda')|\varphi_2(\lambda)\rangle &\approx \int d(\mathbf{p})_4\;\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2;\lambda')}\;\tilde{g}_2(\mathbf{p}_3,\mathbf{p}_4;\lambda)\\ &\quad\times\left\{\frac{4}{\lambda_c^2}\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)+c_4\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\right\}\\ &= \int d(\mathbf{q})_4\;\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2;\lambda')}\;\tilde{g}_2(\mathbf{p}_3,\mathbf{p}_4;\lambda)\\ &\quad\times\left\{\frac{4}{\lambda_c^2}\delta(\mathbf{q}_2-\mathbf{q}_4)+c_4\delta(2\omega(\mathbf{q}_2)-2\omega(\mathbf{q}_4))\right\}\delta(\mathbf{q}_1-\mathbf{q}_3).\end{aligned} \tag{151}$$

Evaluation of each of the $\delta(p_j^2-m^2)$ result in factors $1/(2\omega_j)$ that multiply to unity with the factors ω_j+p_{j0} from state describing functions in $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$. The final line is the change to Jacobi variables (142) with non-relativistic approximation. The non-relativistic approximation includes $\omega_j\approx\lambda_c^{-1}$. The

$$\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$$

"cross" term in the free field contribution to the VEV is negligible for state describing functions with sufficient isolation (119) to have a reliable quantum-classical correspondence. The Pauli-Jordan two-point function has exponential spatial decline in space-like directions with characteristic length λ_c and if separations $\|\mathbf{u}(\lambda)\|$ are large compared to λ_c, the contribution

of $\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$ is negligible. From (145) and in Jacobi coordinates (142), the energy-momentum conservation delta functions become

$$\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) = \delta(2\omega(\mathbf{q}_2) - 2\omega(\mathbf{q}_4))\delta(\mathbf{q}_1 - \mathbf{q}_3)$$

and

$$\begin{aligned}\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4) &= \delta(\tfrac{1}{2}(\mathbf{q}_1-\mathbf{q}_3)+\mathbf{q}_2-\mathbf{q}_4)\delta(\tfrac{1}{2}(\mathbf{q}_1-\mathbf{q}_3)-\mathbf{q}_2+\mathbf{q}_4)\\ &= \delta(\mathbf{q}_1-\mathbf{q}_3)\delta(\mathbf{q}_2-\mathbf{q}_4).\end{aligned}$$

The result is (151).

Substitution of the factored state describing function (141) into (150) in Jacobi coordinates (142) results in

$$\begin{aligned}g_2(\mathbf{p}_1,\mathbf{p}_2;\lambda) &= e^{i(\frac{1}{2}\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{u}(\lambda)}e^{-i(\frac{1}{2}\mathbf{q}_1-\mathbf{q}_2)\cdot\mathbf{u}(\lambda)}\ \tilde{f}_M(\mathbf{q}_1;\lambda)\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(\lambda);\lambda)\\ &= \tilde{f}_M(\mathbf{q}_1;\lambda)\ e^{i2\mathbf{q}_2\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(\lambda);\lambda).\end{aligned}$$

To abbreviate the presentation, designate the generalized function from the VEV in the scalar product (151) as

$$\begin{aligned}T_4(\mathbf{q}_2,\mathbf{q}_4) &= \frac{4}{\lambda_c^2}\delta(\mathbf{q}_2-\mathbf{q}_4) + c_4\delta(2\omega(\mathbf{q}_2)-2\omega(\mathbf{q}_4))\\ &= \frac{4}{\lambda_c^2}\delta(\mathbf{q}_2-\mathbf{q}_4) + \frac{c_4}{\lambda_c}\delta(\mathbf{q}_2^2-\mathbf{q}_4^2).\end{aligned} \tag{152}$$

Then, substitution for (150) into (151) produces

$$\begin{aligned}\langle\varphi_2(\lambda')|\varphi_2(\lambda)\rangle &= \int d(\mathbf{q})_4\ T_4(\mathbf{q}_2,\mathbf{q}_4)\delta(\mathbf{q}_1-\mathbf{q}_3)\ \overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2;\lambda')}\ \tilde{g}_2(\mathbf{p}_3,\mathbf{p}_4;\lambda)\\ &= \int d(\mathbf{q})_4\ T_4(\mathbf{q}_2,\mathbf{q}_4)\delta(\mathbf{q}_1-\mathbf{q}_3)\\ &\qquad\times\overline{\tilde{f}_M(\mathbf{q}_1;\lambda')}\ e^{-i2\mathbf{q}_2\cdot\mathbf{u}(\lambda')}\overline{\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(\lambda');\lambda')}\\ &\qquad\times\tilde{f}_M(\mathbf{q}_3;\lambda)\ e^{i2\mathbf{q}_4\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{q}_4-\mathbf{w}(\lambda);\lambda)\end{aligned} \tag{153}$$

for the scalar product of two state describing functions functions of the form (138) with (141).

The expression (153) for the scalar product suffices to evaluate the three scalar products in the likelihood (137). Substitution of the non-relativistic approximation for time evolution of state describing functions (144) and the identity (146) into the scalar product (153) results in

$$\begin{aligned}\langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \approx \int d\mathbf{q}_1\ |\tilde{f}_M(\mathbf{q}_1;\lambda)|^2\int d\mathbf{q}_2 d\mathbf{q}_4\ T_4(\mathbf{q}_2,\mathbf{q}_4)\\ \times e^{-2i\omega(\mathbf{q}_2)\lambda}e^{-i2\mathbf{q}_2\cdot\mathbf{u}(0)}\overline{\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(0);0)}e^{i2\mathbf{q}_4\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{q}_4-\mathbf{w}(\lambda);\lambda)\end{aligned}$$

designated the mixed scalar product and

$$\langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle \approx \int d\mathbf{q}_1\ |\tilde{f}_M(\mathbf{q}_1;\lambda)|^2 \int d\mathbf{q}_2 d\mathbf{q}_4\ T_4(\mathbf{q}_2,\mathbf{q}_4) \times e^{i2(\mathbf{q}_4-\mathbf{q}_2)\cdot\mathbf{u}(\lambda)} \overline{\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(\lambda);\lambda)}\ \tilde{f}_I(\mathbf{q}_4-\mathbf{w}(\lambda);\lambda)$$

provides the square of both norms, $\|\varphi_2(0)\|$ and $\|\varphi_2(\lambda)\|$. The unimodular phase in (146) provides that the summation

$$\int d\mathbf{q}_1\ |\tilde{f}_M(\mathbf{q}_1;\lambda)|^2 = \int d\mathbf{q}_1\ |\tilde{f}_M(\mathbf{q}_1;0)|^2$$

does not vary with time λ and the description of the center of momentum becomes a common factor in the three scalar products in the likelihood (137). Common factors do not contribute to the likelihood (137). Designate the common factor in the three scalar products composing the likelihood (137)

$$A_g = \int d\mathbf{q}_1\ |\tilde{f}_M(\mathbf{q}_1;\lambda)|^2. \tag{154}$$

The selection for the description of the relative motion of the two bodies is the most classical-like state describing function, a Gaussian.

$$e^{2i\mathbf{q}_2\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{q}_2-\mathbf{w}(\lambda);\lambda) = e^{2i(\mathbf{q}_2-\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}\exp(-L(\lambda)^2(\mathbf{q}_2-\mathbf{w}(\lambda))^2). \tag{155}$$

The initial values of the trajectory $\mathbf{u}(0)$ with $\mathbf{w}(0)$ are selected to satisfy isolation (119) and non-relativistic (120) support. $L(\lambda)$ is a complex length characterizing the breadth of the spatial support. In (155),

$$\mathbb{R}e(L(\lambda)^2) > 0.$$

Real $L(0)^2$ are the most classical particle-like state describing functions, meeting the Heisenberg uncertainty lower bound for simultaneous knowledge of location and momentum. Complex $L(\lambda)^2$ follow for $\lambda > 0$ and for complex $L(\lambda)^2$, the Heisenberg uncertainty lower bound is exceeded due to additional spreading in location over time with a constant spread in momentum. To satisfy the non-relativistic quantum-classical correspondence (126), the support spread parameter $L(\lambda)^2$ must satisfy both upper and lower bounds. An upper bound

$$\frac{|L(\lambda)^2|^2}{\mathbb{R}e(L(\lambda)^2)} \ll \mathbf{u}(\lambda)^2$$

limits spatial extent with respect to separation to satisfy isolation (119). A lower bound suppresses the support on relativistic momenta (120),

$$\mathbb{R}e(L_0(\lambda)^2) \gg \lambda_c^2. \tag{156}$$

The bounds preclude both plane wave and point-like limits.

Time translations of a Gaussian $\tilde{f}_I$ remain Gaussian with the non-relativistic approximation. The Hamiltonian that describes the evolution of the relative motion is

$$\begin{aligned} 2\omega(\mathbf{q}_2) &\approx 2\lambda_c^{-1} + \lambda_c \mathbf{q}_2^2 \\ &= 2\lambda_c^{-1} + \lambda_c \mathbf{w}(\lambda)^2 + 2\lambda_c \mathbf{w}(\lambda)\cdot(\mathbf{q}_2 - \mathbf{w}(\lambda)) + \lambda_c(\mathbf{q}_2 - \mathbf{w}(\lambda))^2 \end{aligned} \tag{157}$$

expanded for $\mathbf{q}_2$ near $\mathbf{w}(\lambda)$ from (261) in appendix 8.14 to second order in small quantities $\lambda_c\|\mathbf{q}_2 - \mathbf{w}(\lambda)\| \ll 1$ and $\lambda_c\|\mathbf{w}(\lambda)\| \ll 1$.

Substitution of the Gaussian state describing function (155) into the scalar product results in

$$\begin{aligned} \langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \approx A_g \int d\mathbf{q}_2 d\mathbf{q}_4 \; T_4(\mathbf{q}_2, \mathbf{q}_4) \\ \times e^{-2i\omega(\mathbf{q}_2)\lambda} e^{-i2(\mathbf{q}_2 - \mathbf{w}(0))\cdot\mathbf{u}(0)} e^{i2(\mathbf{q}_4 - \mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-\overline{L(0)^2}(\mathbf{q}_2 - \mathbf{w}(0))^2} \; e^{-L(\lambda)^2(\mathbf{q}_4 - \mathbf{w}(\lambda))^2} \end{aligned} \tag{158}$$

for the mixed scalar product and

$$\begin{aligned} \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \approx A_g \int d\mathbf{q}_2 d\mathbf{q}_4 \; T_4(\mathbf{q}_2, \mathbf{q}_4) \\ \times e^{-i2(\mathbf{q}_2 - \mathbf{q}_4)\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{q}_2 - \mathbf{w}(\lambda))^2} \; e^{-L(\lambda)^2(\mathbf{q}_4 - \mathbf{w}(\lambda))^2} \end{aligned} \tag{159}$$

for the squared norms. $\mathbb{R}e(L(\lambda)^2) > 0$ and $\mathbf{u}(\lambda), \mathbf{w}(\lambda) \in \mathbb{R}$.

Evaluation of the delta functions in $T_4(\mathbf{q}_2, \mathbf{q}_4)$ leaves convergent summations for the scalar products (158) and (159) due to the rapid decline of Gaussian functions. From the dominated convergence theorem, derivatives of the scalar products are summations of the derivatives of the state describing functions.

5.4.2 Brief interval and limited acceleration approximation

In this section, the quantum-classical correspondence (126) is demonstrated for Gaussian state describing functions (147) with brief interval λ, limited acceleration approximations. Gaussian state describing functions (155) are the most classical particle-like by the criterion that both location and momenta are know as well as possible consistent with the Heisenberg uncertainty bounds. The demonstration applies if relative velocities are non-relativistic. The demonstration follows from expansion of the classical dynamical variables $\mathbf{u}(\lambda)$ and $\mathbf{w}(\lambda)$ in Taylor series for brief λ and a non-relativistic approximation for the Hamiltonian. With these approximations, and identification of the the phase $\phi_I(\lambda)$ and the support spread parameter $L(\lambda)^2$, the two state describing functions in (126), $U(-\lambda)\hat{\varphi}_2(0)$ and $e^{i\phi_I(\lambda)}\hat{\varphi}_2(\lambda)$, are equal. With this equality, the likelihood (137) is unity and when a quantum-classical correspondence applies, the quantum-classical correspondence (126) applies. The phase $\phi_I(\lambda)$ is identified as the classical energy of

two corresponding bodies times λ, and the support spread $L(\lambda)^2$ is the quantum mechanical support spreading of the evolution of two free particles.

Developed in section 5.4.1, in non-relativistic approximation, the functions $U(\lambda)\varphi_2(0)$ and $e^{i\phi_I(\lambda)}\varphi_2(\lambda)$ share a common description of the center of momentum and then the two functions are equal if the descriptions of the internals are equal. From the scalar product (158), the demonstration below is that

$$e^{2i\omega(\mathbf{q}_2)\lambda}e^{i2(\mathbf{q}_2-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{q}_2-\mathbf{w}(0))^2} \approx e^{i\phi_I(\lambda)}e^{i2(\mathbf{q}_2-\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}\ e^{-L(\lambda)^2(\mathbf{q}_2-\mathbf{w}(\lambda))^2}. \tag{160}$$

For sufficiently brief intervals $\lambda > 0$, Taylor theorem polynomial approximation to first order in λ suffice for the classical dynamical variables.

$$\begin{aligned} \mathbf{u}(\lambda) &\approx \mathbf{u}(0) + \lambda\dot{\mathbf{u}}(0) \\ \mathbf{w}(\lambda) &\approx \mathbf{w}(0) + \lambda\dot{\mathbf{w}}(0) \end{aligned} \tag{161}$$

and for non-relativistic momenta,

$$\lambda_c\mathbf{w}(\lambda) \approx \dot{\mathbf{u}}(\lambda).$$

A bound on λ to linearly extrapolate the classical dynamics is in addition to the limitation on λ to apply non-relativistic approximation of time translation.

Neglecting $\mathcal{O}(\lambda^2)$ contributions, the linear extrapolation (161) in the exponent of the classical description of the dynamics provides

$$\begin{aligned} (\mathbf{q}_2-\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda) &\approx (\mathbf{q}_2-\mathbf{w}(0)-\lambda\dot{\mathbf{w}}(0))\cdot(\mathbf{u}(0)+\lambda\dot{\mathbf{u}}(0)) \\ &= (\mathbf{q}_2-\mathbf{w}(0))\cdot\mathbf{u}(0) + \lambda(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{u}}(0) - \lambda\dot{\mathbf{w}}(0)\cdot\mathbf{u}(0). \end{aligned}$$

In the second term in the exponent of the classical description of the dynamics, the identification

$$L(\lambda)^2 = L(0)^2 - i\lambda_c\lambda \tag{162}$$

and linear extrapolation (161) results in

$$\begin{aligned} L(\lambda)^2(\mathbf{q}_2-\mathbf{w}(\lambda))^2 &\approx (L(0)^2 - i\lambda_c\lambda)\,(\mathbf{q}_2-\mathbf{w}(0)-\lambda\dot{\mathbf{w}}(0))^2 \\ &= L(0)^2(\mathbf{q}_2-\mathbf{w}(0))^2 - 2\lambda L(0)^2(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0) - i\lambda_c\lambda(\mathbf{q}_2-\mathbf{w}(0))^2. \end{aligned}$$

This $L(\lambda)^2$ describes the support spread from non-relativistic free propagation [24] of two particles with a separation described by a Gaussian state describing function in the Jacobi coordinates (142).

The Taylor polynomial approximation (157) of the Hamiltonian $\omega(\mathbf{q}_2)$ with the momentum $\mathbf{q}_2$ centered on a non-relativistic $\mathbf{w}(\lambda)$ is

$$2\omega(\mathbf{q}_2) \approx 2\lambda_c^{-1} + \lambda_c\mathbf{w}(\lambda)^2 + 2\lambda_c\mathbf{w}(\lambda)\cdot(\mathbf{q}_2 - \mathbf{w}(\lambda)) + \lambda_c(\mathbf{q}_2 - \mathbf{w}(\lambda))^2.$$

The linear extrapolations in time, division of both sides of (160) by the time translation $e^{i\omega(\mathbf{q}_2)\lambda}$, and the non-relativistic approximation for the Hamiltonian substituted into the exponent of the classical description of the dynamics produces

$$\begin{aligned}
&i\phi_I(\lambda) + i2(\mathbf{q}_2 - \mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) - L(\lambda)^2(\mathbf{q}_2 - \mathbf{w}(\lambda))^2 - 2i\omega(\mathbf{q}_2)\lambda \\
&\quad \approx i\phi_I(\lambda) + i\left\{2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \mathbf{u}(0) + \lambda(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \dot{\mathbf{u}}(0) - i2\lambda\dot{\mathbf{w}}(0) \cdot \mathbf{u}(0))\right\} \\
&\qquad - \left\{L(0)^2(\mathbf{q}_2 - \mathbf{w}(0))^2 - 4\lambda L(0)^2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0) - i\lambda_c\lambda(\mathbf{q}_2 - \mathbf{w}(0))^2\right\} \\
&\qquad - i\lambda\left\{2\lambda_c^{-1} + \lambda_c\mathbf{w}(\lambda)^2 + 2\lambda_c\mathbf{w}(\lambda) \cdot (\mathbf{q}_2 - \mathbf{w}(\lambda)) + \lambda_c(\mathbf{q}_2 - \mathbf{w}(\lambda))^2\right\}.
\end{aligned}$$

Substitution of $\lambda\mathbf{w}(\lambda) = \lambda\mathbf{w}(0)$ to $\mathcal{O}(\lambda^2)$, substitution of the non-relativistic approximation $\lambda_c\mathbf{w}(\lambda) = \dot{\mathbf{u}}(\lambda)$, and collecting terms results in

$$\begin{aligned}
&i\phi_I(\lambda) + i2(\mathbf{q}_2 - \mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) - L(\lambda)^2(\mathbf{q}_2 - \mathbf{w}(\lambda))^2 - 2i\omega(\mathbf{q}_2)\lambda \\
&\quad \approx i\phi_I(\lambda) + 2i(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \mathbf{u}(0) - L(0)^2(\mathbf{q}_2 - \mathbf{w}(0))^2 \\
&\qquad + 2\lambda L(0)^2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0) \\
&\qquad - i\lambda\left(2\lambda_c^{-1} + 2\lambda_c^{-1}\dot{\mathbf{u}}(\lambda)^2 - 2\lambda_c^{-1}\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0))\right)
\end{aligned}$$

for the exponent of the classical description of the dynamics. It follows from identifications of the phase

$$\phi_I(\lambda) = (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0))\frac{\lambda}{\lambda_c} \tag{163}$$

and an envelope correction factor

$$e^{2\lambda L(0)^2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)} \tag{164}$$

that

$$\begin{aligned}
&i\phi_I(\lambda) + i(\mathbf{q}_2 - \mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) - L(\lambda)^2(\mathbf{q}_2 - \mathbf{w}(\lambda))^2 \\
&\quad \approx 2i\omega(\tfrac{\mathbf{q}_2}{2})\lambda + i(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \mathbf{u}(0) - L(0)^2(\mathbf{q}_2 - \mathbf{w}(0))^2 + 2\lambda L(0)^2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0).
\end{aligned}$$

Except for the envelope correction factor, this is the desired equality (160). The identifications (162) for the support spread parameter $L(\lambda)^2$ and (163) for the phase $\phi_I(\lambda)$ demonstrate equality to $\mathcal{O}(\lambda^2)$ of the quantum dynamical description

$$e^{2i\omega(\mathbf{q}_2)\lambda}e^{i2(\mathbf{q}_2 - \mathbf{w}(0)) \cdot \mathbf{u}(0)}e^{-L(0)^2(\mathbf{q}_2 - \mathbf{w}(0))^2}$$

with the classical dynamical description

$$e^{i\phi_I(\lambda)}e^{i2(\mathbf{q}_2 - \mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda)}\ e^{-L(\lambda)^2(\mathbf{q}_2 - \mathbf{w}(\lambda))^2}$$

of the evolution of the support of the two body state if the envelope evolution correction factor (164) is negligible.

Neglecting terms of $\mathcal{O}(\lambda^2)$ in the brief interval extrapolations, the envelope evolution correction factor does not vary with time.

$$e^{-2\lambda L(0)^2(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)} = e^{-2\lambda L(\lambda)^2(\mathbf{q}_2-\mathbf{w}(\lambda))\cdot\dot{\mathbf{w}}(\lambda)}.$$

The envelope evolution correction is negligible if either acceleration is limited, or if the representative momentum (118) approximation, $\mathbf{q}_2 = \mathbf{w}(0)$, applies.

From the Cauchy-Schwarz-Bunyakovsky inequality,

$$|(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)| \leq \|\mathbf{q}_2-\mathbf{w}(0)\|\,\|\dot{\mathbf{w}}(0)\|.$$

If the initial acceleration in the brief interval approximation is limited,

$$\|\dot{\mathbf{w}}(0)\| \leq \frac{\epsilon}{\mathbf{u}(0)^2}, \tag{165}$$

then

$$|(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)| \leq \|\mathbf{q}_2-\mathbf{w}(0)\|\frac{\epsilon}{\mathbf{u}(0)^2}$$

and the envelope evolution correction factor $e^{4L(0)^2(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}$ is negligible with respect to the envelope,

$$4\lambda\mathbb{R}e(L(0)^2)\|\mathbf{q}_2-\mathbf{w}(0)\|\frac{\epsilon}{\mathbf{u}(0)^2} \ll \mathbb{R}e(L(0)^2)(\mathbf{q}_2-\mathbf{w}(0))^2,$$

as long as

$$\frac{4\epsilon\lambda}{\mathbf{u}(0)^2} \ll \|\mathbf{q}_2-\mathbf{w}(0)\|.$$

The envelope evolution correction is already negligible in a neighborhood of the point $\mathbf{q}_2 = \mathbf{w}(0)$. Then, to approximate scalar products (158) and (159), the bound on λ need only apply for $\mathbf{q}_2$ with

$$\sqrt{\mathbb{R}e(L(0)^2)}\,\|\mathbf{q}_2-\mathbf{w}(0)\| \geq \epsilon.$$

The envelope evolution factor is well-approximated by unity if the interval λ is bounded,

$$\lambda \ll \frac{\mathbf{u}(0)^2}{4\sqrt{\mathbb{R}e(L(0)^2)}}.$$

With limited accelerations (165) and for sufficiently brief intervals λ, the envelope evolution correction factor (164) is negligible with respect to the envelope,

$$e^{2L(0)^2(\mathbf{q}_2-\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} \approx 1.$$

Finally, the non-relativistic (120), brief interval, limited acceleration (165) approximation is that the relative motion descriptions in the state describing functions $U(\lambda)\varphi_2(0)$ and $\varphi_2(\lambda)$ are equivalent to $\mathcal{O}(\lambda^2)$.

$$e^{i\phi_I(\lambda)}e^{i(\mathbf{q}_2-\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}e^{-L(\lambda)^2(\mathbf{q}_2-\mathbf{w}(\lambda))^2} \approx e^{-2i\omega(\frac{\mathbf{q}_2}{2})\lambda}\, e^{i(\mathbf{q}_2-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{q}_2-\mathbf{w}(0))^2} \tag{166}$$

if $L(\lambda)^2$ is from (162) and $\phi_I(\lambda)$ is from (163). (166) establishes the quantum-classical correspondence (126). This result illustrates the discussion below (136) in section 5.3. For brief intervals, $|I(\lambda)|^2 = 1$ to $\mathcal{O}(\lambda^2)$. For sufficiently brief intervals, a quantum-classical correspondence (126) generally applies. However, since the approximation (166) neglects terms of $\mathcal{O}(\lambda^2)$, the expansion is inadequate to apply the optimality criterion (132) for selection of the maximum likelihood corresponding trajectory. The correspondence of quantum dynamics with classical dynamics follows from (135) and this correspondence is developed in section 5.4.3.

The phase (163) corresponds to the classical energy of two non-relativistic particles.

$$2mc^2 + \text{K.E.} + \text{V} = mc^2\,(2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0)).$$

$\lambda_c mc^2 = \hbar c$. The rest mass energy is 2, the kinetic energy of the two particles is $4\dot{\mathbf{u}}(0)^2$, and the pair potential V$= 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0)$ is identified as a $-g/r$ pair potential in section 5.4.3 below. If the trajectories satisfy Newton's equation of motion, then the total classical energy is independent of λ.

Substitution of the brief interval, limited acceleration approximation (166) into the scalar products (158) and (159) result in

$$\begin{aligned}
&\langle U(-\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \\
&\quad\approx A_g\int d\mathbf{q}_2 d\mathbf{q}_4\; T_4(\mathbf{q}_2,\mathbf{q}_4)\; e^{-2i\omega(\mathbf{q}_2)\lambda}e^{-i2(\mathbf{q}_2-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-\overline{L(0)^2}(\mathbf{q}_2-\mathbf{w}(0))^2} \\
&\qquad\times e^{-i\phi_I(\lambda)}e^{2i\omega(\mathbf{q}_4)\lambda}\; e^{2i(\mathbf{q}_4-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{q}_4-\mathbf{w}(0))^2} \\
&\quad= A_g e^{-i\phi_I(\lambda)}\int d\mathbf{q}_2 d\mathbf{q}_4\; T_4(\mathbf{q}_2,\mathbf{q}_4)\; e^{-2i(\mathbf{q}_2-\mathbf{q}_4)\cdot\mathbf{u}(0)}e^{-\overline{L(0)^2}(\mathbf{q}_2-\mathbf{w}(0))^2}e^{-L(0)^2(\mathbf{q}_4-\mathbf{w}(0))^2} \\
&\quad= e^{-i\phi_I(\lambda)}Q(1)
\end{aligned} \tag{167}$$

for the mixed scalar product and

$$\begin{aligned}
&\langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle \\
&\quad\approx A_g\int d\mathbf{q}_2 d\mathbf{q}_4\; T_4(\mathbf{q}_2,\mathbf{q}_4)\; e^{i\phi_I(\lambda)}e^{2i\omega(\mathbf{q}_2)\lambda}\; e^{-i2(\mathbf{q}_2-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-\overline{L(0)^2}(\mathbf{q}_2-\mathbf{w}(0))^2} \\
&\qquad\times e^{-i\phi_I(\lambda)}e^{-2i\omega(\mathbf{q}_4)\lambda}\; e^{2i(\mathbf{q}_4-\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{q}_4-\mathbf{w}(0))^2} \\
&\quad= A_g\int d\mathbf{q}_2 d\mathbf{q}_4\; T_4(\mathbf{q}_2,\mathbf{q}_4)\; e^{-2i(\mathbf{q}_2-\mathbf{q}_4)\cdot\mathbf{u}(0)}e^{-\overline{L(0)^2}(\mathbf{q}_2-\mathbf{w}(0))^2}e^{-L(0)^2(\mathbf{q}_4-\mathbf{w}(0))^2} \\
&\quad= Q(1)
\end{aligned} \tag{168}$$

for the squared norms with the definition

$$Q(1) = A_g \int d\mathbf{q}_2 d\mathbf{q}_4 \; T_4(\mathbf{q}_2, \mathbf{q}_4) \, e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{q}_2 - \mathbf{w})^2} \; e^{-L(0)^2(\mathbf{q}_4 - \mathbf{w})^2}. \tag{169}$$

The generalized function $T_4(\mathbf{q}_2, \mathbf{q}_4)$ is from (152) and A_g is from (154). $\mathbf{u}(0)$ is the initial location of particle one, $\mathbf{w}(0)$ is the initial momentum of particle one, and $L(0)^2$ is the initial support spread parameter in the description of relative motion (155).

$$L(0)^2 \in \mathbb{R}$$

is selected to set the initial state describing function as a most classical-like state description. $F(\mathbf{q}_2, \mathbf{q}_4)$ is a Lebesgue summable function of $\mathbf{q}_2, \mathbf{q}_4$. The Hamiltonian derived factors

$$e^{-2i(\omega(\mathbf{q}_2) - \omega(\mathbf{q}_4))\lambda} = 1$$

due either to conservation of momentum, $\mathbf{q}_2 = \mathbf{q}_4$, from the free field VEV or conservation of energy,

$$2\omega(\mathbf{q}_2) = 2\omega(\mathbf{q}_4)$$

from the connected VEV. The unimodular phase factor $e^{-i\phi_i(\lambda)}$ contributed by substitution of (166) distributes out of the summation and therefore does not contribute to likelihood (137). The approximations are to second order in non-relativistic (120) quantities

$$\lambda_c \|\mathbf{q}_2 - \mathbf{w}(0)\|, \lambda_c \|\mathbf{q}_4 - \mathbf{w}(0)\| \text{ and } \lambda_c \|\mathbf{w}(0)\| \ll 1.$$

Factors

$$e^{-2i(\mathbf{q}_2 - \mathbf{w}(0))\cdot \mathbf{u}(0)} e^{2i(\mathbf{q}_4 - \mathbf{w}(0))\cdot \mathbf{u}(0)} = e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}(0)}$$

from the common time in the brief interval approximation (166).

It is convenient to discriminate the contribution of the free field VEV from the contribution of the four-argument connected function to $Q(1)$.

$$Q(1) = Q_F(1) + Q_C(1), \tag{170}$$

with

$$Q_F(1) = \frac{4A_g}{\lambda_c^2} \int d\mathbf{q}_2 d\mathbf{q}_4 \; \delta(\mathbf{q}_2 - \mathbf{q}_4) \; e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}} e^{-\overline{L(0)^2}(\mathbf{q}_2 - \mathbf{w})^2} e^{-L(0)^2(\mathbf{q}_4 - \mathbf{w})^2}$$

and

$$Q_C(1) = \frac{c_4 A_g}{\lambda_c} \int d\mathbf{q}_2 d\mathbf{q}_4 \; \delta(\mathbf{q}_2^2 - \mathbf{q}_4^2) \; e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}} e^{-\overline{L(0)^2}(\mathbf{q}_2 - \mathbf{w})^2} e^{-L(0)^2(\mathbf{q}_4 - \mathbf{w})^2}$$

from the definitions (152) for T_4 and (169) for $Q(F)$, and with an abbreviated notation $\mathbf{u} = \mathbf{u}(0)$ and $\mathbf{w} = \mathbf{w}(0)$ and with the Jacobi sum and difference variables (142). $Q(1)$ is approximated by elementary functions in appendix 8.15.

5.4.3 Energy correspondence and $-g/r$ potentials

In this section, the condition (135) that identifies the classical trajectory that maximizes the likelihood (137) is evaluated. This evaluation relates the classical dynamical variables in the phase (163) to the parameters of the constructed, single neutral scalar field realization of relativistic quantum physics. Maximal likelihood and state describing functions with isolated (119) concentrations of non-relativistic (120) support well-represented by a single location and momentum (118) imply the quantum-classical correspondence (126). The regions of dominant support of the state describing functions are centered on classical dynamical variables.

From the optimality condition (135) and the expression (163) for $\phi_I(\lambda)$,

$$\begin{aligned} \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\langle\varphi_2(0)|\varphi_2(0)\rangle} &= \frac{d\phi_I(\lambda)}{d\lambda} \\ &= (2+\dot{\mathbf{u}}(0)^2+2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0))\frac{1}{\lambda_c}. \end{aligned} \tag{171}$$

The expectation value of the energy H equals the classical energy (163) of the two corresponding classical particles. Equality applies for state describing functions $\varphi_2(0)$ of the form (141) with (146) and (155) within the fidelity of the non-relativistic, brief interval and limited acceleration approximations in section 5.4.2.

For instances with quantum classical correspondences, evaluation of the scalar product (169) in appendix 8.14 and the subsequent evaluation (271) of the expectation of the energy in appendix 8.16 results in

$$\begin{aligned} \frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\|\varphi_2(0)\|^2} &= (2+\dot{\mathbf{u}}^2+2\ddot{\mathbf{u}}\cdot\mathbf{u})\,\frac{1}{\lambda_c} \\ &\approx \frac{2}{\lambda_c}+\frac{\dot{\mathbf{u}}^2}{\lambda_c}+\frac{3\lambda_c}{4L(0)^2}-\frac{\lambda_c^2}{4L(0)^2}\frac{Q_C(1)}{\lambda_c Q(1)}. \end{aligned} \tag{172}$$

The free field VEV contribution $Q_F(1)$ and four-point connected VEV contribution $Q_C(1)$ to $Q(1)$ are from (170) evaluated in appendix 8.15. The expectation of the energy consists of the rest mass of two particles

$$\frac{2}{\lambda_c},$$

the kinetic energy of two particles

$$\frac{\dot{\mathbf{u}}^2}{\lambda_c},$$

a small zero-point energy

$$\frac{3\lambda_c}{4L(0)^2},$$

and the remainder vanishes if the contribution $Q_C(1)$ of the connected VEV that characterizes interaction vanishes

$$-\left(\frac{\lambda_c}{4L(0)^2}\right)\frac{Q_C(1)}{Q(1)} \approx 2\,\frac{\ddot{\mathbf{u}}\cdot\mathbf{u}}{\lambda_c}. \tag{173}$$

The contribution of the potential to the energy is negative. The expected value of the energy of the state described by $\varphi_2(0)$ with an interaction characterized by the single neutral scalar field VEV (148) equals a zero-point energy plus the classical energy of two bodies with a relative motion described by $\mathbf{u}(\lambda)$ and an interaction characterized by the pair potential

$$2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda) = -\frac{g}{\|2\mathbf{u}(\lambda)\|}$$

from (175). From (263) and (268), the dominance in instances with quantum-classical correspondences of $Q_F(1) \gg Q_C(1)$ and the asymptotic $1/r$ variation of $e^{-x^2}\mathrm{erfi}(x)$ from (270) with $x = r/\sqrt{2L(0)^2}$ displays a $1/r^2$ variation of $Q_C(1)/Q(1)$ while the potential has a $1/r$ variation. However, this is no contradiction to a $-g/r$ potential. Characteristic of the quantum-classical correspondences introduced in Schrödinger's 1926 study of non-relativistic linear harmonic oscillators [51], the parameters of the particular state that exhibits the quantum-classical correspondence are determined by the corresponding classical dynamical parameters. $L(0)^2$ is determined from parameters c_4 determined in the scattering likelihoods and g determined by the corresponding classical dynamics. Like the linear harmonic oscillator example, section 5, the states with quantum-classical correspondences are determined by corresponding classical dynamical variables. The state describing functions that exhibit a quantum-classical correspondence are among the most classical particle-like.

The final development within this section verifies that the $2\ddot{\mathbf{u}}\cdot\mathbf{u}$ term in the classical energy is a $-g/r$ pair potential for trajectories $\mathbf{u}(\lambda)$ that satisfy Newton's equation of motion. $\mathbf{u}(\lambda)$ is half the two body separation and Newton's equation for the trajectory $\mathbf{u}(\lambda)$ is then

$$mc^2\ddot{\mathbf{u}}(\lambda) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial(2\mathbf{u}(\lambda))}$$

from (259) in appendix 8.13 and in the units of this note. The terms in $\phi_I(\lambda)$ from (163) and Newton's equation of motion results in the identification

$$\begin{aligned} 2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda) &= \frac{V(2\|\mathbf{u}(\lambda)\|)}{mc^2} \\ &= -\frac{1}{mc^2}\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda). \end{aligned} \tag{174}$$

The corresponding potential satisfies

$$V(2\|\mathbf{u}(\lambda)\|) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda).$$

The solution $V(2\|\mathbf{u}(\lambda)\|)$ is a $-g/r$ pair potential,

$$2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda) = -\frac{g}{\|2\mathbf{u}(\lambda)\|}. \tag{175}$$

The strength of the potential is determined by the length g. The chain rule verifies the identification (175).

$$\begin{aligned} -\left(\nabla_{\mathbf{u}}\frac{1}{\|\mathbf{u}\|}\right)\cdot\mathbf{u} &= -\left(\frac{\partial\|\mathbf{u}\|^{-1}}{\partial\|\mathbf{u}\|}\right)\left(\frac{\partial\|\mathbf{u}\|}{\partial u_x}u_x + \frac{\partial\|\mathbf{u}\|}{\partial u_y}u_y + \frac{\partial\|\mathbf{u}\|}{\partial u_z}u_z\right) \\ &= \frac{1}{\|\mathbf{u}\|^2}\left(\frac{u_x^2}{\|\mathbf{u}\|} + \frac{u_y^2}{\|\mathbf{u}\|} + \frac{u_z^2}{\|\mathbf{u}\|}\right) \\ &= \frac{1}{\|\mathbf{u}\|}. \end{aligned}$$

The trajectory optimizing condition (135) that follows from maximization of the likelihood (137) near $\lambda = 0$,

$$\left.\frac{\partial|I(\lambda)|^2}{\partial\lambda}\right|_{\lambda=0} = 0,$$

provides the corresponding classical trajectory $\mathbf{u}(\lambda)$ as solutions of Newton's equation for a $-g/r$ pair potential. The condition (172) relates the quantum c_4 and classical g parameters.

5.5 Extended interval propagation

In this section, satisfaction of the quantum-classical correspondence (126) over brief intervals is extended to longer intervals.

The quantum-classical correspondence (126)

$$U(-\ell\lambda)\hat{\varphi}_n(\lambda_o;0) \approx e^{i\phi_I(\ell\lambda)}\hat{\varphi}_n(\lambda_o;\ell\lambda)$$

is expressed as a sequence of briefer interval approximations,

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) \approx \left(e^{i\phi_I(\lambda)}\right)^\ell\hat{\varphi}_n(\lambda_o;\ell\lambda)$$

using the group property of time translation and additivity of the phase $\phi_I(\lambda)$. Here, referring to section 5.3, the notation is augmented to explicitly display the time argument λ_o of the state describing function as well as the temporal parameter for the corresponding classical trajectories.

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) = \hat{\varphi}_n(\lambda_o-\ell\lambda;0)$$

from (75). The error in the quantum-classical correspondence (126) for an interval $\ell\lambda$ follows from the approximation errors for each subinterval within $\ell\lambda$. The error at step ℓ is described by state describing functions ϵ_ℓ.

$$U(-\lambda)\hat{\varphi}_n(\lambda_o - \ell\lambda; \ell\lambda) = e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o - \ell\lambda; (\ell+1)\lambda) + \epsilon_{\ell+1}. \tag{176}$$

The error after accumulation of the ℓ steps of duration λ is described by ϵ.

$$U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o; 0) = \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o; n\lambda) + \epsilon.$$

From successive substitution of (176) it follows that

$$\begin{aligned}
U(-\lambda)^\ell\hat{\varphi}_n(\lambda_o; 0) &= U(-\lambda)^{\ell-1}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o; \lambda) + \epsilon_1\right) \\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-1}\hat{\varphi}_n(\lambda_o; \lambda) + U(-\lambda)^{\ell-1}\epsilon_1 \\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-2}\hat{\varphi}_n(\lambda_o - \lambda; \lambda) + U(-\lambda)^{\ell-1}\epsilon_1 \\
&= e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-3}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o - \lambda; 2\lambda) + \epsilon_2\right) + U(-\lambda)^{\ell-1}\epsilon_1 \\
&= \left(e^{i\phi_I(\lambda)}\right)^2 U(-\lambda)^{\ell-4}\hat{\varphi}_n(\lambda_o - 2\lambda; 2\lambda) + e^{i\phi_I(\lambda)}U(-\lambda)^{\ell-3}\epsilon_2 + U(-\lambda)^{\ell-1}\epsilon_1 \\
&= \ddots \\
&= \left(e^{i\phi_I(\lambda)}\right)^\ell U(-\lambda)^{-\ell}\hat{\varphi}_n(\lambda_o - \ell\lambda; \ell\lambda) + \sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1} U(-\lambda)^{\ell+1-2j}\epsilon_j \\
&= \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o; \ell\lambda) + \sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1} U(-\lambda)^{\ell+1-2j}\epsilon_j.
\end{aligned}$$

Then, the error ϵ after ℓ steps is

$$\epsilon = \sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1} U(-\lambda)^{\ell+1-2j}\epsilon_j$$

from the errors ϵ_j for each subinterval. From the triangle inequality, and unitarity of $U(-\lambda)$ and $e^{i\phi_I(\lambda)}$,

$$\|\epsilon\| \leq \sum_{j=1}^{\ell}\|\epsilon_j\|.$$

The accumulated error grows no faster than the number of subintervals. Extension of the interval applies only as long C.1-3 remain valid.

6 Bound states

In the constructed realizations of relativistic quantum physics, interaction is expressed in the vacuum expectation values (VEV) of the fields. The Hamiltonians (103) of the constructions, section 3, satisfy relativistic invariance and exhibit only continuous spectra. In this section, the presence of bound states in the constructed realizations of relativistic quantum physics is demonstrated. There are multiple argument state describing functions with centers-of-momentum described as free particles and stably bound expectations for the differences of arguments. Other than basic bounds, the masses of these bound states are undetermined.

One convenient example construction suffices to establish existence. The selected construction is a single, neutral scalar field with a single elementary particle of finite mass m, and bound states are demonstrated within the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$. The Hamiltonian for evolution of the combined description of the center of momentum and bound state internals is (102) in the two-argument subspace. The bound state describing functions are constructed with this Hamiltonian equivalent to a single particle free particle Hamiltonian for the center of momentum with a periodic evolution for the internals.

Analogously to the many possibilities for bound states within $\mathcal{L}^2$ Hilbert spaces in non-relativistic quantum mechanics, the Hilbert spaces $\mathbf{H}_{\mathcal{P}}$ include many descriptions of bound states. Determination of the masses of the bound states of physical interest would follow from a quantum dynamical principle, in addition to axioms A.1-7 of section 4.1. This as yet unidentified principle is introduced in section 1.

6.1 Preliminaries

Bound states within the two-argument subspace $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$ of the constructed Hilbert space $\mathbf{H}_{\mathcal{P}}$ are demonstrated. Functions $\varphi_2 \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$ have Fourier transforms of the form [31]

$$\tilde{\varphi}_2(p_1, p_2) = (p_{10} + \omega_1)(p_{20} + \omega_2)\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2). \tag{177}$$

The function over momenta $\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2)$ describes the spatial support of the state describing function φ_2 on constant time planes in the reference frame selected by choice of spacetime coordinates x_j. φ_2 has point support over time. Spacetime vectors $x = x_0, \mathbf{x}$ with $x_0 = ct$ and spatial Euclidean vectors $\mathbf{x} \in \mathbb{R}^3$ are lengths, energy-momenta vectors are wavenumbers designated $p = p_0, \mathbf{p}$ with Euclidean momentum vector $\mathbf{p} \in \mathbb{R}^3$. with $\hbar$ Planck's constant h divided by 2π and c the speed of light, momenta are $\mathbf{P} = \hbar\mathbf{p}$ and energies are $E = \hbar c p_0$. Multiple arguments are denoted $(p)_n = p_1, p_2, \ldots p_n$. The wavenumber $\omega(\mathbf{p}_j) = \omega_j$ from (8) is proportional to an energy on the mass shell m.

The scalar product in the two-argument subspace $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$ of a state describing function ψ_2 with the state describing function $A\varphi_2$ that results from a linear operator $A : \mathbf{H}_{\mathcal{P}}(\mathbb{R}^8) \mapsto$

$\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$ applied to the state describing function φ_2 is

$$\begin{aligned}
\langle\psi_2|A\varphi_2\rangle &= \int d(x)_4\, \overline{\psi_2(x_1,x_2)}\langle\Phi(x_1)\Phi(x_2)\Omega|\Phi(x_3)\Phi(x_4)\Omega\rangle A\varphi_2(x_3,x_4)\\
&= \int d(p)_4\, \overline{\tilde{\psi}_2(-p_1,-p_2)}\langle\widetilde{\Phi}(p_1)\widetilde{\Phi}(p_2)\Omega|\widetilde{\Phi}(p_3)\widetilde{\Phi}(p_4)\Omega\rangle A\tilde{\varphi}_2(p_3,p_4)\\
&= \int d(p)_4\, \overline{\tilde{\psi}_2(p_1,p_2)}\langle\widetilde{\Phi}(-p_2)\widetilde{\Phi}(-p_1)\Omega|\widetilde{\Phi}(p_3)\widetilde{\Phi}(p_4)\Omega\rangle A\tilde{\varphi}_2(p_3,p_4)\\
&= \int d(p)_4\, (\widetilde{\mathcal{F}}_{2,2}(-p_1,-p_2,p_3,p_4) + \widetilde{\mathcal{U}}_{2,2}(-p_1,-p_2,p_3,p_4))\\
&\qquad \times(p_{10}+\omega_1)(p_{20}+\omega_2)\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2)}A(p_{30}+\omega_3)(p_{40}+\omega_4)\tilde{f}_2(\mathbf{p}_3,\mathbf{p}_4).
\end{aligned}$$

The evaluation of the scalar product follows from the Parseval identity definition for the Fourier transform of generalized functions, the Bose symmetry of the scalar field VEV, a change of summation variables $\mathbf{p}_1,\mathbf{p}_2 \to -\mathbf{p}_1,-\mathbf{p}_2$, and the substitutions (177) and (148) from section 5.4.1. Both ψ_2 and φ_2 are in the form (177). The state describing function ψ_2 is described by $\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2)$ and φ_2 is described by $\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)$. Substitution of (149) and evaluation of the mass shell generalized functions results in

$$\begin{aligned}
\langle\psi_2|A\varphi_2\rangle &= \int d(\mathbf{p})_4 \left(\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)+\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3) \right.\\
&\qquad \left. +c_4\,\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\frac{\delta(\omega_1+\omega_2-\omega_3-\omega_4)}{4\omega_3\omega_4}\right)\\
&\qquad \times\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2)}\,A\,4\omega_3\omega_4\tilde{f}_2(\mathbf{p}_3,\mathbf{p}_4)\\
&= \int d(\mathbf{p})_2\,\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2)}\,A\,4\omega_1\omega_2(\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde{f}_2(\mathbf{p}_2,\mathbf{p}_1))\\
&\quad +c_4\int d(\mathbf{p})_4\,\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\frac{\delta(\omega_1+\omega_2-\omega_3-\omega_4)}{\omega_3\omega_4}\\
&\qquad \times\overline{\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2)}A\,\omega_3\omega_4\tilde{f}_2(\mathbf{p}_3,\mathbf{p}_4)
\end{aligned} \tag{178}$$

from the equality of $\omega_1\omega_2$ with $\omega_3\omega_4$ in the free field contribution and noting that A may not commute with $\omega_3\omega_4$.

For the single neutral scalar field construction, the unitary temporal translation operator in the two-argument subspace expressed in the Fourier domain is from (103),

$$\tilde{\varphi}_2((p)_2) = e^{-i(\omega_1+\omega_2)\lambda}\tilde{\varphi}_2((p)_2).$$

From Born's rule and with expectations expressed with the function $\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)$ from (177), the temporal evolution of the expectation value of an observable A in the two-argument subspace is

$$E[A; e^{-i(\omega_1+\omega_2)\lambda}\tilde{f}_2] = \frac{\langle U(\lambda)\varphi_2|AU(\lambda)\varphi_2\rangle}{\langle\varphi_2|\varphi_2\rangle}. \tag{179}$$

6.2 The state describing functions of bound states

Appropriately supported and temporally evolving functions describe bound states. The temporal evolution of the center of momentum of the bound complex of two mass m elementary particles is described as a free particle of mass $m_b < 2m$. The Hamiltonian for evolution of the combined description of the center of momentum and bound state internals is the Hamiltonian generator of time translation from (103) for the two-argument subspace, and this evolution is assumed to be equivalent to a description of the center of momentum as a free particle with a periodic evolution of the internals. The period of evolution for the internals is determined by an energy offset necessary to conserve energy for a stable bound state. The description of the center of momentum will spread with time consistently with classical correspondence and the Heisenberg uncertainty principle.

6.2.1 A relativistic description of bound states

Functions $\tilde{\varphi}_2(p_1, p_2) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$, the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$, of the form (177) with

$$\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2) = \delta(H_B)\, \tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2) \tag{180}$$

are selected to describe bound states. The delta function limits the support of bound state describing functions to include only those momenta $\mathbf{p}_1, \mathbf{p}_2$ that satisfy

$$\begin{aligned} H_B &= H_B(\mathbf{p}_1+\mathbf{p}_2, \frac{\mathbf{p}_1-\mathbf{p}_2}{2}) \\ &= \omega(\mathbf{p}_1) + \omega(\mathbf{p}_2) - \omega_b(\mathbf{p}_1 + \mathbf{p}_2) - E \\ &= 0 \end{aligned} \tag{181}$$

with

$$\omega_b(\mathbf{p}) = \sqrt{\lambda_b^{-2} + \mathbf{p}^2} \tag{182}$$

similarly to $\omega(\mathbf{p})$ from (8) but with

$$\lambda_b = \frac{\hbar}{m_b c},$$

the reduced Compton wavelength (9) for a bound state mass $m_b < 2m$. For these state describing functions (180), the two-argument subspace Hamiltonian equals a free particle Hamiltonian for a mass m_b composite particle with momentum $\mathbf{p}_1 + \mathbf{p}_2$ plus a binding energy E.

$$\omega(\mathbf{p}_1) + \omega(\mathbf{p}_2) = \omega_b(\mathbf{p}_1 + \mathbf{p}_2) + E. \tag{183}$$

For these bound state describing functions, time evolution is the same as the evolution of a free particle with the bound state mass m_b. The binding energy E is the minimal additional energy

required to separate the bound state into two arbitrarily greatly separated, rest mass m free elementary particles. The bound state has a mass defect

$$0 \leq m_B < 2m$$

that ensures its stability. Conservation of energy provides that the binding energy E is the difference of the rest mass energies of two free elementary particles and the bound complex.

$$\begin{aligned} E &= 2\lambda_c^{-1} - \lambda_b^{-1} \\ &= \frac{c}{\hbar}(2m - m_b) \\ &> 0. \end{aligned} \tag{184}$$

Below, $\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2)$ is selected as an even function,

$$\tilde{h}_2(-\mathbf{p}_1, -\mathbf{p}_2) = \tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2).$$

The selection of an $\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2)$ as an even function suffices to set the expectations of locations and momenta to zero. Zero expectations ensure that the description (180) of the bound state is in a rest frame of the bound complex with the origin centered on the location of the bound complex. In nonrelativistic physics, the description of the center of momentum can be selected as independent of the description of the bound state internals. Independence is implemented by a factored description

$$\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2) = \tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\, \tilde{u}_1(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2}).$$

But, the descriptions of center of momentum and internals are not generally unentangled in a relativistic description. Classically, relativistic time and length dilation due to motion of the center of momentum affects the description of internals. Indeed, below it will be developed that descriptions of the center of momentum and internals disentangle in non-relativistic approximations.

The norms $\|\varphi_2\|$ of bound state describing functions of the form (180) diverge. From the scalar product (178) in section 6.1, substitution of an operation $A = 1$ and state describing functions $\psi_2 = \varphi_2$ results in the norm of a φ_2.

$$\begin{aligned} \langle \varphi_2 | \varphi_2 \rangle \;=\; & \int d(\mathbf{p})_2 \; \delta(H_B(\mathbf{p}_1 + \mathbf{p}_2, \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)))\, \overline{\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2)} \\ & \quad \times 4\omega_1\omega_2 \; \delta(H_B(\mathbf{p}_1 + \mathbf{p}_2, \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)))\, (\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2) + \tilde{h}_2(\mathbf{p}_2, \mathbf{p}_1)) \\ & + c_4 \int d(\mathbf{p})_4 \, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ & \quad \times \delta(H_B(\mathbf{p}_1 + \mathbf{p}_2, \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)))\, \overline{\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2)} \\ & \quad \times \delta(H_B(\mathbf{p}_3 + \mathbf{p}_4, \tfrac{1}{2}(\mathbf{p}_3 - \mathbf{p}_4)))\, \tilde{h}_2(\mathbf{p}_3, \mathbf{p}_4). \end{aligned} \tag{185}$$

The free field VEV contribution to the scalar product (185) has a $(\delta(H_B))^2$ divergence. The four-argument connected VEV contribution has the same divergence. From conservation of energy,

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

and from the conservation of momentum $\mathbf{p}_1+\mathbf{p}_2 = \mathbf{p}_3+\mathbf{p}_4$. Then,

$$\omega_b(\mathbf{p}_1+\mathbf{p}_2) = \omega_b(\mathbf{p}_3+\mathbf{p}_4)$$

and the equality (183) setting $H_B = 0$ provides that

$$\begin{aligned}\delta(H_B(\mathbf{p}_1+\mathbf{p}_2, \tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)))\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4)\\ = \delta(H_B(\mathbf{p}_3+\mathbf{p}_4, \tfrac{1}{2}(\mathbf{p}_3-\mathbf{p}_4)))\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4)\end{aligned} \tag{186}$$

in scalar products (185). The contributions of the free field VEV and the four-argument connected VEV diverge as $\delta(H_B)^2$. Similarly to that momentum eigenfunctions have divergent norms, φ_2 of the form (180) are not elements of $\mathbf{H}_\mathcal{P}$ but are arbitrarily well approximated by elements within the Hilbert space. One convenient mollification of the form (180) substitutes

$$\delta(H_B) \mapsto \sqrt{\frac{L}{\pi}}\, e^{-LH_B} \in \mathbf{H}_\mathcal{P}(\mathbb{R}^8) \tag{187}$$

with arbitrarily large, finite $L \in \mathbb{R}$. It follows that with a mollification (187), $\delta(H_B)$ limits of expectations (179) are finite.

The mollification (187) can be considered a linear combination of bound states for varying binding energies E. For determined $\mathbf{p}_1, \mathbf{p}_2$,

$$g(H_B) = \int ds\; g(s)\delta(H_B - s)$$

that becomes $\delta(H_B)$ as $g(s) \to \delta(s)$. Then, slightly ill-defined binding energies E implement mollifications.

6.2.2 The support of bound state describing functions

For even $\tilde{h}_2(\mathbf{p}_1, \mathbf{p}_2)$ in the state describing functions of the form (180), the expectations of both momenta $\mathbf{p}_1, \mathbf{p}_2$ are zero. From (181), H_B is an even function,

$$H_B(\mathbf{p}_1+\mathbf{p}_2, \frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)) = H_B(-\mathbf{p}_1-\mathbf{p}_2, \frac{-\mathbf{p}_1+\mathbf{p}_2}{2})$$

and then the product $\delta(H_B)\tilde{h}_2$ is even. In the expectation (179),

$$\begin{aligned} E[\mathbf{p}_j; e^{-i(\omega_1+\omega_2)\lambda}\delta(H_B)\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)] &= E[\mathbf{p}_j; e^{-i(\omega_b(\mathbf{q}_C)+E)\lambda}\delta(H_B)\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)] \\ &= E[\mathbf{p}_j; \delta(H_B)\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)] \\ &= 0 \end{aligned}$$

for $j = 1, 2$ from commutation of the momentum operators with the Hamiltonian in (185), and that even VEV, even $\delta(H_B)\tilde{h}_2$, and an odd A results in integration of a odd function over an even interval in the expectation (185). It follows from the linearity of expectations that the sum and difference of the momenta are zero.

$$E[\mathbf{p}_1 + \mathbf{p}_2; e^{-i(\omega_1+\omega_2)\lambda}\delta(H_B)\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)] = E[\mathbf{p}_1 - \mathbf{p}_2; \delta(H_B)\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)] = 0.$$

The selection of an even function $\tilde{h}_2$ provides that the reference frame of the state description (180) is a rest frame in the sense that expectations of the momenta are zero. In addition, the state description is stable in the sense that the expectations of the momenta and their products do vary over time due to the commutation of the Hamiltonian and $\mathbf{p}_j$.

The Fourier domain representations of relativistic single particle location operators [44] are

$$\hat{X}_\nu = -i\,\omega^{1/2}\frac{d}{d\mathbf{p}_\nu}\,\omega^{-1/2}$$

with $\nu = x, y, z$ and a vector operator $\hat{\mathbf{X}}_j$ for each argument $\mathbf{p}_1, \mathbf{p}_2$. If the relativistic single particle location operators are extended into the two-argument subspace, then an argument identical to the development of expectations of the momenta demonstrates that the locations associated with both arguments are zero.

However, the description of location and spatial extent of the state describing functions (180) are less evident. In free field theory, extension of the single particle relativistic location operators [44] are Hermitian and apply to multiple argument state describing functions when the supports of the arguments are spatially isolated [31]. However, with interaction and lacking spatially isolated support, a two-argument subspace state description does not necessarily correspond to two mass m elementary particles. Two-argument state descriptions are not generally two particles of known species. To examine the spatial support of the state describing functions (180), the likelihoods of perception of two mass m elementary particles are evaluated.

The support of a state describing function $\varphi((x)_2) \in \mathbf{H}_\mathcal{P}(\mathbb{R}^8)$ follows from Born's rule that characterizes likelihoods of observation. The likelihood that the two-argument state describing function $\varphi_n((x)_2)$ is observed located near points $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^6$ at time λ is

$$\frac{|\langle d_2(\lambda, (\mathbf{y})_2)|\varphi_2\rangle|^2}{\|d_2\|^2\|\varphi_2\|^2}. \tag{188}$$

The spatial support of the two-argument function $d_2(\lambda,(\mathbf{y})_2)$ is localized in neighborhoods of the points $\mathbf{y}_1,\mathbf{y}_2$ and the temporal support is concentrated on a point at time λ, $x_{j0}=\lambda$ for $j=1,2$. For example, the Fourier transform domain function

$$\widetilde{d}_2((p)_2;\lambda,(\mathbf{y})_2)=\prod_{j=1}^{2}(p_{j0}+\omega_j)\,e^{-i\omega_j\lambda+i\mathbf{p}_j\cdot\mathbf{y}_j}e^{-\epsilon\mathbf{p}_j^2}$$

with arbitrarily small $\epsilon\in\mathbb{R}_{>0}$ is a convenient choice within $\mathbf{H}_{\mathcal{P}}(\mathbb{R}^8)$ with dominant spatial support limited to neighborhood of $(\mathbf{y})_2$ and point support on λ.

From (177), the $\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)$ that characterizes the example function $\widetilde{d}_2((p)_2;\lambda,(\mathbf{y})_2)$ is

$$\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)=e^{-i(\omega_1+\omega_2))\lambda}\,e^{i(\mathbf{p}_1\cdot\mathbf{y}_1+\mathbf{p}_2\cdot\mathbf{y}_2)}e^{-\epsilon(\mathbf{x}_1^2+\mathbf{p}_2^2))}. \tag{189}$$

Substitution of the $\epsilon\to 0$ limit of (189) into the scalar product (178) and reorganization results in

$$\begin{aligned}\langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \;=\; &\int d(\mathbf{p})_2\; e^{i(\omega_1+\omega_2))\lambda}\,e^{-i(\mathbf{p}_1\cdot\mathbf{y}_1+\mathbf{p}_2\cdot\mathbf{y}_2)}\\ &\times\Big(4\omega_1\omega_2\delta(H_B(\mathbf{p}_1+\mathbf{p}_2,\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)))\,(\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde{h}_2(\mathbf{p}_2,\mathbf{p}_1))\\ &\quad+c_4\int d\mathbf{p}_3\mathbf{p}_4\;\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4)\\ &\qquad\times\delta(H_B(\mathbf{p}_3+\mathbf{p}_4,\tfrac{1}{2}(\mathbf{p}_3-\mathbf{p}_4)))\,\tilde{h}_2(\mathbf{p}_3,\mathbf{p}_4)\Big).\end{aligned} \tag{190}$$

A change of variables is convenient for evaluation of the energy conservation and H_B delta functions.

An invertible transformation designating primed and unprimed energy-momentum variables is a Lorentz transformation

$$\mathbf{p}_j=\mathbf{p}_j(\mathbf{p}'_j) \tag{191}$$

with

$$\begin{bmatrix}\omega(\mathbf{p}_j)\\ \mathbf{p}_j\end{bmatrix}=\Lambda^{-1}\begin{bmatrix}\omega(\mathbf{p}'_j)\\ \mathbf{p}'_j\cdot\end{bmatrix}$$

Λ is a Lorentz transform. $j=1,2,3,4$. The energy-momenta p_j and p'_j are on mass shell ms, each $p_{j0}=\omega(\mathbf{p}_j)$. The Lorentz invariance of p^2 provides that Lorentz transforms Λ map mass shell m energy-momenta vectors to mass shell m energy-momenta vectors. The determinant of the Lorentz transformation Λ is unity. With the change of variables p_3,p_4 to p'_3,p'_4, the scalar

product (190) becomes

$$\begin{aligned}\langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \;&= \int d(\mathbf{p})_2\; e^{i(\omega_1+\omega_2))\lambda}\, e^{-i(\mathbf{p}_1\cdot\mathbf{y}_1+\mathbf{p}_2\cdot\mathbf{y}_2)} \\ &\quad\times\Big(4\omega_1\omega_2\delta(H_B(\mathbf{p}_1+\mathbf{p}_2,\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)))\,(\tilde h_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde h_2(\mathbf{p}_2,\mathbf{p}_1)) \\ &\qquad +c_4\int d\mathbf{p}_3' d\mathbf{p}_4'\; J\,\delta(p_1+p_2-\Lambda^{-1}(p_3'+p_4')) \\ &\qquad\qquad\times\delta(H_B(\mathbf{p}_3+\mathbf{p}_4,\tfrac{1}{2}(\mathbf{p}_3-\mathbf{p}_4)))\,\tilde h_2(\mathbf{p}_3,\mathbf{p}_4)\Big)\,.\end{aligned}$$

with $\mathbf{p}_3(\mathbf{p}_3'),\mathbf{p}_4(\mathbf{p}_4')$ from (191), and each p_{j0} and p_{j0}' is on a mass m shell. The Jacobian determinant

$$J=\frac{\omega(\mathbf{p}_3)\,\omega(\mathbf{p}_4)}{\omega(\mathbf{p}_3')\,\omega(\mathbf{p}_4')} \tag{192}$$

follows from

$$\begin{aligned}\int d\mathbf{p}\; f(\mathbf{p}) \;&= \int dp\; 2\omega(\mathbf{p})\delta(p^2-m^2)f(\mathbf{p}) \\ &= \int dp'\; 2\omega(\mathbf{p})\delta(p'^2-m^2)f(\mathbf{p}) \\ &= \int d\mathbf{p}'\; 2\omega(\mathbf{p}(\mathbf{p}'))\frac{1}{2\omega(\mathbf{p}')}f(\mathbf{p}(\mathbf{p}')),\end{aligned}$$

the unity Jacobian determinant of a Lorentz transformation, the Lorentz invariance of p^2, and (191).

Lorentz invariance of delta functions for the components of a Lorentz four-vector p,

$$\delta(p)=\delta(\Lambda p),$$

provides that the scalar product (190) equals

$$\begin{aligned}\langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \;&= \int d(\mathbf{p})_2\; e^{i(\omega_1+\omega_2))\lambda}\, e^{-i(\mathbf{p}_1\cdot\mathbf{y}_1+\mathbf{p}_2\cdot\mathbf{y}_2)} \\ &\quad\times\Big(4\omega_1\omega_2\delta(H_B(\mathbf{p}_1+\mathbf{p}_2,\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)))\,(\tilde h_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde h_2(\mathbf{p}_2,\mathbf{p}_1)) \\ &\qquad +c_4\int d\mathbf{p}_3' d\mathbf{p}_4'\; J\,\delta(\Lambda(p_1+p_2)-p_3'-p_4') \\ &\qquad\qquad\times\delta(H_B(\mathbf{p}_3+\mathbf{p}_4,\tfrac{1}{2}(\mathbf{p}_3-\mathbf{p}_4)))\,\tilde h_2(\mathbf{p}_3,\mathbf{p}_4)\Big)\,.\end{aligned}$$

With the $\mathbf{p}_3',\mathbf{p}_4'$ summations preceding the $\mathbf{p}_1,\mathbf{p}_2$ summations, for each $\mathbf{p}_1,\mathbf{p}_2$ there is a Lorentz transformation Λ to the center of momentum frame for each $\mathbf{p}_1,\mathbf{p}_2$. This Lorentz transformation Λ is evaluated in appendix 8.11. With the transformation $\Lambda p_j=p_j'$ for $j=1,2,3,4$, $\mathbf{p}_2'=-\mathbf{p}_1'$, $\omega_2'=\omega_1'$ and the energy-momentum conservation delta functions become

$$\delta(\Lambda(p_1+p_2)-p_3'-p_4')=\delta(\mathbf{p}_3'+\mathbf{p}_4')\,\delta(2\omega_1'-2\omega_3').$$

The transformation Λ is to the center of momentum frame for each $\mathbf{p}_1, \mathbf{p}_2$.

A change to Jacobi sum and difference variables is convenient for evaluation of the H_B delta functions. The changes of variables

$$\begin{aligned} \mathbf{q}_{S_1} &= \mathbf{p}_1 + \mathbf{p}_2 \\ \mathbf{q}_{D_1} &= \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) \\ \mathbf{q}_{S_3} &= \mathbf{p}_3 + \mathbf{p}_4 \\ \mathbf{q}_{D_3} &= \tfrac{1}{2}(\mathbf{p}_3 - \mathbf{p}_4) \end{aligned} \tag{193}$$

have Jacobian determinants of unity and inversions

$$\begin{aligned} \mathbf{p}_1 &= \tfrac{1}{2}\mathbf{q}_{S_1} + \mathbf{q}_{D_1}, & \mathbf{p}_2 &= \tfrac{1}{2}\mathbf{q}_{S_1} - \mathbf{q}_{D_1} \\ \mathbf{p}_3 &= \tfrac{1}{2}\mathbf{q}_{S_3} + \mathbf{q}_{D_3}, & \mathbf{p}_4 &= \tfrac{1}{2}\mathbf{q}_{S_3} - \mathbf{q}_{D_3}. \end{aligned} \tag{194}$$

The similar relations apply for the prime energy-momenta variables using the transformations (191). The magnitudes of momenta are designated

$$\rho_j = \|\mathbf{p}_j\|, \qquad \rho_{D_3} = \|\mathbf{q}_{D_3}\| \tag{195}$$

and similarly for the primed variables. In the center of momentum frame that applies for each $\mathbf{p}_1, \mathbf{p}_2$,

$$\begin{aligned} \mathbf{p}'_3 &= \mathbf{q}'_{D_3} \\ \mathbf{p}'_4 &= -\mathbf{q}'_{D_3}. \end{aligned}$$

$\rho'_{D_3} = \rho'_3$ from the momentum conservation delta $\delta(\mathbf{q}'_{S_3})$, (193) and (194). With the change to the primed frame sum and difference variables, and from (252) in appendix 8.11, the energy-momentum conservation delta functions become

$$\delta(p'_1 + p'_2 - p'_3 - p'_4) = \delta(\mathbf{q}'_{S_3}) \, \frac{\omega(\rho'_{D_3})}{2\rho'_{D_3}} \, \delta(\rho'_{D_3} - \rho'_1). \tag{196}$$

The factors $\omega_1\omega_2$ in the free field contribution to the VEV, and $\omega_3\omega_4$ and $\omega'_3\omega'_4$ that determine J in the connected VEV contribution expressed with the change of variables are

$$\begin{aligned} 2\omega_1\omega_{2j} &= (\omega_1 + \omega_{2j})^2 - \omega_1^2 - \omega_{2j}^2 \\ &= (\omega_1 + \omega_2)^2 - \mathbf{p}_1^2 - \mathbf{p}_2^2 \\ &= (\omega_b(\mathbf{q}_{S_1}) + E)^2 - \tfrac{1}{2}\mathbf{q}_{S_1}^2 - 2\mathbf{q}_{D_1}^2 \end{aligned} \tag{197}$$

and similarly for $\omega_3\omega_4$ in terms of $\mathbf{q}_{S_3}, \mathbf{q}_{D_3}$ and $\omega_3'\omega_4'$ in terms of $\mathbf{q}'_{S_3}, \mathbf{q}'_{D_3}$. In nonrelativistic instances, each $\mathbf{p}_k^2 \ll \lambda_c^{-2}$, and these factors are approximated by functions solely of the center of momentum variables $\mathbf{q}_{S_j}$, $j = 1, 3$. In non-relativistic instances, the motion of the center of mass decouples from the internals motion. A convenient identity,

$$\mathbf{p}_1 \cdot \mathbf{y}_1 + \mathbf{p}_2 \cdot \mathbf{y}_2 = \mathbf{q}_{S_1} \cdot \mathbf{y}_{S_1} + \mathbf{q}_{D_2} \cdot \mathbf{y}_{D_2} \tag{198}$$

follows from designation of sum and difference position variables.

$$\begin{aligned} \mathbf{y}_{S_1} &= \tfrac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2) \\ \mathbf{y}_{D_1} &= \mathbf{y}_1 - \mathbf{y}_2. \end{aligned}$$

The delta function $\delta(H_B(\mathbf{q}_S, \mathbf{q}_D))$ is evaluated in appendix 8.10. The momentum arguments of H_B in the evaluation are designated $\mathbf{q}_S, \mathbf{q}_D$ with magnitudes denoted ρ_S, ρ_D. With

$$\mathbf{q}_S \cdot \mathbf{q}_D = \rho_S \rho_D \cos\phi, \tag{199}$$

the H_B delta functions equal

$$\delta(H_B(\mathbf{q}_S, \mathbf{q}_D)) = \frac{\delta(\rho_D - \rho_o)}{\left|\frac{\partial H_B}{\partial \rho_D}\right|}. \tag{200}$$

with

$$\begin{aligned} \rho_o &= \sqrt{\frac{\beta}{\alpha}} \\ \alpha &= 4\left(\omega_b(\rho_S) + E\right)^2 - 4\rho_S^2 \cos^2\phi \\ \beta &= \left(\omega_b(\rho_S) + E\right)^2 \left(\left(\omega_b(\rho_S) + E\right)^2 - 4\omega(\tfrac{1}{2}\rho_S)^2\right). \end{aligned}$$

β is nonnegative and α is positive definite. $\left|\frac{\partial H_B}{\partial \rho_D}\right|$ is evaluated in (249) in appendix 8.10 and it is demonstrated that

$$\frac{\rho_D}{\left|\frac{\partial H_B}{\partial \rho_D}\right|}$$

is infinitely differentiable.

Finally, with the changes of variables (191) and (193), and substitutions (183) for the Hamiltonian, (196) for energy-momentum conservation, (198) for the complex exponential, and

(200) for $\delta(H_B)$, the scalar product (190) becomes

$$\begin{aligned}\langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \;&= \int d\mathbf{q}_{S_1} d\mathbf{q}_{D_1}\; e^{i(\omega_b(\rho_{S_1})+E)\lambda}\, e^{-i(\mathbf{q}_{S_1}\cdot\mathbf{y}_{S_1}+\mathbf{q}_{D_1}\cdot\mathbf{y}_{D_1})} \\ &\quad\times\left(4\omega_1\omega_2\frac{\delta(\rho_{D_1}-\rho_o)}{\left|\frac{\partial H_B}{\partial\rho_{D_1}}\right|}\,(\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde{h}_2(\mathbf{p}_2,\mathbf{p}_1))\right. \\ &\quad\left.+c_4\int d\mathbf{q}'_{D_3}\; J\,\frac{\omega(\mathbf{q}'_{D_3})}{2\rho'_{D_3}}\,\delta(\rho'_{D_3}-\rho'_1)\,\frac{\delta(\rho_{D_3}-\rho_o)}{\left|\frac{\partial H_B}{\partial\rho_{D_3}}\right|}\tilde{h}_2(\mathbf{p}_3,\mathbf{p}_4)\right)\end{aligned} \tag{201}$$

with J from (192) and $\omega_1\omega_2$ from (197). In terms of the summation variables, $\mathbf{p}_1 = \frac{1}{2}\mathbf{q}_{S_1}+\mathbf{q}_{D_1}$, $\mathbf{p}_2 = \frac{1}{2}\mathbf{q}_{S_1}-\mathbf{q}_{D_1}$, $\mathbf{p}_3 = \mathbf{p}_3(\mathbf{p}'_3)$, $\mathbf{p}_4 = \mathbf{p}_4(\mathbf{p}'_4)$ and $\mathbf{p}'_3 = -\mathbf{p}'_4 = \mathbf{q}'_{D_3}$. ρ'_1 is the magnitude of the spatial components of $p'_1 = \Lambda p_1$ from (191). It is convenient to evaluate (201) with spherical coordinates for $\mathbf{q}_{S_1}, \mathbf{q}_{D_1}$ and $\mathbf{q}'_{D_3}$. A factor of ρ_{D_1} from $d\mathbf{q}_{D_1}$ expressed in spherical coordinates regularizes the divergence from the H_B delta function in the contribution of the free field VEV. In the contribution of the four-argument connected VEV, two factors of ρ'_{D_3} remove the divergence from the H_B delta function and the divergence from the energy conservation delta function. The squared magnitude of this scalar product (201) provides the likelihoods of perception at the locations $\mathbf{y}_1, \mathbf{y}_2$ expressed in the sum and difference variables $\mathbf{y}_{S_1}, \mathbf{y}_{D_1}$ from (198).

The support of the difference variable $\mathbf{y}_{D_1} = \mathbf{y}_1 - \mathbf{y}_2$ is localized and, except for relativistic effects that couple center of momentum and internals motion, independent of time λ. With $r_{D_1} = \|\mathbf{y}_{D_1}\|$ and $G_2(\mathbf{q}_{S_1}, \mathbf{q}_{D_1})$ equal to the function of $\mathbf{q}_{S_1}, \mathbf{q}_{D_1}$ within the parentheses in (201),

$$\begin{aligned}&r_{D_1}^k \cos^k\phi_{D_1}\langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \\ &\quad= \int d\mathbf{q}_{S_1} d\mathbf{q}_{D_1}\; e^{i(\omega_b(\rho_{S_1})+E)\lambda}\, e^{-i\mathbf{q}_{S_1}\cdot\mathbf{y}_{S_1}}\left(i^k\frac{\partial^k}{\partial\rho_{D_1}^k}e^{-i\mathbf{q}_{D_1}\cdot\mathbf{y}_{D_1}}\right)G_2(\mathbf{q}_{S_1},\mathbf{q}_{D_1}) \\ &\quad= (-i)^k\int d\mathbf{q}_{S_1} d\mathbf{q}_{D_1}\; e^{i(\omega_b(\rho_{S_1})+E)\lambda}\, e^{-i(\mathbf{q}_{S_1}\cdot\mathbf{y}_{S_1}+\mathbf{q}_{D_1}\cdot\mathbf{y}_{D_1})}\left(\frac{\partial^k}{\partial\rho_{D_1}^k}G_2(\mathbf{q}_{S_1},\mathbf{q}_{D_1})\right)\end{aligned}$$

with the dot product designated

$$\mathbf{q}_{D_1}\cdot\mathbf{y}_{D_1} = \rho_{D_1} r_{D_1}\cos\phi_{D_1}.$$

From (201),

$$G_2(\mathbf{q}_{S_1},\mathbf{q}_{D_1}) = \left(4\omega_1\omega_2 \frac{\delta(\rho_{D_1}-\rho_o)}{\left|\frac{\partial H_B}{\partial \rho_{D_1}}\right|} \left(\tilde{h}_2(\mathbf{p}_1,\mathbf{p}_2)+\tilde{h}_2(\mathbf{p}_2,\mathbf{p}_1)\right) \right.$$
$$\left. +c_4 \int d\mathbf{q}'_{D_3}\; J\, \frac{\omega(\mathbf{q}'_{D_3})}{2\rho'_{D_3}}\, \delta(\rho'_{D_3}-\rho'_1)\, \frac{\delta(\rho_{D_3}-\rho_o)}{\left|\frac{\partial H_B}{\partial \rho_{D_3}}\right|} \tilde{h}_2(\mathbf{p}_3,\mathbf{p}_4) \right),$$

k is limited by the differentiability of G_2. If $\tilde{h}_2$ is infinitely differentiable, then G_2 will be infinitely differentiable from the infinitely differentiability of ω and Λ. The last line of results from continued integration by parts justified by the vanishing of $G_2(\mathbf{q}_{S_1},\mathbf{q}_{D_1})$ and its derivatives at the limits of the integration. The decline of G_2 at large values of the arguments is controlled by the decline of $\tilde{h}_2$. Bounding the right hand side,

$$\left| r_{D_1}^k \cos^k \phi_{D_1} \langle d_2(\lambda,(\mathbf{y})_2)|\varphi_2\rangle \right| \leq \int d\mathbf{q}_{S_1} d\mathbf{q}_{D_1} \left| \frac{\partial^k}{\partial \rho_{D_1}^k} G_2(\mathbf{q}_{S_1},\mathbf{q}_{D_1}) \right| < \infty$$

and the finite integral demonstrates that the support of the internals declines as $r_{D_1}^{-k}$, the support is effectively centered on $\mathbf{y}_1 = \mathbf{y}_2$, and this bound on the support is independent of time λ. These properties are all properties of a bound state.

The support of the location of the center of momentum $\mathbf{y}_{s_1} = \frac{1}{2}(\mathbf{y}_1+\mathbf{y}_2)$ is similarly localized but spreads with increasing time *lambda* similarly to the evolution of a free particle [31].

This argument for the existence of bound states applies even without inclusion of the four-argument connected VEV that describes interaction. However, without interaction, there is no mechanism to produce bound states and bound states could never be observed. Existence of bound state did not require knowledge of the VEV, but the likelihood of creating or perceiving a bound state is zero without $n \geq 2$, n-argument connected contributions to VEV.

The demonstration above establishes the existence of state describing functions with properties of bound states within the constructed models of relativistic quantum physics.

7 Technical concerns with canonical quantization

Inquiry into the mathematical development of quantum mechanics was pursued notably by John von Neumann. The inquiry has been extended by Huzihiro Araki, Nikolay Bogolubov, Hans-Jürgen Borchers, Rudolf Haag, Jan Raphael Høegh-Krohn, Léon van Hove, Res Jost, Franco Strocchi, Arthur Wightman and many others [9, 69]. Developments use the concept of Hilbert space advanced notably by David Hilbert, Erhard Schmidt, Frigyes Riesz, Marshall Stone and John von Neumann. This section discusses technical concerns with the canonical formalism for quantum mechanics.

Quantum mechanics has provided an expanded and successful description of nature. Nevertheless, issues remain even in the development of non-relativistic quantum mechanics [62]. The canonical formalism [13, 24, 60, 63] "quantizes" classical dynamics. If there is a "quantization" of the corresponding classical dynamics, is it a concern that while functions of classical dynamical variables are classical dynamical variables, products of Hermitian operators are not necessarily Hermitian operators? Are some classical quantities distinguished by having canonical quantizations while other quantities do not? Or is it a concern that the (generalized) eigenfunctions of operators associated with observables by the canonical formalism are not always elements of the Hilbert space, that is, do not describe states of nature? In such cases, what state results from "collapse to an eigenfunction of the observable?" The Schrödinger representation of location X_ν and momentum P_ν operators satisfy the Born-Heisenberg-Jordan relation $[X_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$ and serve as archetypes for a canonical quantization of classical dynamical variables. These X_ν and P_ν apply in the $\mathcal{L}^2$ Hilbert spaces suitable for non-relativistic quantum mechanics. In the following few paragraphs, the discussion includes that: the elevation of location X_ν fails to be a Hermitian operator in relativistic physics; the eigenfunctions of the operators X_ν and P_ν are not elements of $\mathcal{L}^2$ Hilbert spaces; locations and momenta associated with state describing functions are well-defined as observable features even when expectations of the corresponding operators X_ν and P_ν diverge; and the elevations of products of locations $\mathbf{x}$ and momenta $\mathbf{p}$ are not necessarily Hermitian operators even in non-relativistic quantum mechanics. Each of these points is a difficulty or ambiguity for canonical quantization.

Introduced in section 1, extrapolation of the location operator X_ν to relativistic physics does not succeed [44, 68]. With consideration of relativity, the X_ν that result from elevation of the classical dynamical variables x_ν are not Hermitian operators. Illustrated in appendix 8.2.6, operators whose eigenvalues are observable quantities must be Hermitian. Transition likelihoods are determined by the scalar product and these likelihoods describe events independently of inertial observer. As a consequence, the scalar product is a Lorentz invariant in relativistic physics. The assumption that the elevation of location X_ν is a densely defined Hermitian operator results in the contradiction

$$\begin{aligned}\langle X_\nu^* \psi_1 | g_1 \rangle &= \langle \psi_1 | X_\nu g_1 \rangle \\ &= \int dxdy\ \Delta^+(y-x)\, \overline{\psi_1(y)}\, x_\nu g_1(x) \\ &\neq \langle X_\nu \psi_1 | g_1 \rangle \\ &= \int dxdy\ \Delta^+(y-x)\, y_\nu \overline{\psi_1(y)}\, g_1(x).\end{aligned}$$

The two-point function $\Delta^+(x)$ determines the scalar product (13) in the one-argument subspace. Relativistic two-point functions are Källén-Lehmann forms [9]. Since Källén-Lehmann forms are not of point support over spacetime [56], $x_\nu \Delta^+(y-x) \neq y_\nu \Delta^+(y-x)$. Then $X_\nu^* \neq X_\nu$ for $\psi_1, g_1 \in \mathcal{D}_{X_\nu}$. The elevation of $\mathbf{x}_\nu$ to Hilbert space operator X_ν is not densely defined Hermitian

in relativistic quantum mechanics. This "localization problem" is one of many problems that develop with the canonical formalism [7, 8]. In contrast, the energy-momentum operators,

$$\sum_{\nu=1}^{n} P_\nu$$

in the n-argument subspace, are generators of translations in a unitary realization of the translation group and are densely defined and Hermitian as a consequence of Stone's theorem [24, 41]. A unitary realization of the translation group results from translation invariance of the scalar product. Location, an observable of evident importance in physics, does not precisely correspond to the elevation of $\mathbf{x}_\nu$ in relativistic quantum mechanics. The Hermitian operator associated with relativistic location is not the canonical quantization of x_ν. Suitable location operators have eigenfunctions that are the relativistically invariant localized functions. The relativistically invariant localized functions describe the states that most closely correspond to localized bodies in relativistic quantum physics, [44] and appendix 8.3.

In appropriate instances, classical and quantum state descriptions correspond but a canonical quantization is not necessary to achieve a quantum-classical correspondence. For relativistic location, a canonical quantization is not possible [44, 68]. Relativistic location provides an archetype for realizable, relativistic quantum-classical correspondences, and motivates consideration of non-Hermitian field operators. The elevations X_ν of location $\mathbf{x} \in \mathbb{R}^3$,

$$X_\nu = -i\frac{d}{d\mathbf{p}_\nu},$$

are distinct from the Hermitian relativistic location operators[1] [44]

$$\hat{X}_\nu = -i\,\omega^{1/2}\frac{d}{d\mathbf{p}_\nu}\,\omega^{-1/2}. \tag{202}$$

The operators X_ν and $\hat{X}_\nu$ are expressed in the Fourier transform, momentum domain and $\nu = x, y, z$. Despite the lack of Hermiticity, the elevated operators X_ν conditionally approximate the Hermitian operators $\hat{X}_\nu$ in the sense that the support of the eigenfunctions of the Hermitian location operators $\hat{X}_\nu$, appendix 8.3, are often well represented by the point support of the eigenfunctions of X_ν. The non-Hermitian X_ν approximate the Hermitian $\hat{X}_\nu$ if applied to appropriate state descriptions. That is,

$$X_\nu \approx \hat{X}_\nu$$

[1]The Newton-Wigner location operators $\hat{X}_\nu$ canonically commute with the Hermitian elevations of momenta and are Hermitian for the relativistic free field scalar product, appendix 8.3. From the Baker-Campbell-Hausdorff relations, momentum generates spatial translations.

when applied to functions dominantly supported on non-relativistic momenta, those with

$$\hbar^2\mathbf{p}^2 \ll (mc)^2$$

within the dominant support.[m]

Study of Hilbert space operators, particularly unbounded operators in infinite dimensional Hilbert spaces, is a subtle and elegant subject [41] that illustrates many of the "paradoxes of infinity." In appendix 8.2.5, a Hilbert space operator A is introduced as a mapping of elements from a Hilbert space $\mathbf{H}$ back into $\mathbf{H}$, $A : \mathbf{H} \mapsto \mathbf{H}$. If

$$|A\psi\rangle = |g\rangle$$

then $|\psi\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$, the domain of A, and $|g\rangle \in \mathcal{R}_A \subseteq \mathbf{H}$, the range of A. The domains of bounded operators can be extended to the entire $\mathbf{H}$. A is bounded if $\|A\psi\| \leq c\|\psi\|$ for some $c \in \mathbb{R}$ independently of the element $|\psi\rangle \in \mathbf{H}$. The least upper bound c defines an operator norm $\|A\|$. The domain of an unbounded A is necessarily a proper subset of the Hilbert space but the domain may be dense in $\mathbf{H}$. A set of elements $|e_n\rangle$ is dense in $\mathbf{H}$ if every $|\psi\rangle \in \mathbf{H}$ is within an arbitrarily small neighborhood of a finite linear combination of the $|e_n\rangle$,

$$\|\psi - \sum_{n=1}^{N} \alpha_n e_n\| < \epsilon$$

for $N > N_\epsilon$ with the $\alpha_n \in \mathbb{C}$ and $\epsilon \to 0$ as N grows without bound. A separable Hilbert space has a denumerable (finite or infinite) dense set of elements $|e_n\rangle$. An example is that functions in the Schwartz function space $\underline{\mathcal{S}}$ [20] are dense in the square-summable functions $\mathcal{L}^2$. $\mathcal{L}^2$ is a separable Hilbert space. For an unbounded operator B, a subsequence from a diverging sequence $|Bu_n\rangle$ can be selected and relabeled such that $\|Bu_n\| > n\|u_n\|$. Then, the neighborhood of each element $|\psi\rangle$ with $|B\psi\rangle \in \mathbf{H}$ contains a sequence constructed below with $\|B(\psi - \psi_n)\| > n\|\psi - \psi_n\|$ for n growing without bound and $\|\psi - \psi_n\| \to 0$. That is, an unbounded operator is not continuous anywhere in the Hilbert space. Indeed, using the divergent sequence $|u_n\rangle$ selected above, set

$$|v_n\rangle = \frac{|u_n\rangle}{n\|u_n\|}, \quad \text{and then} \qquad \|Bv_n\| = \frac{\|Bu_n\|}{n\|u_n\|} > 1.$$

[m] For non-relativistic momenta, $\omega \approx mc/\hbar = \lambda_c^{-1}$, a constant. For example, in the $\mathcal{L}^2(\mathbb{R}^3)$ norm for functions over space,

$$\frac{\|(X_\nu - \hat{X}_\nu)\psi\|}{\|\psi\|} \leq \frac{\lambda_c^2}{\sqrt{8}\,\alpha} \ll \lambda_c$$

if $\lambda_c \ll \alpha$ with α characterizing the extent of the spatial support and $\hbar/\alpha$ characterizing the extent of the momentum support of $\psi(\mathbf{x}) = \exp(-\mathbf{x}^2/(2\alpha^2))$. λ_c is the Compton wavelength (9) for the finite mass m particle. The condition $\lambda_c \ll \alpha$ provides that the dominant support of $\psi(\mathbf{x})$ is non-relativistic. On states dominantly supported on nonrelativistic momentum, the expected values of X_ν and $\hat{X}_\nu$ are nearly equal neglecting location differences small with respect to the Compton wavelength.

The sequence $|\psi_n\rangle = |\psi\rangle + |v_n\rangle$ is a convergent sequence, $\|\psi - \psi_n\| = \|v_n\| = 1/n \to 0$, but $\|B(\psi - \psi_n)\| = \|Bv_n\| > 1$. Then generally, neighboring states within a Hilbert space do not have nearly the same observables when observables are associated with unbounded Hilbert space operators.

In contrast, states exhibit observable features even when the expectation values of corresponding operators diverge or when the operation is undefined. For the example of location,

$$\psi(x) = e^{-\mathbf{x}^2/(4\sigma^2)} + \frac{\epsilon}{(1+\mathbf{x}^2)^{1-\delta}}$$

is dominantly supported near $\mathbf{x} = 0$ for $0 < \epsilon, \delta \ll 1$ with the length σ determining the size of the neighborhood. The σ, ϵ, δ can be selected to provide an arbitrarily large likelihood that an observation of location is near $\mathbf{x} = 0$. Here, likelihoods apply to finite volumes. The likelihood that the perceived location will be within a finite volume including the dominant support is much greater that the likelihood the perceived location will be from any other disjoint, equal volume. In this interpretation, the likely location is near $\mathbf{x} = 0$ for the state described by $\psi(x)$ even though $\langle\psi|\mathbf{x}\psi\rangle$ diverges. For location, $x_\nu \mapsto X_\nu$ is the Hilbert space location operator, and the expectation value $\langle\psi|X_\nu\psi\rangle$ diverges even though the likely perception of the body described by $\psi(x)$ is that the body will be observed near the origin. At large $\mathbf{x}^2$ (the unobservable "far side of the moon"), there is a small likelihood of detection within any finite volume but a sufficient volume for the mean value to diverge. There is no actual divergence since we are not capable of detecting location over infinite volumes. Our measurements are always localized and finite resolution; location X_ν and momentum P_ν are idealized observables. Observations are limited to finite, localized volumes: without an omniscient classical observer, no knowledge of infinite volumes is available. The likelihoods of physical interest are relative likelihoods of detection comparing finite volumes. The unknowable, distant support of functions only hypothetically affects our considerations. In the example, $\|\psi(x) - e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is small, they are neighboring states, but $\|X_\nu\psi(x) - X_\nu e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is divergent in the $\mathcal{L}^2$ norm.

Correspondences with the classical concepts of location and momentum is provided by the support of the state describing functions. Considering support, the likelihoods of observations of location and momenta are nearly the same for state describing functions that have nearly the same support. The supports of state describing functions provide more robust and consistent observable features than expected values of unbounded, densely defined Hermitian Hilbert space operators such as X_ν and P_ν.

The observables, location, momentum and field strength, are anticipated to have eigenvalues for every real number. As a consequence, their eigenfunctions cannot be elements of $\mathbf{H}_\mathcal{P}$. No Hermitian operator in a separable Hilbert space has an eigenfunction for every real number. A Hermitian operator in a rigged Hilbert space may have a generalized eigenfunction for any real number, but these generalized eigenfunctions are not elements within the Hilbert space of states. There is a countable number of mutually orthogonal elements in a separable Hilbert space, and the eigenfunctions of a Hermitian operator with distinct eigenvalues are necessarily

orthogonal [3]. Observations of location, momentum or field strength can not "collapse" to an eigenfunction of the observable: generalized eigenfunctions are idealizations and not state describing functions.

Quantizations of classical dynamical variables can be excluded as Hermitian operators even in ordinary (non-relativistic) quantum mechanics. The product x^3p can be considered a classical dynamical variable and in ordinary quantum mechanics with a single spatial dimension, the corresponding operator in $\mathcal{L}^2$ should be the Hermitian

$$i\hbar x^{3/2}\frac{d}{dx}x^{3/2} = i\hbar(\frac{x^3}{2}\frac{d}{dx} + \frac{d}{dx}\frac{x^3}{2}).$$

The classical dynamic variables x and p correspond to unbounded, self-adjoint operators x and $i\hbar d/dx$ in this one spatial dimensional $\mathcal{L}^2$ example. However, this densely defined, formally Hermitian operator that corresponds with x^3p has square-summable eigenfunctions

$$s_\lambda(x) = \sqrt{2\lambda}\ \frac{\exp\left(-\lambda/(2x^2)\right)}{x^{3/2}}$$

with imaginary eigenvalues $-i\hbar\lambda$ [9]. $0 < \lambda \in \mathbb{R}$. As a consequence, $X^{3/2}PX^{3/2}$ is not a densely defined Hermitian operator in the $\mathcal{L}^2$ Hilbert space. This $s_\lambda(x) \in \mathcal{L}^2$ is defined for $x > 0$ and equals zero otherwise, or $s_\lambda(x)$ can be extended to negative x. The formally Hermitian operator $X^{3/2}PX^{3/2}$ that corresponds to the classical dynamical variable x^3p is not Hermitian for $\mathcal{L}^2$. This example illustrates Groenewold's theorem [24]. Nevertheless, for the example of linear harmonic motion and minimum uncertainty (Gaussian) states $s_t(x)$ with small spatial variances, the trajectory of x^3p given by Newtonian mechanics approximates $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ from quantum dynamics [35, 51]. This establishes that there are particular states with real $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ that agree with the classical approximations x^3p even though $X^{3/2}PX^{3/2}$ is not a Hermitian operator. The Hermitian operator most closely corresponding to x^3p need not be the elevation of x^3p. Once again, the support of functions for appropriate state describing functions exhibit quantum-classical correspondences while a correspondence of classical dynamical variables with operators fails. Even when the operator that is the canonical quantization of the classical dynamical variable is not Hermitian, classical dynamics may approximate the quantum dynamics for appropriate states.

8 Appendices

8.1 The Dirac-von Neumann axioms for quantum mechanics

The Dirac-von Neumann axioms describe quantum mechanics. Dirac-von Neumann axioms I-III describe quantum mechanics in general, and axioms IV and V describe the canonical quantization of non-relativistic physics.

Here, the statement of the Dirac-von Neumann axioms is adapted from F. Strocchi's discussion [60].

I. *Observables*: The Hermitian operators A corresponding to the observables of a quantum mechanical system are within the algebra of bounded self-adjoint operators $\mathcal{B}(\mathbf{H})$ for a separable Hilbert space $\mathbf{H}$.

II. *States*: The pure states of a quantum mechanical system are described by rays $a|\underline{s}\rangle$, $|s\rangle \in \mathbf{H}$, $a \in \mathbb{C}$ and $a \neq 0$. More generally, a state is described by a nonnegative, unit trace, state density operator $\rho \in \mathcal{B}(\mathbf{H})$.

III. *Expectations*: If a state is represented by the normalized pure state $|s\rangle \in \mathbf{H}$, then, for the observable corresponding to $A \in \mathcal{B}(\mathbf{H})$, the experimental expectation is $\langle s|As\rangle$. More generally, experimental expectations are Trace$(A\rho)$. If A has a complete set of normalized eigenvectors $|e_n\rangle \in \mathbf{H}$, then

$$\langle s|As\rangle = \sum_n \lambda_n |\langle e_n|s\rangle|^2$$

and from Born's rule, the likelihood of observing λ_n is $|\langle e_n|s\rangle|^2$. More generally, A is described by the spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [41], chapters 7-10 [24]).

Axioms I-III describe a Hilbert space realization of quantum mechanics and are implicit in the development of section 3. Conditions A.1 and A.2 of section 4.1 imply a Hilbert space realization. State density operators $\rho \in \mathcal{B}(\mathbf{H})$ are discussed in section 8.2.6. States are described by elements of the Hilbert space: limits such as the eigenfunctions of location and momentum are in the dual to the basis function spaces but depart from the rigged Hilbert spaces of interest [9]. Superselection sectors in the Hilbert spaces of interest illustrate that not all Hilbert space elements represent states of nature. Hilbert spaces of interest are represented as direct sums of superselection sectors and the observables are associated with operators limited to within sectors. The limitation of observables to bounded operators departs from Dirac's development but reflects that only finite values are observable, for example, locations within an accessible, finite volume. Limiting consideration to bounded operators provides several technical conveniences [24, 60] as well as conceding to physical reality.

Axioms I-III describe a general Hilbert space realization of quantum mechanics. Characterization of the observables remains. Dirac-von Neumann axioms IV-V determine properties of particular operators in non-relativistic physics.

IV. *Dirac canonical quantization*: The Hermitian operators that describe canonical coordi-

nates q_i and momenta p_j , $i, j = 1, \ldots N$, obey canonical commutation relations

$$\begin{aligned} [q_i, q_j] &= 0 \\ [p_i, p_j] &= 0 \\ [q_i, p_j] &= -i\hbar\delta_{ij}. \end{aligned}$$

V. *Schrödinger representation*: The canonical commutation relations are realized in the Hilbert space

$$\mathbf{H} = \mathcal{L}^2 = \{\psi(x) | \int d\mathbf{x}\, |\psi(ct, \mathbf{x})|^2 < \infty, x \in \mathbb{R}^4, \mathbf{x} = x_1, x_2, x_3\}$$

by:

$$\begin{aligned} |q_i\psi\rangle &= x_i\, |\psi\rangle \\ |p_i\psi\rangle &= -i\hbar\, |\frac{\partial\psi}{\partial x_i}\rangle. \end{aligned}$$

In the Dirac-von Neumann axioms, canonical variables q_i, p_i are canonical quantizations of classical dynamical variables. If the canonical variables q_i, p_i are location and momentum, respectively, the canonical commutation relations are known as the Born-Heisenberg-Jordan relations. In the $\mathcal{L}^2$ Hilbert space applicable to non-relativistic physics, these elevations result in unbounded Hermitian Hilbert space operators q_i, p_i, violating I. The correspondence in axiom V is the "elevation of c-number to q-number" for q_i. Unboundedness introduces the consideration that sums of observables are not necessarily observables (if necessarily limited domains are sufficiently disjoint) [24]. And, the (generalized) eigenfunctions of location and momentum are not elements of the Hilbert space. These and additional difficulties with the canonical formalism are discussed in section 7. Axiom V describes a particular, non-relativistic quantum-classical correspondence for location and momentum. The canonical formalism extrapolates axioms IV-V with fields as canonical coordinates [63]. Axiom IV remains valid for location and momentum in the constructions.

> The lack of a clear distinction between the role of the two sets of axioms, I, II, III and IV,V, is at the origin of the widespread point of view, adopted by many textbooks, by which all of them are characteristic of quantum systems. The distinction between classical and quantum systems is rather given by the mathematical structure of [the algebra of observables] and it will have different realizations depending on the particular class of systems. – Franco Strocchi [60].

8.2 Hilbert spaces and quantum mechanics

8.2.1 Hilbert spaces

A Hilbert space **H** is characterized by the number of linearly independent elements. For every two elements in the complex Hilbert spaces of interest here, there is a complex number $\langle \underline{f}|\underline{g}\rangle$,

$$\underline{f}, \underline{g} \in \mathbf{H} \mapsto \langle \underline{f}|\underline{g}\rangle \in \mathbb{C}$$

designated the scalar product of the elements. Properties of this scalar product include that $\langle \underline{g}|\underline{f}\rangle$ is the complex conjugate of $\langle \underline{f}|\underline{g}\rangle$ and the scalar product is linear in the second argument, $\langle \underline{f}|\alpha\underline{g} + \beta\underline{h}\rangle = \alpha\langle \underline{f}|\underline{g}\rangle + \beta\langle \underline{f}|\underline{h}\rangle$ for $\alpha, \beta \in \mathbb{C}$, and as a consequence, complex conjugate linear in the first argument. Scalar products are nonnegative, $\langle \underline{f}|\underline{f}\rangle \geq 0$, and this provides satisfaction of the Cauchy-Schwarz-Bunyakovsky inequality,

$$|\langle \underline{g}|\underline{f}\rangle|^2 \leq \langle \underline{g}|\underline{g}\rangle\langle \underline{f}|\underline{f}\rangle.$$

The zero element is unique in a Hilbert space. If $\langle \underline{f}|\underline{f}\rangle = 0$, then $\underline{f} = 0$ and $\langle \underline{g}|\underline{f}\rangle = 0$ for every element $|\underline{g}\rangle$ of the Hilbert space. A degenerate scalar product has all the properties of a scalar product except that there is more than one element of zero length. A degenerate scalar product (13) is defined for pairs of function sequences from the basis spaces $\underline{\mathcal{P}}$ considered as a linear vector space. The Hilbert spaces of interest are the completions of linear vector spaces with elements that are equivalence classes of vectors labeled by function sequences [12]. The elements of the Hilbert space may be characterized by any function sequence in an equivalence class. An isometry extends the degenerate scalar product to a scalar product: the elements of the Hilbert space are equivalence classes of functions equal in the norm (25) and $\langle \underline{f}|\underline{f}\rangle = 0$ states that the function sequence $\underline{f}$ is an element of the equivalence class of zero. The separation of two Hilbert space elements $\underline{g}, \underline{f}$ is $\|\underline{g} - \underline{f}\|$ using the norm (25). The distance is zero for two equivalent function sequences, and two is the maximum separation of normalized state descriptions ($\|\underline{f}\| = \|\underline{g}\| = 1$). The separation is $\sqrt{2}$ for orthogonal, normalized states. Two states $\underline{f}, \underline{g}$ are orthogonal if $\langle \underline{g}|\underline{f}\rangle = 0$. In quantum mechanics, every element on a ray $|a\underline{f}\rangle$ with finite $a \in \mathbb{C}$ describes the same physical state: only relative phase and amplitude within linear combinations are significant. Linearity and completeness are characteristic of Hilbert spaces. A Hilbert space **H** is complete: the limit of every Cauchy sequence of elements $|\underline{g}_\nu\rangle \in \mathbf{H}$ is included. That is, if

$$\|\underline{g}_\nu - \underline{g}_n\| \to 0$$

for $\nu, n > N \to \infty$, then there is an element $|\underline{g}\rangle \in \mathbf{H}$ such that $|\underline{g}\rangle = \lim_{\nu\to\infty} |\underline{g}_\nu\rangle$. In a separable Hilbert space, every element is arbitrarily well-approximated by a denumerably indexed sum of N linearly independent elements.

$$|\underline{f}\rangle = \sum_\ell \alpha_\ell |e_\ell\rangle$$

with $\ell \in \mathbb{N}$, the natural numbers, and $\alpha_\ell \in \mathbb{C}$. A (closed) subspace of a Hilbert space is the linear span of a subset of elements. If there are only N linearly independent elements, the Hilbert space or subspace is finite dimensional of dimension N, and if the number of linearly independent elements is unbounded but includes a denumerable, dense set of elements, the Hilbert space or subspace is denoted infinite dimensional and separable. The rigged Hilbert spaces of interest here are separable [10].

8.2.2 Rigged Hilbert spaces

Rigged (equipped) Hilbert spaces are appropriate settings for quantum mechanics. VEV that provide a Poincaré invariant scalar product (13) must be generalized functions: the VEV can not be summable functions [9]. The basis function spaces $\underline{\mathcal{P}}$ used in the constructions include only those elements from $\underline{\mathcal{S}}$ with Fourier transforms that vanish on the negative energy mass shells. Three classes of functions, denoted a Gelfand triple after Israel Gelfand, describe a rigged Hilbert space. The elements of a countably normed basis function space are denoted test functions [9, 19, 20]. A particularly useful space of test functions are the Schwartz tempered functions $\underline{\mathcal{S}}$: the tempered functions are smooth (infinitely differentiable) and exhibit rapid decline for large values of their arguments [20]. The space of Fourier transforms of $\underline{\mathcal{S}}$ coincides with $\underline{\mathcal{S}}$. The associated class of generalized functions $\underline{\mathcal{S}}'$ are the linear distributions (generalized functions) dual to $\underline{\mathcal{S}}$. Linear distributions $T(x)$ map functions to complex numbers.

$$T(x) \in \underline{\mathcal{S}}' : \psi(x) \in \underline{\mathcal{S}} \mapsto T(\psi) \in \mathbb{C}.$$

$\underline{\mathcal{S}}'$ is usefully conceived as limits of functions $T(x)$ such that

$$T(\psi) = \int dx\, T(x)\psi(x)$$

is finite when $\psi(x) \in \underline{\mathcal{S}}$ with acknowledgment that this concept includes limits that are not summable using Lebesgue measure.[n] Indeed, the generalized functions dual to the functions of bounded support can be represented

$$T(\psi) = \sum_n \int d\mu_k(x)\, \frac{d^n\psi(x)}{dx^n}$$

with the summation over a finite number of terms using measures μ_k on the real numbers [20]. $\mathbf{H}_\mathcal{S}$ consists of $\underline{\mathcal{S}}$ plus the Cauchy sequences convergent in a Hilbert space norm (25). As a consequence, the Gelfand triple $(\mathcal{S}, \mathbf{H}_\mathcal{S}, \mathcal{S}')$ satisfies

$$\underline{\mathcal{S}} \subset \mathbf{H}_\mathcal{S} \subset \underline{\mathcal{S}}'.$$

[n] One perspective on the distinction between functions and generalized functions is illustrated by generalized functions with a single point of support. Both $\delta(x)$ and $\delta'(x)$ can be considered limits of test functions and supported solely on the point $x = 0$, but $\delta(x)$ and $\delta'(x)$ are distinct generalized functions.

The eigenfunctions of location and momentum are not elements of $\mathbf{H}_{\mathcal{S}}$ but are elements of $\underline{\mathcal{S}}'$. Generalized eigenfunctions of a linear operator A defined in $\underline{\mathcal{S}}$ are the generalized functions $T_\lambda(x) \in \underline{\mathcal{S}}'$ such that $T_\lambda(Af) = \lambda T_\lambda(f)$ for any $f \in \mathcal{S}$, [21].

The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are a union of nuclear, countably normed spaces [21]. The elements of $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are test functions and their limits that label elements within $\mathbf{H}_{\mathcal{P}}$ include generalized functions.

8.2.3 The Cauchy-Schwarz-Bunyakovsky inequality

The Cauchy-Schwarz-Bunyakovsky [47] inequality is that

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$$

for elements u, v within a complex linear vector space $\mathbf{H}$. The Cauchy-Schwarz-Bunyakovsky inequality applies if the vector space has a product $\langle u|v\rangle : u, v \in \mathbf{H} \mapsto \langle u|v\rangle \in \mathbb{C}$ with properties

$$\begin{aligned} \langle v|v\rangle &\geq 0 \\ \langle u|v\rangle &= \overline{\langle v|u\rangle} \\ \langle w|\alpha u + \beta v\rangle &= \alpha\langle w|u\rangle + \beta\langle w|v\rangle \end{aligned} \tag{203}$$

for all u, v, w in the vector space $\mathbf{H}$ and $\alpha, \beta \in \mathbb{C}$. This product is designated a complex scalar product if $\langle v|v\rangle = 0$ implies $v = 0$, and otherwise it is a degenerate scalar product.

For any two elements in the vector space, choose one element to label as v if $\langle v|v\rangle > 0$ and label the remaining element u. This includes all cases except for both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Using the Gram-Schmidt construction, the element

$$z = u - \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v$$

is orthogonal to v.

$$\langle z|v\rangle = 0.$$

From the properties (203) and the construction of z,

$$\begin{aligned} \langle u|u\rangle &= \langle z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v|z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v\rangle \\ &= \langle z|z\rangle + \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle} \\ &\geq 0. \end{aligned}$$

But $\langle z|z\rangle \geq 0$ and it follows that

$$\langle u|u\rangle \geq \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle}.$$

This demonstrates the inequality unless both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. If both are zero, the properties (203) provide that

$$\begin{aligned}\langle u+v|u+v\rangle &= \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle u|u\rangle \\ &= 2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \\ &\geq 0.\end{aligned}$$

Similarly, $\langle u-v|u-v\rangle \geq 0$ provides that

$$2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \leq 0.$$

As a consequence, $\mathbb{R}\mathrm{e}(\langle u|v\rangle) = 0$. Similarly, $\langle u+iv|u+iv\rangle \geq 0$ and $\langle u-iv|u-iv\rangle \geq 0$ provide that $\mathbb{I}\mathrm{m}(\langle u|v\rangle) = 0$. Then,

$$\langle u|v\rangle = 0$$

if both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Summarizing, the Cauchy-Schwarz-Bunyakovsky inequality applies to a linear vector space with elements u, v and product $\langle u|v\rangle$ with properties (203).

If $\langle u|v\rangle$ is a scalar product and neither u nor v are the zero element, then

$$|\langle u|v\rangle|^2 = \langle u|u\rangle\langle v|v\rangle$$

if and only if

$$u = \alpha v$$

for a nonzero $\alpha \in \mathbb{C}$. For nonzero u and v and a scalar product, the equality applies if and only if $z = 0$ for the z constructed above and then

$$u = \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v.$$

8.2.4 Entanglement

Quantum mechanics includes descriptions of entangled states. Descriptions of states are elements of Hilbert spaces and elements are expressible as linear combinations of other elements. The linear expansion of states is a property of quantum physics distinct from classical physics.

A simple example of entanglement has four states: |up, up⟩, |up, down⟩, |down, up⟩, and |down, down⟩ that span the spin states of two spin one-half bodies. The states describe the four possibilities for the measurement of spin polarizations along a particular axis. Any linear combination of these four states describes a state. For example,

$$\frac{1}{\sqrt{2}}|\text{up, down}\rangle + \frac{1}{\sqrt{2}}|\text{down, up}\rangle$$

is one possible state. For this example state, the polarizations are entangled: up for the first body occurs only with down for the second body, and down for the first body occurs only with up for the second body. The state of the first body is determined by the state for the second and total spin adds to zero in any observation. However, the states are not determined, the first body can be detected as either spin up or spin down with equal likelihoods in this example. But, knowledge of one of the spins implies knowledge of the other due to an entanglement of states. Entanglement of the spins is established when the particles are causally related; the entanglement persists as the particles become acausally separated. An entanglement of bodies need not be perfect. For example,

$$a\,|\text{up, up, down}\rangle + b\,|\text{up, down, up}\rangle + c\,|\text{down, up, up}\rangle$$

is an example with a particular angular momentum but no perfectly entangled pair. In appendix 8.2.8, the states that result from an observation are described by linear combinations of states of an observer entangled with the possible results of observation, $|\psi\rangle = \sum_k \alpha_k\,|o_k, s_k\rangle$. The o_k are orthogonal descriptions of the observer's state, and the s_k describe the entangled, observed state. In the example of Schrödinger's cat thought experiment [53], o_1 would be "observed a live cat," o_2 would be "observed a dead cat," s_1 would be "a live cat," s_2 would be "a dead cat" and these descriptions are entangled $\alpha_1|o_1, s_1\rangle + \alpha_2|o_2, s_2\rangle$. An appeal to experience indicates that $|o_1, s_2\rangle$ and $|o_2, s_1\rangle$ do not persist in the evolution of states.

8.2.5 Hilbert space operators

Hilbert space operators are linear maps of elements of a Hilbert space $\mathbf{H}$ to elements within the Hilbert space [3, 41]. A is a Hilbert space operator if

$$A : |\underline{f}\rangle \in \mathbf{H} \mapsto |A\underline{f}\rangle \in \mathbf{H}$$

for a subset of elements $|\underline{f}\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$ denoted the domain of A. For a bounded operator,

$$\|A\psi\| \leq \|A\|\,\|\psi\|$$

with

$$\|A\| = \sup(\|A\psi\| : \|\psi\| \leq 1) < \infty.$$

A bounded operator is continuous, and the domain can be extended to the entire Hilbert space. Unitary operators are bounded and preserve the scalar product, that is, for unitary U, $\langle Ug|U\psi\rangle = \langle g|\psi\rangle$. Illustrated in section 7, an unbounded operator is not continuous, its domain is not a (closed) subspace, and the domain necessarily does not include the entire Hilbert space although the domain may be dense in the Hilbert space. The adjoint operator A^* of a Hilbert space operator A is defined by the property that $\langle A^*h|\psi\rangle = \langle h|A\psi\rangle$ for $|\psi\rangle \in \mathcal{D}_A$. The set of $h \in \mathbf{H}$ such that there is a $g \in \mathbf{H}$ with $\langle h|A\psi\rangle = \langle g|\psi\rangle$ is the domain $\mathcal{D}_{A^*}$ of A^*. An operator A

with domain $\mathcal{D}_A$ is Hermitian if $\langle \underline{u}|A\underline{v}\rangle = \langle A\underline{u}|\underline{v}\rangle$ for every $\underline{u}, \underline{v} \in \mathcal{D}_A$ [3], a Hermitian operator is symmetric if $\mathcal{D}_A$ is dense in the Hilbert space, and a symmetric operator is self-adjoint if $\mathcal{D}_A = \mathcal{D}_{A^*}$ and $A\underline{u} = A^*\underline{u}$ for every $\underline{u} \in \mathcal{D}_A$. These designations correspond to Hermitian, maximal Hermitian, and hypermaximal respectively in von Neumann's designation [62].

8.2.6 Operators in quantum mechanics

Every closed subspace of states corresponds with a projection operator [41]. Fundamental results for Hilbert space operators include the spectral theory for rigged Hilbert space operators [21, 24, 41]; and Stone's theorem [24] for unitarily realized groups of Hilbert space transformations. Spectral theory displays the Hermitian operators that represent observables as limits of linear combinations of projections weighted by representative values of the observable associated with each subspace of states. Projection operators E are bounded, self-adjoint and idempotent, $E = E^* = E^2$, $\|E\psi\| \leq \|\psi\|$. For projections, the range $\mathcal{R}_E = E\mathbf{H} \subset \mathbf{H}$, a proper subset unless $E = \mathbb{I}$. From a projection E, any state $|\psi\rangle \in \mathbf{H}$ may be decomposed as an element $|E\psi\rangle$ within the subspace $E\mathbf{H}$ and an element $|(\mathbb{I} - E)\psi\rangle$ in the orthogonal complement of the subspace, Riesz's theorem.

The states of nature are described by nonnegative ($\langle\psi|\rho\psi\rangle \geq 0$), self-adjoint ($\rho = \rho^*$, $\mathcal{D}_\rho = \mathcal{D}_{\rho^*}$ dense in $\mathbf{H}$), trace-class, normalized operators ρ (Trace$[\rho] = 1$).

$$\text{Trace}[\rho] = \sum_k \langle e_k|\rho e_k\rangle \tag{204}$$

if the $|e_k\rangle$ are an orthonormal basis for the separable Hilbert space $\mathbf{H}$. This basis is not unique and every unitary operator maps a basis to a basis, $|Ue_k\rangle = |e'_k\rangle$. $U^* = U^{-1}$ for a unitary operator and unitary operators are bounded. These operators ρ are designated state density operators [62] and generalize the vector states generally discussed above. Vector states have state density operators $\rho = E$ for E the projection onto a single element $|\psi\rangle$ ($\mathcal{R}_E = \{|a\psi\rangle : a \in \mathbb{C}\}$). Born's rule includes: if a state is described by the state density operator ρ, then the likelihood of observing a state in the subspace $E\mathbf{H}$ corresponding to a projection E given an initial state described by ρ is Trace$[E\rho] \leq 1$.

Operators associated with observables have mean values

$$E[A] = \text{Trace}[A\rho] \in \mathbb{R}$$

for Hermitian operators A and state density operators ρ. Measurements are real numbers and the expectation values of Hermitian operators are real. If A is Hermitian, $A = A^*$ on the domain of A, then for every $\underline{f} \in \mathcal{D}_A$,

$$\begin{aligned} \langle A^*\underline{f}|\underline{f}\rangle &= \langle A\underline{f}|\underline{f}\rangle \\ &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle \underline{f}|A\underline{f}\rangle \end{aligned}$$

from the properties of scalar products and definition of the adjoint operator. Then $A = A^*$ implies that the expectation values $\langle \underline{f}|A\underline{f}\rangle$ are real. And, if the expectation value is real and $\underline{f} \in \mathcal{D}_A$, then

$$\begin{aligned}\langle \underline{f}|A\underline{f}\rangle &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle A\underline{f}|\underline{f}\rangle\end{aligned}$$

and $A = A^*$ for $\underline{f} \in \mathcal{D}_A$. As a consequence, observables are limited to Hermitian operators.

Significant examples of observables in non-relativistic quantum mechanics are the Hermitian location operators X_ν. In $\mathcal{L}^2$,

$$\langle \psi|\psi\rangle = \int d\mathbf{x}\ |\psi(x)|^2$$

is finite at each time t with $x = ct, \mathbf{x}$. The summation is over three-dimensional space, $\mathbb{R}^3$. Three location operators X_ν, one for each of the three spatial dimensions, are given by

$$X_\nu \psi(x) = x_\nu \psi(x).$$

There is no bound on the values assumed by location in a Euclidean space and consequently the location operator is an unbounded operator. $x_\nu\psi(x)$ is not necessarily square-integrable for square-integrable $\psi(x)$ and as a consequence, $\mathcal{D}_{X_\nu} \subset \mathbf{H}$, a proper subset. Functions $\psi(x) \in \mathcal{L}^2$ such that $x_\nu\psi(x) \in \mathcal{L}^2$ are dense in $\mathcal{L}^2$ but elements of slow decline are excluded from $\mathcal{D}_{X_\nu}$ (e.g., $\psi(x) = (1 + \mathbf{x}^2)^{-1+\delta} \in \mathcal{L}^2$ for $1 \gg \delta > 0$ in three dimensions but $x_\nu\psi(x) \notin \mathcal{L}^2$).

8.2.7 Interpretation of the functions $f_n((x)_n)_{(\kappa)_n}$ in $\underline{f}$

The representation of state (2) is an expansion in powers of fields, and consequently the n-argument functions are not coefficients of n particle states. In the case of free fields, both creation and annihilation contributions are manifest in the expansion (2). An understanding of the functions $f_n((x)_n)_{(\kappa)_n}$ in the state describing function sequences $\underline{f}$ of Borchers' notation (1) for the constructions based on the support limited functions $\underline{\mathcal{P}}$ varies significantly from the descriptions familiar in free field theories and RQFT. Nevertheless, for a limited class of particle-like "macroscopic" state describing functions, interpretation of the state describing functions (1) follows the description of particles from free field theory [23, 24, 54, 63]. When the supports of every argument of a state describing function are "macroscopic," localized and widely space-like separated, then cluster decomposition A.6 provides that the constructed field is approximated by a free field: contributions of the VEV equal free field VEV. In this appendix, these results are illustrated.

The familiar free field example applies when $\underline{\mathcal{S}}$ is the basis space of function sequences. Presented in section 3.2, a free field $\Phi_o(\underline{f})$ is Hermitian (10). For a free field, the scalar product is given by a Wightman functional (15)

$$\langle \underline{h}|\underline{f}\rangle = \mathcal{F}(\underline{h}^* \times \underline{g});$$

the field expands into creation and annihilation operator components

$$\Phi_o(\underline{f}) = \Phi_o^+(\underline{f}) + \Phi_o^-(\underline{f})$$

that have commutation/anti-commutation relations

$$[\Phi_o^-(\underline{f}^*), \Phi_o^+(\underline{g})]_\pm = W_2(\underline{f}^* \times \underline{g})$$

and commute otherwise; and the vacuum state is cyclic with

$$\Phi_o^-(\underline{f})\Omega_o = 0$$

[23, 24, 54, 63]. Notation includes that Φ_o^+ is the creation and Φ_o^- is the annihilation operator, and the only nonzero functions in the arguments $\underline{f}, \underline{g}$ of the field are single argument functions $f_1(x)_\kappa, g_1(x)_\kappa$. $\underline{f}^*$ is from (11). Whether the commutator or anti-commutator is applied depends on the field components labeled by indices κ_j. The statistics of field components are boson or fermion, section 3.2. The algebraic evaluation of the VEV of a free field results in (225),

$$\mathcal{F}_{k,2n-k}((x)_n)_{(\kappa)_n} = \sum_{pairs} \sigma(S,(\kappa)_n) \prod_{j=1}^{n} W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}},$$

a summation over all $(2n)!/(2^n n!)$ ways of pairing indices $i_j, \ell_j \in \{1, 2n\}$, and within each $W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}$ the indices are ordered $i_j < \ell_j$. The $\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ with an odd number of arguments n are zero and $\mathcal{F}_{0,0} = 1$.

Limiting discussion to product functions

$$f_n((x)_n)_{(\kappa)_n} = \prod_{j=1}^{n} f_{1,j}(x_j)_{\kappa_j},$$

elements (3) of the Hilbert space generated from the vacuum by applying products of the field are

$$\begin{aligned} |f_{n,(\kappa)_n}\rangle &= \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{j=1}^{n} \Phi_o(x_j)_{\kappa_j}\Omega\rangle \\ &= | \prod_{k=1}^{n} \Phi_o(f_j)\Omega\rangle \end{aligned}$$

with

$$\Phi_o(f_\alpha) = \int dx \; \Phi_o(x)_{\kappa_\alpha} f_{1,\alpha}(x)_{\kappa_\alpha}$$

and in the abbreviated notation (28) for function sequences $\underline{f}$ with the j-argument functions $f_j = 0$ for $j \neq n$. Expression of the free field in terms of the creation and annihilation operators, annihilation of the vacuum, and the commutation relations result in

$$\begin{aligned} |f_{n,(\kappa)_n}\rangle &= |\prod_{k=1}^{n} \Phi_o(f_j)\Omega\rangle \\ &= |\prod_{k=1}^{n-2} \Phi_o(f_j)(\Phi_o^-(f_{n-1}) + \Phi_o^+(f_{n-1}))\Phi_o^+(f_n)\Omega\rangle \\ &= |\prod_{k=1}^{n-2} \Phi_o(f_j)(\Phi_o^+(f_{n-1})\Phi_o^+(f_n) + W_2(f_{n-1} \times f_n))\Omega\rangle. \end{aligned}$$

In RQFT, products of n creation operators applied to the vacuum describe n particles [23, 24, 54, 63]. Continuation of the evaluation for $|f_n\rangle$ results in an expansion of the element described by state describing functions $f_n((x)_n)_{(\kappa)_n}$ as a summation of descriptions for $n - 2\ell$ particles, $0 \leq n - 2\ell \leq n$ with $\ell \in \mathbb{N}_0$. The n-argument functions $f_n((x)_n)_{(\kappa)_n}$ in (1) for elements of the Hilbert space are the expansion coefficients for products of the quantum field and are not generally interpretable as descriptions of n particles. The functions $f_n((x)_n)_{(\kappa)_n}$ generate a description of a state vector that is a linear combination of descriptions for $k \leq n$ particles and the vacuum if n is even.

However, the interpretation is quite different for the constructions of section 3. Generally there is no unambiguous interpretation of $n \geq 2$, n-argument state describing functions as particles. Applying Born's rule, for the constructions, a state description generally has finite projections onto to all particle descriptions that are kinematically included. If momentum, angular momentum and charges are conserved, then generally two states are not orthogonal due to the connected components of the VEV in the scalar product (6). However, when the supports of the state describing functions satisfy localization (116) and isolation (119), then the states are well approximated by the states of free fields, $\Phi(\underline{f}) \approx \Phi_o(\underline{f})$. The states of free fields have the established interpretation illustrated above. And, if the $f_1(x)_\kappa \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)_\kappa$, then the field $\Phi_o(\underline{f})$ is supported only on positive energies due to the zero of $f_1(x)_\kappa$ on the negative energy mass shells, and then $\Phi_o(\underline{f}) = \Phi_o^+(\underline{f})$. $\Phi_o(\underline{f}^*) = \Phi_o^-(\underline{f}^*) = \Phi_o^+(\underline{f})^*$, the operator adjoint. From the lack of support on the negative energy mass shells,

$$W_2(\underline{f}^* \times \underline{g}^*) = W_2(\underline{f} \times \underline{g}) = 0$$

and only $W_2(\underline{f}^* \times \underline{g})$ contribute in an expansion of free field states. The interpretation of state describing functions $f_n((x)_n)_{(\kappa)_n}$ as descriptions of n free particles follows from the limited support of the functions in $\underline{\mathcal{P}}$ for "macroscopic" state describing functions that satisfy localization (116) and isolation (119). The cluster decomposition condition A.6 provides that the scalar products of the fields equal the scalar products of free fields if localization and isolation are

satisfied, section 4.3. More generally, the states described by functions $f_n((x)_n)_{(\kappa)_n}$ may have finite projections onto states with distinct numbers and types of particles due to the connected contributions in the constructed VEV.

8.2.8 Observation in Hilbert spaces

The Everett-Wheeler-Graham relative state interpretation of the formalism of quantum mechanics [11] escapes the measurement paradoxes and ad hoc assertions of earlier understandings. The development describes observation from the premise that the "mathematical formalism of quantum mechanics is sufficient as it stands" [11]. Concepts such as "collapse of the wave function to an eigenfunction" providing a classical description of the observable are inconsistent with relativity. It would take a period of infinite duration for the support of a typical state describing function such as an energy eigenfunction of the hydrogen atom to collapse to a single point. The nonzero support of the energy eigenfunction is infinite: the speed of light is finite. And, the elevation of location to quantum mechanical operator is not Hermitian, [44] and section 7. Causality requires that the time evolution of state describing functions is expressed with operators that satisfy the Poincaré invariance of likelihoods. Time evolution includes measurements. With the observer as well as the observed included in the quantum mechanical description, time evolution is continuous, unitary time translation and there is no need to distinguish which interactions constitute measurement. The relative state description is consistent with our experience as well as consistent with the quantum mechanical description of nature, although it violates longstanding classical intuition.

The process of observation is fundamental in quantum mechanics: observers are not omniscient and the interactions that result in perceptions affect the relevant description of state. This process explains how the perception of nature can differ from the description. In classical physics, the state can be observed without disruption: an observer is external to the systems under consideration and is essentially omniscient. However, in quantum mechanics, observation is an interaction within the quantum description: an observation entangles descriptions of the observer with descriptions of the observed. Observers are also described by quantum mechanics. There is no "macroscopic" or "classical" domain governed by distinct physical principles. A quantum description of state that describes a composite of system under observation and observer evolves into a linear combination that describes distinct alternative possibilities from an observation. For example, an initially localized state spreads over space with time and a distinct state of the observer becomes entangled with each possible outcome of a subsequent localizing interaction. It is inconsistent with our experience that there is only one observer state: observers perceive results from among possibilities. To be included in a quantum description and avoid the difficulties described in the Schrödinger cat, EPR, and Wigner's friend measurement paradoxes [53, 16, 67], a distinct state of the observer must be entangled with each of the possibilities for observation. Distinct perceptions are described by distinct states. Within linear expansions of state descriptions, "we" are described by one of the possible states. This relative

state or "many worlds" description [11] derives naturally from the quantum mechanical description of state as elements of Hilbert spaces. Although inherent to quantum mechanics, this natural interpretation is very different from classical concepts and it took several decades after the initial formulations of quantum mechanics for the relative state interpretation to emerge. Hugh Everett III, John Archibald Wheeler, Neill Graham and Bryce DeWitt developed the relative state interpretation of quantum mechanics [11]. Earlier interpretations maintained classical descriptions for observers. Even the designation as "many worlds" betrays a classical predisposition: many classical worlds but one consistent quantum description. That state describing functions expand in linear combinations of other state descriptions is characteristic of Hilbert space descriptions of nature. A result of relative state interpretation [11] is that the expansion is equivalent to a mixture, a linear combination of physically independent descriptions. However, none of these descriptions are classically described. The description of a quantum state as a statistical distribution over classical state descriptions is a flawed concept inconsistent with nature, sections 1 and 7. Consideration of the state as a superposition over many classical worlds, each with a state described by classical dynamical variables is an instance of a persistent classical predisposition. The relative state interpretation uses the natural quantum mechanical description of states by functions with entanglement leading to an equivalent decomposition into a mixture of relative states labeled by the observer's perceptions.

The quantum mechanical description of observations also explains our comfort with the classical view: often, within the precision of our perceptions, the resulting quantum mechanical description appears to replicate a classical description. However, a richer range of descriptions is included among the descriptions of state in quantum mechanics. A localized interaction results in observation of a localized state; two slits in a screen followed by localizing detectors results in observation of wave-like interference when the initial states are widely supported and propagate toward the screen. In common instances, the perception of a quantum description is indistinguishable from a classical description although in general, in instances such as the uniformly illuminated double slit, a quantum description of state deviates substantially from a classical description. Quantum mechanics supersedes classical descriptions.

The states of an observer are labeled by their perceptions. The relevant final state of the observed is determined by the entangled observer state, the result of an interaction characterized as an observation. From Born's rule, the likelihoods of the various possibilities are determined by scalar products. The scalar products follow from the initial state description, the description of the perceived state, and chance. Chance determines our particular perception from among the possibilities. Perceptions such as the track of a planet may be the overwhelmingly likely, or the likelihoods may deviate substantially from any classical descriptions such as for Young's double slit or the energy levels of an atom. Observation results in a random selection of perceived state from within a linear expansion of the temporal propagation of initial state. If $|\underline{\psi}\rangle$ describes an initial state, the description includes our history of perceptions and an interaction of the observer with an observed will result in a linear expansion of the evolved state. The states in the linear expansion are labeled by the possibilities for the updated histories of the observer's

perceptions.

$$|U(t)\underline{\psi}\rangle = \sum_{\theta} |E_\theta U(t)\underline{\psi}\rangle = \sum_{\theta}\sum_{k} \langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle\ |\underline{g}_{k,\theta}\rangle. \tag{205}$$

The initial state $|\underline{\psi}\rangle$ jointly describes the observer and observed, and the final state expands in entangled descriptions for each of the possible perceived results θ. Entanglement was introduced in appendix 8.2.1 and entanglement develops in the time evolution of a state that initially has independent descriptions for an observer and an observed. Consistently with separability of the Hilbert space, the notation in (205) is that this summation is denumerably indexed, our observations have finite resolution. And, the notation applies when the orthogonal subspaces associated with projections E_θ are spanned by a basis of elements $|\underline{g}_{k,\theta}\rangle$with $\|\underline{g}_{k,\theta} = 1$,

$$E_\theta = \sum_{k} |\underline{g}_{k,\theta}\rangle\langle \underline{g}_{k,\theta}|.$$

The $|\underline{g}_{k,\theta}\rangle$ jointly describe states of the observer and observed. The orthogonal projections E_θ are onto subspaces of the Hilbert space with descriptions of observers with a history of perceptions θ. Observers often interpret the observed's state as an eigenfunction of the elevation of a corresponding classical variable. That is, for example, in a well designed measurement, the observer interprets the perception of a state dominantly supported within a small neighborhood of $\mathbf{x}_o$ as a point located at $\mathbf{x}_o$.

We never observe that recorded observations spontaneously change, that further observations or communications change the record of past perceptions or that the current state is inconsistent with past perceptions. This experience indicates that states labeled by distinct perceptions are orthogonal: the Hilbert space decomposes into orthogonal subspaces labeled by the distinct histories of observations. This decomposition provides no likelihood of transitions from our history of perceptions to a distinct history. The observation may be in error or imperfect, that is, the actual state entangled with an observer may not be the perceived state, but distinct histories of perceptions lie in orthogonal subspaces. Then,

$$E_\theta E_\vartheta = 0 \qquad \text{if} \qquad \theta \neq \vartheta,$$

and $\sum_\theta\ E_\theta = \mathbb{I}$ at least within an orthogonal subspace that includes the initial state $|\underline{\psi}\rangle$ and the states $|\underline{g}_{k,\theta}\rangle$ of interest. Whether this summation includes all possible states in the Hilbert space is not of issue. Our concern is the subspace $|E_\theta U(t)\underline{\psi}\rangle$ that includes our particular history of perceptions, "us."

The likelihood of adding the observation described by $\underline{g}_{k,\theta}$ to the history of the observer is given by Born's rule,

$$\text{Likelihood=Trace}(E_\theta \rho) = \sum_{k} |\langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle|^2$$

in the case of a normalized vector state $\rho = |\underline{\psi}\rangle\langle\underline{\psi}|$ with $\|\underline{\psi}\| = 1$ and with E_θ the projection onto the subspace with orthonormalized basis $|\underline{g}_{k,\theta}\rangle$.

The Everett-Wheeler-Graham (EWG) interpretation includes a virtual collapse of the full description of nature to a relative state entangled with an observer's perception. Demonstrated in [11] and illustrated below, conditional predictions based solely on this relative state agree with predictions using the full description of state. This agrees with our experience; we need only know what we have observed and other possibilities are of no consequence to our ability to predict future outcomes. Alternative "worlds" have no reality for us. The alternatives do not affect our predictions although our experience is one from among many possibilities. This development resolves one of the mysteries of quantum mechanics: why do we never perceive a state in a superposition over contradictory states like a superposition of 'live cat' and 'dead cat' in Schrödinger's cat thought experiment [53]? Our observations entangle us with one or the other state. The formalism of quantum mechanics is adequate to describe our experience. There is no need to augment quantum mechanics with state collapse. Early interpretations of quantum mechanics included ad hoc assertions like "collapse of the wave packet" or "hidden variables" in attempts to preserve a classical understanding. Further development has demonstrated that our observations of nature are adequately described by quantum mechanics without such ad hoc augmentations to preserve classical intuition.

The EWG interpretation of quantum mechanics develops the concept of relative (entangled) state. A Hermitian operator A corresponding to an observable has real expectation values. The conditional expectation is the expected value with states limited to a specified subspace of the Hilbert space. A Hermitian rigged Hilbert space operator A is represented[o] as

$$A = \sum_k a_k E_k$$

with orthogonal projections E_k from a resolution of unity and $a_k \in \mathbb{R}$ [21, 41]. Eigenvalues a_k are the observed values that follow for states in the subspaces $E_k\mathbf{H}$. The conditional expectation value of the observable A conditioned upon an observer's perception θ is

$$\begin{aligned} E^\theta[A] &= \frac{E[E_\theta A E_\theta]}{E[E_\theta]} \\ &= \frac{\text{Trace}(E_\theta A E_\theta \rho)}{\text{Trace}(E_\theta \rho)} \end{aligned} \tag{206}$$

with ρ the nonnegative, unit trace, state density operator [62] discussed in appendix 8.2.6. The E_θ are the orthogonal projections (205) that project onto subspaces of states that include the

[o]Appropriate limits of these summations are included using the spectral theorem for self-adjoint operators, theorem 1, appendix to section 4 [21], lemma 5.6.7 [41], chapters 7-10 [24]. The eigenfunctions of common Hermitian operators such as position and momentum lie outside the rigged Hilbert space, the eigenvalues are lie in a continuum.

observer perceptions labeled by history θ entangled with states of the observed. The E_θ project onto composites of observed plus observer states. Two example constructions of composite states are discussed below.

From the idempotence of projections and transposition invariance of the trace (204) [26], Trace(AB)=Trace(BA),

$$\text{Trace}(E_\theta \rho) = \text{Trace}(E_\theta \rho E_\theta)$$

with $E_\theta \rho E_\theta$ the projection of the state density operator into the subspace labeled by θ. The relative state density operator [11], relative to the observer history labeled θ, is

$$\rho^\theta = \frac{E_\theta \rho E_\theta}{E[E_\theta]} \tag{207}$$

normalized to unit trace. The relative state density operators are orthogonal, $\rho^\theta \rho^\vartheta = 0$ if $\theta \neq \vartheta$, and remain orthogonal with time evolution implemented by unitary time translation, $U(t)^* U(t) = 1$. In terms of this relative state density operator, the conditional expectation (206) is

$$E^\theta[A] = \text{Trace}(A\rho^\theta).$$

It was argued on physical grounds that states labeled by different histories of perceptions were orthogonal. As a consequence, there will be a resolution of unity into projection operators E_θ labeled by the perception histories θ.

$$\sum_\theta E_\theta = 1$$

with the θ labeling the possible histories of perceptions, plus one additional orthogonal subspace of all remaining state descriptions. Distinguish this latter projection as E_0. Decomposition of the Hilbert space into orthogonal complements follows from the orthogonality of states with distinct histories, and Riesz's theorem on orthogonal subspaces of a Hilbert space [49].

If the operator A corresponding to an observable quantity is in the commutant of the E_θ from (206), that is, if

$$[A, E_\theta] = 0$$

for each E_θ except possibly for E_0, then a mixture of the relative state density operators ρ^θ is equivalent to the state density operator ρ. The resolution of unity $\sum E_\theta = 1$, the idempotence of projections, commutation of A and the E_θ, transposition invariance of the trace (204) and

the definition of relative state results in the equivalence.

$$\begin{aligned} E[A] &= \mathrm{Trace}(A\rho) \\ &= \sum_\theta \mathrm{Trace}(E_\theta A\rho) \\ &= \sum_\theta \mathrm{Trace}(E_\theta^2 A\rho) \\ &= \sum_\theta \mathrm{Trace}(E_\theta A E_\theta \rho) \\ &= \sum_\theta \mathrm{Trace}(A E_\theta \rho E_\theta) \\ &= \sum_\theta \mathrm{Trace}(A\rho^\theta)\, E[E_\theta] \\ &= \sum_\theta E^\theta[A]\, E[E_\theta]. \end{aligned}$$

Then, using the definition of relative state density operator (207), for all operators A in the commutant of the E_θ, the mixture of relative states,

$$\rho^{eq} = \sum_\theta E[E_\theta]\, \rho^\theta$$

has the same observable properties as the complete state description ρ. Then,

$$\mathrm{Trace}(A\rho) = \mathrm{Trace}(A\rho^{eq}). \tag{208}$$

The relative state density operators ρ^θ provide the correct expectation values conditioned on a history of observer perceptions θ. This equivalence of a mixture of the mutually orthogonal relative state density operators with the complete description of the state ρ explains why the quantum mechanical description of state is not in contradiction with a classical concept for an observer. Each term in the mixture evolves independently. All our future observations evolve either form our current relative state description ρ^θ, or the complete description of state ρ. We need not know or account for the "other branches" to predict the future that is relevant to us, the future conditioned on our history of perceptions. This suggests a "collapse of the wave function" described in early developments of quantum mechanics [62], but there is no discernible physical distinction between a virtual and actual collapse of the state description ρ to ρ^θ upon observation. Likelihood conservation, unitary implementation of time translation, is preserved if collapse is considered virtual. There is no need for an ad hoc collapse. The equivalence remains with forward time translation.

$$U(t)\rho\, U(t)^* = \sum_\theta E[E_\theta]\, U(t)\rho^\theta\, U(t)^*.$$

The first example of observables A and projections E_θ that commute is the tensor product of two Hilbert spaces. The states of the observed are elements of a Hilbert space $\mathbf{H}_1$ and the states of the observer are elements of another Hilbert space $\mathbf{H}_2$. The Hilbert space of interest is the tensor product, $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$. A second example based on the cluster decomposition property of the VEV is developed in appendix 8.2.9. The second example describes a observed that is initially distantly spatial-like separated from the observer and the commutation is with arbitrarily great likelihood. In this second example, the orthogonality is to arbitrary precision but inexact since there are no state descriptions of bounded support within $\mathbf{H}_\mathcal{P}$. In both cases, the observables A commute with the projections E_θ.

A tensor product is the composition of Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From descriptions of the body $\psi, g \in \mathbf{H}_1$ with scalar product $\langle \psi | g \rangle_1$, and states of the observer described $u, v \in \mathbf{H}_2$ with scalar product $\langle u | v \rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ has states labeled $\psi \otimes u, g \otimes v$ with scalar product

$$\langle \psi \otimes u | g \otimes v \rangle = \langle \psi | g \rangle_1 \langle u | v \rangle_2.$$

If ${}^1e_k$ and ${}^2e_k$ are orthonormal bases for separable $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively, then an orthonormal basis for the composite Hilbert space has labels

$$e_{kj} = {}^1e_k \otimes {}^2e_j$$

and $\dim(\mathbf{H}) = \dim(\mathbf{H}_1) \times \dim(\mathbf{H}_2)$. Operators defined in $\mathbf{H}_1 \otimes \mathbf{H}_2$ include extensions of the operators from $\mathbf{H}_1$ and $\mathbf{H}_2$. Hilbert space operators in the tensor product include $C = A \otimes B$,

$$\langle \psi \otimes u | C\ g \otimes v \rangle = \langle \psi | Ag \rangle_1 \langle u | Bv \rangle_2.$$

Then any operators $A \otimes \mathbb{I}$ commute with any $\mathbb{I} \otimes B$. If $\mathbf{H}_1$ includes the descriptions of the observed and $\mathbf{H}_2$ includes the descriptions of the observer, then this example satisfies the assertions above in discussion of the physical equivalence of the mixture of relative states ρ^θ to ρ. For an observation to occur, the time translation $U(t)$ from (205) must couple the constituent Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$. Tensor products demonstrate that the assumed concept of observables for a body in the commutant of the projections onto observer states is realizable. $\mathcal{B}(\mathbf{H}_1) \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{B}(\mathbf{H}_2)$ demonstrate that there are commuting sets of operators of interest. $\mathcal{B}(\mathbf{H})$ is the set of all bounded operators for the Hilbert space $\mathbf{H}$. That the tensor product composition suffices to define a Hilbert space is discussed further in appendix 8.2.10. A simple example illustrating entanglement and the commutation of observables with projections onto the observer states is in appendix 8.18.

The physical equivalence of ρ^{eq} and ρ provides that there is no need for a collapse of state description upon measurement. The relative state interpretation is the natural understanding of quantum mechanics, bizarre images summoned by "many worlds" notwithstanding.

8.2.9 Localized observables, separation and independence

The intuitive notion that separation implies independence provides the second example of Hermitian operators A associated with the observed that (nearly) commute with projections E_θ onto states that include a particular history of observer perceptions. Cluster decomposition (69) provides that scalar products factor when the support of the functions that describe the observed and observer are greatly space-like separated. This is suggestive of the tensor product decomposition of Hilbert spaces.

This intuitive, satisfying property that separation implies independence is enabled by consideration of localized observations. If the observed is greatly spatially isolated from the observer, then these observables (arbitrarily nearly) commute with projections onto states that describe the observer. The approximate independence is that

$$0 \approx \|[E_\theta, A]\| \ll \|A\|$$

with A a Hermitian operator representing the localized observable and for projections $\|E_\theta\| = 1$. This commutation is the property used to develop the equivalence of the mixture of relative states with a general state descriptions in the EWG development of observation discussed in appendix 8.2.8. The commutation is only approximate because the supports of state describing functions are not of bounded support: the functions within $\underline{\mathcal{P}}$ are anti-local. In these instances, the equivalence (208) of the mixture of relative states ρ^θ with ρ is to arbitrarily great likelihood.

In the constructions, the cluster decomposition condition (69) provides that the truncated functions defined by cluster expansion (66) are connected functions,

$$^C\mathcal{W}_{k,n-k} = {}^T\mathcal{W}_{k,n-k}.$$

The vanishing of connected functions $^C\mathcal{W}_{k,n-k}$ as the supports of arguments are greatly space-like separated, and satisfaction of cluster decomposition (69) provides that

$$\mathcal{W}_{n,m}((y)_{n+m}) = \mathcal{W}_{j,\ell}((x)_P)\mathcal{W}_{n-j,m-\ell}((x)_{P'})$$

as $\lambda \to \infty$ if $y_{i_k} = x_k$ for $i_k \in P$ and $y_{i_k} = x_k + (0, \lambda\mathbf{a})$ for $i_k \in P' = \{1, n+m\}/P$, the set complement of the $j+\ell$-element set of integers P from $\{1, n+m\}$, with the evident embellishment of the $(x)_n$ notation.

The demonstration is limited to a selected instance. The observed is an elementary particle described by a one-argument function $\psi(x)$. In the final, post measurement phase, the support of $\psi(x)$ is distantly spatially isolated from the support of the description of the observer. $\mathbf{x}_o$ is the time dependent location of the observed. A finite spherical volume centered on $\mathbf{x}_o$ is designated by $V_{\mathbf{x}_o}$. $V_{\mathbf{x}_o}$ includes all but a negligible amount of the support of $\psi(x)$. Assert that there is a choice of eigenfunctions $\{e_\ell(x)\}$ such that: the support of every $e_\ell(x)$ outside of $V_{\mathbf{x}_o}$ is uniformly negligible; and the Hermitian operator $\hat{A}$ associated with the observable is

$$|\hat{A}\underline{f}\rangle = (0, A_1 f(x_1), \ldots, \sum_{j=1}^{n} A_j f_n((x)_n), \ldots) \tag{209}$$

with

$$A_j f_n((x)_n) = \sum_\ell a_\ell e_\ell(x_j) \int dy_1 dy_2 \; W_2(y_1, y_2) \overline{e_\ell(y_1)} f_n(x_1, \ldots x_{j-1}, y_2, x_{j+1}, \ldots x_n).$$

W_2 is the two-point function, and $j \in \{1, n\}$. The a_ℓ are the eigenvalues of the one-argument subspace observable A associated with the eigenfunctions $e_\ell(x)$.

$$A = \sum_\ell a_\ell |e_\ell\rangle\langle e_\ell|$$

and the state describing function expands in a linear combination of eigenvectors

$$\psi(x) \approx \sum_\ell \langle \underline{e}_\ell | \underline{\psi} \rangle \; e_\ell(x). \tag{210}$$

The functions $e_\ell(x)$ are orthonormal,

$$\langle \underline{e}_\ell | \underline{e}_{\ell'} \rangle = \delta_{\ell,\ell'}. \tag{211}$$

The single argument operator A_j is A applied to the jth argument in the n-argument subspace. A is asserted to be bounded and essentially localized within the volume $V_{\mathbf{x}_o}$. A is designated essentially localized within $V_{\mathbf{x}_o}$ if cluster decomposition (69) implies that

$$\langle \underline{f} | \hat{A} \underline{f} \rangle \to 0 \tag{212}$$

if the dominant support of a localized state $|\underline{f}\rangle$ is arbitrarily greatly space-like separated from $V_{\mathbf{x}_o}$. These properties restrict the class of observables. For example, the linear harmonic oscillator energy eigenfunctions are localized but not uniformly negligible outside of a bounded volume: the greater the energy, the broader the support. The volume $V_{\mathbf{x}_o}$ is selected so that the contribution of any $\psi(x)$ or $e_\ell(x)$ of interest beyond $V_{\mathbf{x}_o}$ is arbitrarily negligible. Examples of localized, one argument observables include location with the $e_\ell(x) \approx \delta(\mathbf{x} - \mathbf{x}_\ell)$ and $x_\ell \in V_{\mathbf{x}_o}$.

The second set of operators of interest are projections E_θ onto states that include particular descriptions of the observer. The initial description of the observer is selected to be (nearly) independent of the description of the observed: initially the observer and observed are unentangled with distantly space-like separated supports. E_θ projects a joint, final, post-observation description of observer and observed (205) onto a description with the observer described by a particular new perception added to the history θ. The final state of the evolved observer state is labeled $\underline{h}_0$ and the final observer states with the evolved perceptions are labeled $\underline{h}_\theta$. These descriptions of state do not include description of the observed. The supports of both the $\underline{h}_0$ and $\underline{h}_\theta$ within $V_{\mathbf{x}_o}$ are arbitrarily negligible. Initial and final are long before and long after the interaction that constitutes an observation. The projections onto the final observer states labeled $\underline{h}_\theta$ are

$$E_\theta = \text{orthogonal projection onto the union of the ranges of } E_{\underline{g}_{\ell\theta}} \tag{213}$$

defined from projections

$$|E_{\underline{g}_{\ell\theta}}\ \underline{f}\rangle = \frac{\langle \underline{g}_{\ell\theta}|\underline{f}\rangle}{\langle \underline{g}_{\ell\theta}|\underline{g}_{\ell\theta}\rangle}\,|\underline{g}_{\ell\theta}\rangle$$

onto the product states defined

$$|\underline{g}_{\ell\theta}\rangle = |\underline{e}_\ell \times \underline{h}_\theta\rangle \tag{214}$$

with $\underline{e}_\ell = (0, e_\ell(x_1), 0, \ldots)$ the eigenfunctions of the observable.

Estimates for the commutator of the observed's essentially localized observable A with the projections E_θ onto particular observer states follow from the initial independence, the final distant spatial separation, and cluster decomposition (69). Demonstrated immediately below, the definitions for the observables (209), (211), (210), and (212), projections (213), and joint observer-observed states (214) in the rigged Hilbert spaces of relativistic quantum physics provide that

$$\begin{aligned} |E_\theta\,\underline{\psi} \times \underline{h}_0\rangle &\approx \langle \underline{h}_\theta|\underline{h}_0\rangle\ |\underline{\psi} \times \underline{h}_\theta\rangle \\ |\hat{A}\,\underline{\psi} \times \underline{h}_0\rangle &\approx |(A\underline{\psi}) \times \underline{h}_0\rangle. \end{aligned} \tag{215}$$

The projection onto particular observer states does not affect the description of the observed. The likelihood of a particular final state of the observed is determined by the likelihood that the evolved $\underline{h}_0$ has a nonzero projection onto $\underline{h}_\theta$. The likelihoods of observed values for the localized observable $\hat{A}$ is determined only by the description of the observed.

The demonstration of (215) follows from the scalar product (13).

$$\begin{aligned} \langle \underline{g}_{\ell\theta}|\underline{\psi} \times \underline{h}_0\rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta|\underline{\psi} \times \underline{h}_0\rangle \\ &\approx \langle \underline{e}_\ell|\underline{\psi}\rangle\ \langle \underline{h}_\theta|\underline{h}_0\rangle \end{aligned}$$

as a consequence of the distant space-like final separation of supports and cluster decomposition (69) of the VEV into connected functions. The separation of the volume $V_{\mathbf{x}_o}$ from the final support of the observer also provides that

$$\begin{aligned} \langle \underline{g}_{\ell\theta}|\underline{g}_{\ell\theta}\rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta|\underline{e}_\ell \times \underline{h}_\theta\rangle \\ &\approx \langle \underline{e}_\ell|\underline{e}_\ell\rangle\langle \underline{h}_\theta|\underline{h}_\theta\rangle \\ &= 1 \end{aligned}$$

with the normalization $\|\underline{h}_\theta\| = 1$. Similarly, $\langle \underline{g}_{\ell\theta}|\underline{g}_{\mu\theta}\rangle \approx 0$ if $\ell \neq \mu$. This approximate orthogonality then provides that

$$E_\theta \approx \sum_\ell E_{\underline{g}_{\ell\theta}} \tag{216}$$

from (213). As a consequence,

$$\begin{aligned}
|E_\theta \underline{\psi} \times \underline{h_0}\rangle &\approx \sum_\ell |E_{\underline{g}_{\ell\theta}} \underline{\psi} \times \underline{h_0}\rangle \\
&= \sum_\ell \frac{\langle \underline{g}_{\ell\theta} | \underline{\psi} \times \underline{h_0}\rangle}{\langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta}\rangle} \, |\underline{g}_{\ell\theta}\rangle \\
&= \sum_\ell \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{\psi} \times \underline{h_0}\rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\
&\approx \sum_\ell \langle \underline{e}_\ell | \underline{\psi}\rangle \, \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\
&= \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\underline{\psi} \times \underline{h}_\theta\rangle
\end{aligned}$$

from substitution of the expansion (210) of the observed state. E_θ approximates a projection operator with the desired property of projecting onto the observer state of interest independently of the descriptions of the observed.

The form (209) of the local observable $\hat{A}$ provides

$$\begin{aligned}
\hat{A}\,\underline{\psi} \times \underline{h_0} &= (0, A\psi(x_1) f_{o,0}, \ldots \sum_{j=1}^{n} A_j \psi(x_1) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\
&\approx (0, A\psi(x_1) f_{o,0}, \ldots (A\psi(x_1)) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\
&= (A\,\underline{\psi}) \times \underline{h_0}
\end{aligned}$$

as a consequence of the distant spatial separations of the support of the $e_\ell(x)$ and $\underline{h_0}$, and cluster decomposition. This completes the demonstration of (215).

From (209), the commutators of the $\hat{A}$ and E_θ follow from

$$\begin{aligned}
|\hat{A} E_\theta \, \underline{\psi} \times \underline{h_0}\rangle &\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\hat{A} \, \underline{\psi} \times \underline{h}_\theta\rangle \\
&\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |(A\underline{\psi}) \times \underline{h}_\theta\rangle
\end{aligned}$$

and

$$\begin{aligned}
|E_\theta \hat{A} \, \underline{\psi} \times \underline{h_0}\rangle &\approx |E_\theta \, (A\underline{\psi}) \times \underline{h_0}\rangle \\
&\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |(A\underline{\psi}) \times \underline{h}_\theta\rangle.
\end{aligned}$$

Then,

$$[E_\theta, \hat{A}] \approx 0$$

for states that describe an observed and observer that are initially unentangled and distantly space-like separated both initially and finally.

This development is not general but illustrates that separation and a lack of entanglement provide independence. This demonstration is an alternative to the tensor product example. This result is enabled within relativistic quantum physics by consideration of observations that are essentially localized, the likely association of locations with the observed, and satisfaction of cluster decomposition (69) in A.6.

8.2.10 Tensor products of linear vector spaces

Discussed in [62] and appendix 8.2.8, a tensor product is the composition of linear vector spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From elements described $g_1, g_2 \in \mathbf{H}_1$ with degenerate scalar product $\langle g_1|g_2\rangle_1$, and elements described $f_1, f_2 \in \mathbf{H}_2$ with degenerate scalar product $\langle f_1|f_2\rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ includes states labeled $g_1 \otimes f_1, g_2 \otimes f_2$ with degenerate scalar product

$$\langle g_1 \otimes f_1|g_2 \otimes f_2\rangle = \langle g_1|g_2\rangle_1 \langle f_1|f_2\rangle_2. \tag{217}$$

In this appendix, the sufficiency of this assignment to specify a degenerate scalar product on the complete tensor product space is addressed. (217) is evidently a degenerate scalar product of product elements $g_1 \otimes f_1, g_2 \otimes f_2$, but does this degenerate scalar product extend to all elements of $\mathbf{H}$?

First it is established that the Cauchy-Schwarz-Bunyakovsky inequality applies for all product states, and then this result is used to demonstrate that the degenerate scalar product applies to linear combinations of product states. Then, if $\mathbf{H}$ is defined as the completion of the linear span of product functions, then the degenerate scalar product applies to $\mathbf{H}$.

The Cauchy-Schwarz-Bunyakovsky inequality applies for all product states.

$$\begin{aligned} |\langle g_1 \otimes f_1|g_2 \otimes f_2\rangle|^2 &= |\langle g_1|g_2\rangle_1|^2 |\langle f_1|f_2\rangle_2|^2 \\ &\leq \langle g_1|g_1\rangle_1 \langle g_2|g_2\rangle_1 \langle f_1|f_1\rangle_2 \langle f_2|f_2\rangle_2 \\ &= \langle g_1 \otimes f_1|g_1 \otimes f_1\rangle \langle g_2 \otimes f_2|g_2 \otimes f_2\rangle \end{aligned}$$

from (217) and the Cauchy-Schwarz-Bunyakovsky inequality in the constituent linear vector

spaces. As a consequence, (217) extends to linear combinations of product functions.

$$\begin{aligned}|\langle u+v|u+v\rangle|^2 &= \langle u|u\rangle + \langle v|v\rangle + \langle u|v\rangle + \langle v|u\rangle \\ &= \langle u|u\rangle + \langle v|v\rangle + 2\Re e\langle u|v\rangle \\ &\geq \langle u|u\rangle + \langle v|v\rangle - 2\sqrt{\langle u|u\rangle\langle v|v\rangle} \\ &= \left(\sqrt{\langle u|u\rangle} - \sqrt{\langle v|v\rangle}\right)^2 \\ &\geq 0\end{aligned}$$

if the Cauchy-Schwarz-Bunyakovsky inequality applies to the elements u, v. The Cauchy-Schwarz-Bunyakovsky applies for product elements $g \otimes f$ and then nonnegativity applies to linear combinations of product elements. If the sum of two product elements has a nonnegative scalar product, then the sum of three also does by a similar argument. Hence, all linear combinations of product elements are nonnegative. The degenerate scalar product extends to an complete tensor product space if the space is defined as follows.

A denumerable basis of elements e_{nm} is defined from products of basis elements for the constituent Hilbert spaces

$$e_{nm} = {}^1e_n \otimes {}^2e_m$$

with ${}^1e_n$ a basis for the separable $\mathbf{H}_1$ and ${}^2e_m$ a basis for the separable $\mathbf{H}_2$. Then (217) provides that

$$\langle e_{nm}|e_{k\ell}\rangle = \delta_{n,k}\delta_{m,\ell}$$

with Kronecker deltas. The e_{nm} are a basis for $\mathbf{H}_1 \otimes \mathbf{H}_2$ defined as the completion $\mathbf{H}$ of all linear combinations of product states $g \otimes f$ in the norm from (217). Nonnegativity extends to all convergent limits.

$$\langle u|u\rangle = \langle \sum_{n,m} a_{nm}e_{nm}| \sum_{k,\ell} a_{k\ell}e_{k\ell}\rangle = \sum_{n,m}\sum_{k,\ell} \overline{a_{nm}}a_{k\ell}\delta_{n.k}\delta_{m,\ell} = \sum_{n,m} |a_{nm}|^2$$

that is manifestly nonnegative.

8.3 Translations, location operators and relativistically invariant localized states

Hermitian Hilbert space operators that correspond to location are discussed in this appendix.

Discussed below, spatial translations of the Fourier transforms $\tilde{f}_n((p)_n)$ of functions are generated by p_ν and momentum translations of $f_n((x)_n)$ are generated by x_ν. Translation invariance of the scalar product and Stone's theorem leads to the conclusion that momenta are densely defined Hermitian operators. However, in relativistic physics, the scalar product is not invariant to translations in momenta. In a non-relativistic development, the $\mathcal{L}^2$ scalar product

is invariant to translations in momenta but in a relativistic development, the universality of the speed of light dictates a Lorentz invariant scalar product. As a consequence, x_ν does not generate a unitarily implemented symmetry in a relativistic development. If multiplication by x_ν were a densely defined Hermitian operator, then translations in momentum would be unitarily implemented and the scalar product would be momentum translation invariant. Multiplication of functions by x_ν can not correspond to the location operator in relativistic physics since it cannot be Hermitian, and as developed in appendix 8.2.6, only Hermitian operators are associated with observables. It is demonstrated in section 7 that the operator X_ν that elevates x_ν is not Hermitian.

From the properties (23) of the Fourier transform, translation of a function corresponds to multiplication of the Fourier transform by e^{-ipa}. Recall that $px = p_0ct - \mathbf{p}\cdot\mathbf{x}$, a Lorentz invariant. For both free fields and the constructions, the Hamiltonian is $H = \hbar c p_0$ [34]. From the scalar product (13), translation of the function to $x-a$ corresponds to translation of a field by $x+a$. Fourier transforms correspond

$$e^{-ipa}\tilde{\psi}(p) \leftrightarrow \psi(x-a)$$

and then for sufficiently small $\|a\|$, Taylor theorem polynomial approximation of the exponential function and $\psi(x)$ provide that

$$(1-ipa)\tilde{\psi}(p) \leftrightarrow \psi(x) - a\cdot\frac{d\psi(x)}{dx}.$$

Then the operator P_ν that corresponds to multiplication of the Fourier transform by p_ν is

$$P_\nu = -i\hbar g_{\nu\nu}\frac{d}{dx_\nu}.$$

The Minkowski signature g is from (24). With

$$U(a)\psi(x) = \psi(x-a),$$

the translation invariance of the scalar product provides that translations $U(a)$ are unitary and the generators,

$$U(a) = \exp(-iap),$$

are the energy-momentum operators $P_\nu = \hbar p_\nu$. From Stone's theorem, the energy-momentum operators are densely defined and Hermitian. P_ν is evidently unbounded. Similarly, multiplication of functions by x_ν generates translations in energy-momenta. Fourier transforms correspond

$$e^{iqx}\psi(x) \leftrightarrow \tilde{\psi}(p-q)$$

and then for sufficiently small $\|q\|$,

$$(1+iqx)\psi(x) \leftrightarrow \tilde{\psi}(p) - q\cdot\frac{d\tilde{\psi}(p)}{dp}.$$

Then the operator X_ν that corresponds to multiplication of a function by x_ν is

$$X_\nu = ig_{\nu\nu}\frac{d}{dp_\nu}.$$

The correspondence of X_ν and P_ν with location and momentum respectively is established in the interpretation of the support of the arguments of functions as spacetime coordinates and of the Fourier transforms as energy-momenta in units of wavenumber. Considering only spatial coordinates, $\nu = 1, 2, 3$,

$$X_\nu = x_\nu, -i\frac{d}{dp_\nu} \qquad \text{and} \qquad P_\nu = i\hbar\frac{d}{dx_\nu}, \hbar p_\nu \tag{218}$$

that apply to functions or the Fourier transforms of functions, respectively. Signs are determined by the convention for the Fourier transform (21). The X_ν and P_ν given by (218) canonically commute. But the properties of X_ν and P_ν as Hilbert space operators depend on the Hilbert space realizations, that is, depend on the scalar product. In the Hilbert space $\mathcal{L}^2$ appropriate for non-relativistic physics [54], the scalar product is both spacetime and energy-momentum translation invariant and X_ν and P_ν are both densely defined Hermitian operators. For a relativistic scalar product, only the P_ν are Hermitian since in relativistic physics, there is only translation invariance of the scalar product. Demonstrated in section 7, multiplication by x_ν is not realized as a Hermitian Hilbert space operator for the relativistically invariant scalar products.

The identification of X_ν as the location operator and P_ν as the momentum operator in non-relativistic physics results in the Born-Heisenberg-Jordan relation for their commutator.

$$[X_\nu, P_\nu] = -i\hbar.$$

This commutation implies the Heisenberg uncertainty relation as discussed in appendix 8.6 and a Baker–Campbell–Hausdorff identity implies that similarity transforms of the location operators translate the eigenvalues.

$$\begin{aligned} e^{-ia_\nu P_\nu/\hbar} X_\nu e^{ia_\nu P_\nu/\hbar} &= X_\nu + \frac{ia_\nu}{\hbar}\,[X_\nu, P_\nu] \\ &= X_\nu + a_\nu. \end{aligned} \tag{219}$$

As a consequence, there is a simultaneous eigenfunction of the X_ν for every $\mathbf{a} \in \mathbb{R}^3$ given a function that is in the union of the null spaces of the X_ν, $\nu = 1, 2, 3$. From $X_\nu\psi_0(\mathbf{x}) = 0$,

$$\psi_{\mathbf{a}}(\mathbf{x}) = e^{ia_1P_1/\hbar}e^{ia_2P_2/\hbar}e^{ia_3P_3/\hbar}\psi_0(\mathbf{x}) \qquad \text{satisfies} \qquad X_\nu\psi_{\mathbf{a}}(\mathbf{x}) = a_\nu\psi_{\mathbf{a}}(\mathbf{x}).$$

This result followed solely from the Born-Heisenberg-Jordan relation and there are many other differential operators $\widehat{X}_\nu$ that also canonically commute with the momentum operators P_ν. These differential operators $\widehat{X}_\nu$ also have eigenfunctions associated with every location in $\mathbb{R}^3$.

From Clairaut's theorem, the operations

$$\widehat{X}_\nu = -iu(p)\frac{d}{dp_\nu}u^{-1}(p) \tag{220}$$

mutually commute as long as $u(p)$ is twice continuously differentiable. The $\widehat{X}_\nu$ canonically commute with the energy-momentum operators,

$$[\widehat{X}_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$$

with $\nu, \mu = 1, 2, 3$. Then, there is a class of differential operators $\widehat{X}_\nu$ parameterized by twice continuously differentiable functions $u(p)$ that satisfy the Born-Heisenberg-Jordan commutation relations $[\widehat{X}_\nu, P_\nu] = -i\hbar$ (224) and that mutually commute, $[P_\nu, P_\mu] = [\widehat{X}_\nu, \widehat{X}_\mu] = 0$. The operations potentially correspond with observables only if they are Hermitian in the scalar product of the Hilbert space realization of interest.

Location operators $\widehat{X}_\nu$ are determined by the Hilbert space realization of quantum mechanics. Three distinct sets of location operators are developed in this appendix: the elevation of x_ν that applies in a canonical quantization of non-relativistic physics (218); an $\widehat{X}_\nu$ for the relativistic free field realized in Fock space; and an $\widehat{X}_\nu$ for the relativistic constructions based upon the basis function spaces $\underline{\mathcal{P}}$ from section 3.4. The eigenfunctions of these operators are labeled by locations $\mathbf{a} \in \mathbb{R}^3$, are orthogonal for distinct locations as a consequence of Hermiticity [3], and are dominantly supported near each location $\mathbf{a}$ although they are not of point support in the relativistic cases. In relativistic physics, the lack of negative energy support and Lorentz covariance of the states implies that Dirac delta functions are not generalized eigenfunctions of a Hermitian operator within the Hilbert space.

An evident correspondence of particles with state descriptions provides that location operators apply for descriptions of free fields [32, 63]. A free field decomposes into canonically commuting particle creation and annihilation operators; every state can be represented as a linear combination of particle creation operators applied to the vacuum state. When there is interaction, as discussed in section 3.1, these location operators apply to states in the one particle subspace and more generally provide a location for indeterminant bodies. If high order connected functions contribute significantly, then locations do not associate with specific species or numbers of particles. The extension of a single particle operator to multiple particle states (1) is a second quantization that follows (209) for the constructions when interaction is lacking. The discussion of section 3.1 illustrates that general state descriptions are not associated with determined numbers nor types of particles. A particle interpretation is a classically inspired perception of the quantum description of state.

In (220), $u(p) = 1$ is the Hermitian location operator X_ν in the Hilbert space $\mathcal{L}^2$ applicable in non-relativistic physics. The eigenfunctions of the X_ν are Dirac delta functions and ideally associated with classical locations. More generally, in non-relativistic instances, $\|\hbar\mathbf{p}\| \ll mc$, $\widehat{X}_\nu \approx X_\nu$ if $u(p) \approx u(mc/\hbar, 0, 0, 0)$.

The Hermitian operator associated with location is in the form (223) derived from the association of classical bodies with states that are dominantly supported within small isolated volumes. From the discussion of Heisenberg's uncertainty principle in appendix 8.6, product states with factors

$$\tilde{f}(p) = \frac{(p_0 + \omega)\tilde{\varphi}(\mathbf{p})}{\sqrt{2\omega}}$$

have scalar products that coincide with the $\mathcal{L}^2$ scalar product of the $\tilde{\varphi}(\mathbf{p})$ in the one-particle subspace or more generally for states that are well-described by classical particles. Using the translates (219), and a selection of Dirac delta sequences

$$\tilde{\psi}_0(\mathbf{p}) = \omega^{1/2} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \omega^{1/2}$$

with $L \to 0$ for $\varphi(\mathbf{x})$ result in the relativistically invariant localized functions of [44]. For the relativistic free field construction based upon the basis function spaces $\underline{\mathcal{P}}$ [32], the Hermitian operator that corresponds to location has the additional constraint that the eigenfunctions must not be supported on negative energy mass shells.

$$\tilde{\psi}_0(\mathbf{p}) = \frac{(p_0 + \omega)}{\sqrt{2\omega}} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \frac{(p_0 + \omega)}{\sqrt{2\omega}}.$$

That is, $\tilde{\psi}_0(\mathbf{p}) \in \underline{\mathcal{P}}$ is required. The translations (219) produce state describing functions labeled by distinct locations $\mathbf{a} \neq \mathbf{a}'$ and these functions become orthogonal as $L \to 0$. There is a (generalized) eigenfunction associated with every location $\mathbf{a} \in \mathbb{R}^3$. In both cases, the functions $\tilde{\psi}_{\mathbf{a}}(\mathbf{p})$ are functions that label states in the constructed Hilbert space $\mathbf{H}_{\mathcal{P}}$ when L is finite and nonnegative.

If $u(p) = \omega^{1/2}$ or $u(p) = (p_0 + \omega)/\omega^{1/2}$ with ω from (8), then the resulting $\widehat{X}_\nu$ is Hermitian in the one particle subspace for the relativistic scalar product. From (32),

$$\begin{aligned}
\langle f_1 | \widehat{X}_\nu g_1 \rangle &= \int \frac{dp}{2\omega}\, \delta(p_0 - \omega) \overline{\tilde{f}_1}(p) \left(-i\omega^{1/2} \frac{d}{dp_\nu}\, \omega^{-1/2} \tilde{g}_1(p) \right) \\
&= -i \int \frac{d\mathbf{p}}{2}\, \omega^{-1/2} \overline{\tilde{f}_1}(\omega, \mathbf{p}) \frac{d}{dp_\nu}\, \omega^{-1/2} \tilde{g}_1(\omega, \mathbf{p}) \\
&= i \int \frac{d\mathbf{p}}{2}\, \left(\frac{d}{dp_\nu}\, \omega^{-1/2} \overline{\tilde{f}_1}(\omega, \mathbf{p}) \right) \omega^{-1/2} \tilde{g}_1(\omega, \mathbf{p}) \\
&= \langle \widehat{X}_\nu f_1 | g_1 \rangle
\end{aligned} \tag{221}$$

from integration by parts.

An alternative method to associate $\widehat{X}_\nu$ with location is to set $\widehat{X}_\nu$ equal to a Hermitian operator derived from X_ν.

$$\widehat{X}_\nu = \frac{1}{2}(X_\nu + X_\nu^*)$$

with the adjoint operators X_ν^* defined for the free field scalar product (221). Then

$$\begin{aligned}\widehat{X}_\nu \tilde{f}_1(p) &= -\frac{i}{2}\left(\frac{d}{dp_\nu} + \omega \frac{d}{dp_\nu}\omega^{-1}\right)\, \tilde{f}_1(p) \\ &= -ir\left(\tilde{f}_1'(p) - \frac{\omega'}{2\omega}\,\tilde{f}_1(p)\right) \\ &= -i\omega^{1/2}\frac{d}{dp_\nu}\,\omega^{-1/2}\tilde{f}_1(p)\end{aligned}$$

from $\langle X_\nu^* f_1 | g_1\rangle = \langle f_1 | X_\nu g_1\rangle$, (221), integration by parts, and with the prime designating differentiation with respect to p_ν. However, this method is not general. In this case of location, the adjoint shares a dense domain with X_ν. The adjoint of the field $\Phi(\underline{f})^*$ from (29) is generally undefined for the constructions and $\widehat{\Phi}(\underline{f}) = \frac{1}{2}(\Phi(\underline{f}) + \Phi(\underline{f})^*)$ is then not Hermitian.

The relativistic location eigenfunctions, the relativistically invariant localized functions, are orthogonal. The two-point function for a free field (32) is expressed using the Fourier transforms,

$$W_2(\psi^* g) = \int \frac{d\mathbf{p}}{2\omega}\; \overline{\tilde{\psi}(\omega, \mathbf{p})}\, \tilde{g}(\omega, \mathbf{p})$$

and then the relativistically invariant localized functions centered on distinct locations $\mathbf{a}$ and $\mathbf{a}'$ are orthogonal.

$$\begin{aligned}W_2(\psi_{\mathbf{a}}^* \psi_{\mathbf{a}'}) &= \tfrac{1}{2}\int d\mathbf{p}\; e^{-i\mathbf{p}\cdot(\mathbf{a}'-\mathbf{a})} \\ &= \tfrac{1}{2}(2\pi)^3\, \delta(\mathbf{a}-\mathbf{a}').\end{aligned}$$

To characterize the spacetime support of $\tilde{\psi}_{\mathbf{a}}(\mathbf{p})$, [44] uses a transform with a Lorentz covariant measure supported solely on the mass shell,

$$\begin{aligned}f_{\mathbf{a}}(x) &= 2\lambda_c^2 \int \frac{dp}{(2\pi)^{3/2}}\, \theta(E)\delta\left(p^2 - \lambda_c^{-2}\right)\, e^{ipx}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) \\ &= \lambda_c^2 \int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\, \frac{1}{\omega}\, e^{i\omega ct - i\mathbf{p}\cdot\mathbf{x}}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}).\end{aligned}$$

In the limit $L \to 0$ and for $t = 0$, $\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) = \sqrt{\lambda_c \omega}\, e^{i\mathbf{p}\cdot\mathbf{a}}$ and

$$f_{\mathbf{a}}(x) = F(r/\lambda_c)$$

with

$$F(r/\lambda_c) = c_F\, (\lambda_c/r)^{5/4} H_{5/4}^{(1)}(ir/\lambda_c), \tag{222}$$

a Hankel function $H^{(1)}_{5/4}(z)$, dimensionless real constant c_F, and $r^2 = (\mathbf{x} - \mathbf{a})^2$ [1, 44]. The dominant support of $F(r/\lambda_c)$ is near $r = 0$ due to divergence as $r^{-5/2}$ at the origin and a rapid decline bounded by $\exp(-r/\lambda_c)$ at large r. λ_c is the reduced Compton wavelength. The relativistically invariant localized functions $f_{\mathbf{a}}(x)$ do not equal zero within any spatial neighborhood.

By the characterizations developed in section 2, states such as relativistically invariant localized states would typically be perceived as localized despite their lack of bounded support. The relativistically invariant localized states are essentially localized within a volume of radius proportional to the Compton wavelength of the body, 3.0×10^{-13} m for an electron.

Hermitian location operators $\widehat{X}_\nu$ are described by spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [41], and chapters 7-10 [24]). Restricted to the one-argument subspace within $\mathbf{H}_\mathcal{P}$, the three Hermitian location operators are

$$\widehat{X}_\nu = \int_{\mathbb{R}^3} d\mathbf{x}\; x_\nu E_x. \tag{223}$$

$E_x \sim |\psi_x\rangle\langle\psi_x|$ projects one-argument functions to the relativistically invariant localized functions located near $\mathbf{x}$ at time $x_0 = ct$. $\nu = 1, 2, 3$, $x = x_0, \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^3$ and x_ν are the three Cartesian components of $\mathbf{x}$. The relativistically invariant localized functions are mutually orthogonal, $\langle\psi_{y_1}|\psi_{y_2}\rangle = 0$ if $\mathbf{y}_1 \neq \mathbf{y}_2$ and $y_{10} = y_{20}$. The relativistically invariant localized functions are generalized eigenfunctions of the $\widehat{X}_\nu$.

$$\widehat{X}_\nu \psi_x(x_1) = x_\nu \psi_x(x_1),$$

and

$$Q_\chi = \int_\chi d\mathbf{x}\; E_x$$

are projection operators that provide a resolution of unity. $d\mathbf{x}$ is Lebesgue measure on $\mathbb{R}^3$, and the χ refer to measurable volumes within $\mathbb{R}^3$. From the orthogonality of the $\psi_x(x_1)$, $E_x E_{x'} = 0$ if $x_0 = x'_0$ and $\mathbf{x} \neq \mathbf{x}'$. The Q_χ are projections (idempotent, self-adjoint operators)[p] in the one particle subspace of $\mathbf{H}_\mathcal{P}$. The functions $\psi_x(x_1)$ are essentially localized but not strictly localized. The ranges of the projections Q_χ are subspaces of positive energy, Poincaré covariant, essentially localized states and orthogonal projection operators are Birkhoff and von Neumann's experimental propositions [6, 62].

[p] $E_x E_{x'} = 0$ if $\mathbf{x} \neq \mathbf{x}'$ and $t = t'$, and then $Q_\chi Q_{\chi'} = 0$ if $\chi \cap \chi' = \emptyset$ for spatial volumes χ, χ' within a constant time plane. The functions $\psi_x(x_1)$ and $\psi_{x'}(x_1)$ are orthogonal at coincident times. However, evaluated at distinct times $t \neq t'$, $E_x E_{x'} \neq 0$. If there were $\mathcal{Q}_\Delta$ such that $\mathcal{Q}_\Delta \mathcal{Q}_{\Delta'} = 0$ for projections onto spacetime volumes $\Delta, \Delta' \subset \mathbb{R}^4$ with $\Delta \cap \Delta' = \emptyset$ and space-like separated, then those $\mathcal{Q}_\Delta = 0$ [68]. There are no such projections. The volumes with $Q_\chi Q_{\chi'} = 0$ are insufficient to conclude from Bogolubov's edge of the wedge theorem that $Q_\chi = 0$.

8.4 Inconsistency of the classical description with nature

Quantum mechanics is a striking change from classical descriptions. Several competing schools of thought persist on whether quantum mechanical descriptions actually depict nature. The difficulty for many is that quantum mechanics forces us to abandon well-developed classical intuition.

The considerations listed below require a general abandonment of classical concepts. The conflict of classical concepts with nature is illustrated by many considerations including:

1. Planck's calculation of the spectrum of black body radiation
2. the heat capacity of solids
3. the photoelectric effect
4. Stokes fluorescence and Compton scattering
5. Gibb's paradox
6. the creation and annihilation of particles
7. interference patterns in a Michelson interferometer at very low "single quantum at a time" light intensity
8. discrete energy bands in the radiation spectra from atoms
9. the Einstein-Podolsky-Rosen paradox for conserved, quantized quantities such as angular momentum, discussed in appendix 8.5.

Taken together, these considerations do not rectify with a classical world view. Each is a motivation for quantum mechanics, and together with principles of simplicity and universality, a quantum mechanical description for nature is indicated.

For consistency with observations, Max Planck's calculation for the spectrum of black body radiation included that the energy in electromagnetic radiation is quantized in discrete particle-like amounts with an energy E proportional to photon frequency ν, $E = h\nu$. h became known as Planck's constant. Planck's revelation and Albert Einstein's insight that the photoelectric effect is also explained by a quantized photon energy produced agreement with observation. The observed photoelectric effect is strong evidence for the interpretation of electromagnetic radiation as photons. In the photoelectric effect, electrons are not emitted from a surface until the light frequency is sufficient, that is, until the light quanta include sufficient energy that the dominant reaction, interaction of an electron with a single photon, results in an electron with sufficient energy to escape the surface. This observation is (nearly) independent of the amplitude of the incident radiation. The exception is nonlinear optics, if the electron absorbs

the energy of multiple photons. These results contradict the classical description of electromagnetism as waves and energy as a freely specified real parameter. If the illumination is a wave, we should anticipate that once the amplitude of the wave was sufficient that electrons would escape the surface. In his identification of the quantized energy $E = h\nu$ of an electromagnetic field, Einstein was also motivated by the observed heat capacity of solids, in particular, contradictions to the equipartition of energy in classical statistical mechanics, and Stokes fluorescence [57], that the frequency of a photon emitted by a body was less than the illuminating frequency. (Stokes fluorescence has since been supplemented with observations of anti-Stokes radiation, when vibrational modes of an emitter contribute to the energy of the re-emitted photon.) Together with Joseph John Thomson's observations of the electron, Robert Milliken and Harvey Fletcher's measurement of a discrete electric charge, and Jean Perrin's observations and Albert Einstein's description of Brownian motion, these findings provided motivation for the description of matter as a composition of atoms.

The indistinguishability of bodies of the same mass, spin, polarization and charges naturally resolves Gibb's paradox. Gibb's paradox arises if entropy is not an extensive quantity. In statistical physics, an extensive quantity is proportional to the amount of substance, that is, for an extensive entropy, twice the volume of gas should have twice the entropy. Were entropy not extensive, then Gibbs paradox is a violation of the second law of thermodynamics, that entropy of isolated systems does not decrease. Other solutions have been suggested to resolve Gibb's paradox, but the indistinguishability of bodies is a natural resolution consistent with the Bose-Einstein or Fermi-Dirac statistics of similarly described bodies in quantum mechanics. Note that this indistinguishability is in contradiction to the classical concept that the trajectories of individual bodies can be distinguished and followed.

The creation and annihilation of particles is another contradiction to the classical concept of continuous trajectories. In these cases, trajectories disappear and trajectories with distinct descriptions appear. Such transformations of identifiable bodies are not described by classical physics. With creation and annihilation, one cannot follow the trajectory of a single identifiable entity, nor generally even determine the number of bodies in a relativistic state description.

A Michelson interferometer consists of two light paths split and later recombined using a half-silvered mirror, two reflecting mirrors, and an optical path matching slab of clear glass. An interference pattern of light and dark intensity rings is visible on detectors at an end of the optical path, for example, on photographic film. This pattern persists even as the intensity of the light is lowered. From the model of light quanta established in the photoelectric effect and Planck's calculation of black body radiation spectra, it follows that the illumination can be reduced to less than a single photon at a time within the interferometer on average. This establishes that each photon interferes with itself, and since the arms of the interferometer are separated and can be of unequal length, that the photon takes both paths. This contradicts a classical description of the photon as a classical particle with a definite trajectory. The interference pattern is consistent with the classical concept of electromagnetism as a wave, but that description is contradicted by the photoelectric effect. The interferometer is a strong

argument for the reality of the quantum description of state. It is difficult to understand how at low intensities the interference pattern can be responsive to changes in arm path length without a physical presence in each of the two separate arms for each photon.

Discrete line spectra observed in emissions of light, for example, from a gas of hydrogen or sodium atoms, contradict that energy is a real number parameter, an initial condition for classical equations of motion. Line spectra contradict that any in a range of real numbers can provide the initial conditions that result in a hydrogen atom. There is no mechanism within classical descriptions that results in the observed atomic spectra (nor the observed heat capacities of solids at low temperatures). One of the predictions of non-relativistic quantum mechanics is line spectra, and observations of the Lamb shift of hydrogen atom energy levels is one of the most precise tests in physics. The Lamb shift is correctly estimated by the Feynman series rules for quantum electrodynamics.

Flaws in classical physics such as a non-causal radiation reaction force in classical electrodynamics [28] can be tolerated since classical descriptions only approximate the more fundamental, and causal, quantum descriptions. Indicated in Feynman's quote in section 5, it could be anticipated that the transition from the well-established and successful practice of classical physics to quantum mechanics would be slow. Even though it was observation that necessitated the development of the new mechanics, adoption of quantum mechanics is inhibited by the well established reliance on classical concepts. Although the classical perspective is contradicted by nature and quantum mechanics is the more comprehensive and unified model, the classical perspective provide an accurate approximation of the quantum description in common experience.

8.5 The Einstein-Podolsky-Rosen paradox

The Einstein-Podolsky-Rosen (EPR) [16] paradox and confirmation of Bell's inequalities [5] illustrate that classical descriptions are inconsistent with nature. The quantum mechanical description of nature has been described as "bizarre" but, this description often reflects that quantum mechanics demands rejection of long established classical concepts. The EPR paradox [16] illustrates that the quantum mechanical description of nature contradicts classical concepts.[q] Einstein, Podolsky and Rosen develop the argument that quantum mechanics must be incomplete because the quantum description conflicts with a classical concept of state. Of course, the alternative conclusion is that the quantum description suffices but a classical description is inconsistent with nature. To develop the EPR paradox, it is observed that spin angular momentum is quantized with the same discrete values on any axis of observation. The paradox arises if a spin zero particle decays into two spin one-half particles that subsequently fly

[q]Causality violation due to an instantaneous collapse was a criticism of quantum mechanics attributed to Einstein at the 5th Solvay conference. The Einstein-Podolsky-Rosen paradox [16] emphasizes the inconsistency of a classical concept for objects, e.g., distinguishable points characterized by determined classical dynamical variables, with the state describing functions of quantum mechanics.

apart. Sufficiently separated, we can determine the spin state of an arbitrarily distant particle by observing the paired particle: the distant spin is the one that paired with the near spin conserves angular momentum. The paradox is that we determine the spin of the non-causally related distant particle differently as determined by our selection of measurement axis. If the distant particle is classically described, then it has a determined spin unaffected by our observation of the nearby particle. The resolution to the conflict is entanglement, a concept in quantum mechanics that is not supported in a classical description. That is, the resolution of the EPR paradox is not a contradiction to quantum mechanics but a contradiction to classical concepts, to the classical description of state. Entanglement is natural in the Hilbert space description of multiple particle states [52]. The paradox is resolved by the realization that a classical description is not consistent with nature. Although Einstein motivated quantum mechanics with studies of the photoelectric effect and the heat capacity of solids [57], he is perhaps most celebrated for his understanding of time and motion described in special relativity and geometrodynamical gravity. Einstein's derivation of gravity relies on classical description of nature. Expressed in the EPR paradox, Einstein did not embrace implications of quantum mechanics. I presume that Einstein was persuaded by his immensely successful insights into time and motion using classical concepts that are in conflict with quantum mechanics. Also, more consistent perspectives on quantum mechanics [11] developed later. The principle of equivalence, that acceleration is equivalent to a gravitational force, relies on classical description of a body. In a quantum mechanical description, the characterization of the velocity of a body deteriorates with enhancements in location accuracy: both location and velocity are never known with a precision that exceeds the Heisenberg uncertainty bound. We cannot arbitrarily precisely associate the acceleration of a body with its locations.[r] It is a great irony that just as problematic aspects of Newtonian mechanics were resolved with dynamical, relativistic time and the equivalence of acceleration with gravitation, atomic physics necessitated the new mechanics that supersedes classical developments.

8.6 Heisenberg uncertainty

One of the great insights in the development of quantum mechanics is that location and momentum are not independently specified descriptions of bodies. Both location and momenta derive from the state describing function $\psi(x)$. The supports of functions $\psi(x)$ are associated with the likelihoods of locations and the supports of the Fourier transforms $\tilde{\psi}(p)$ are associated with the likelihoods of momenta. Likelihoods are provided by Born's rule from the Hilbert space scalar product. Mean values of location and momentum are independently specified but variances are constrained as described by the Heisenberg uncertainty principle. The Heisenberg uncertainty principle follows from the quantum mechanical description of states. If the scalar

[r] Although for typical non-relativistic Hamiltonians, acceleration and location commute, $[X_\nu, [H, P_\nu]] = 0$, a precise estimate for acceleration from a sequence of localizing measurements fails due to the lack of a precise association of trajectory with a quantum description.

product is $\mathcal{L}^2$, appropriate in non-relativistic physics, then the location operator corresponds to the Hermitian operator realized by multiplication of the function $\psi(x)$ by the value of an argument $\mathbf{x}_\nu$ and the momentum operator corresponds to the Hermitian operator that is the similarly elevated multiplication of $\tilde{\psi}(p)$ by $\mathbf{p}_\nu$ in the momentum domain. Using the Fourier transform relations (23) and linearity, in $\mathcal{L}^2$ location corresponds to $-i\hbar d/dp_\nu$ and momentum to $i\hbar d/dx_\nu$. This is discussed further in appendix 8.3. If at a time t the support of $\psi(x)$ is dominantly supported in the neighborhood of a point $\mathbf{y}(t)$, then

$$\langle\psi|X_\nu\psi\rangle = \int d\mathbf{x}\ \mathbf{x}_\nu|\psi(x)|^2 \approx \mathbf{y}_\nu(t)\int d\mathbf{x}\ |\psi(x)|^2 = \mathbf{y}_\nu(t)$$

for a normalized $\psi(x)$. $\mathbf{y}(t)$ is any suitable representative of the neighborhood of support. From Parseval's equality (22), the definition of Fourier transform (21) and implied identities, if at a time t the support of the Fourier transform $\tilde{\psi}(\mathbf{p},t)$ is dominantly supported near a momentum $\mathbf{q}(t) = \hbar\mathbf{p}(t)$, then with $\nu = 1,2,3$,

$$\langle\psi|P_\nu\psi\rangle = \hbar\int d\mathbf{p}\ \mathbf{p}_\nu|\tilde{\psi}(\mathbf{p},t)|^2 = i\hbar\int d\mathbf{x}\ \overline{\psi(\mathbf{x},t)}\ \frac{d\psi(\mathbf{x},t)}{dx_\nu} \approx \mathbf{q}_\nu(t).$$

There is no knowledge of momentum for a location eigenfunction, and no knowledge of location for a momentum eigenfunction. Are there classical particle-like selections for the functions that describe quantum mechanical states? Here, a nearly classical state is one with both location and velocity known precisely. For finite masses, the time derivative of location is velocity $\mathbf{v}$ and in the non-relativistic approximation $\mathbf{p} \approx m\mathbf{v}$. There is no bound on the precision of knowledge of the mass. Are there states that approximate both location and momentum arbitrarily well? The result is that there is an optimal accuracy that can be simultaneously obtained for knowledge of location and momentum, and a choice of state describing functions that provides the optimal simultaneous knowledge. This result is the Heisenberg uncertainty principle. In quantum mechanics, not only is dynamics no longer described by a smooth trajectory specified by an initial location and velocity, but we cannot know both the location and the time derivative of the location of a body sufficiently well to specify a trajectory. That is, our description of state does not support the concept of Newtonian mechanics. The Heisenberg uncertainty principle is a lower bound on the breadth of the supports in location and momentum in a quantum mechanical description. We can approximate the location and time derivative of location for a body, and this approximation becomes better the heavier the body, but the concept of Newtonian mechanics is not supported by the quantum description of state. Also, in quantum mechanics, we cannot necessarily identify a particular body to propagate forward in time. Similarly described bodies are indistinguishable and cannot be labeled to identify a trajectory except while a single body is sufficiently isolated from other bodies to reliably identify the body. While isolated, a dominant volume in the support of a state describing function can be identified and followed with a high likelihood until it approaches other similar bodies. At this point, its identity becomes lost in ambiguity.

The Heisenberg uncertainty principle is derived in ordinary, non-relativistic quantum mechanics. It is a result that follows from the $\mathcal{L}^2$ scalar product applicable in non-relativistic physics. In $\mathcal{L}^2$, three location operators $X_\nu = x_\nu$ and three momentum operators $P_\nu = i\hbar d/dx_\nu$, one for each spatial dimension, are self-adjoint in $\mathcal{L}^2$ and satisfy the Born-Heisenberg-Jordan relation

$$[X_\nu, P_\nu] = -i\hbar, \tag{224}$$

a canonical commutation relation (CCR). For an arbitrary state $|\psi\rangle$ labeled by a function $\psi(x) \in \mathcal{L}^2$ in the intersection of the domains of X_ν and P_ν (with convergent $|\langle\psi|X_\nu\psi\rangle|$ and $|\langle\psi|P_\nu\psi\rangle|$), define operators

$$A = X_\nu - \langle\psi|X_\nu\psi\rangle, \quad \text{and} \qquad B = P_\nu - \langle\psi|P_\nu\psi\rangle.$$

$\nu = 1, 2$ or 3 and the intersection of the domains includes states labeled by the Schwartz tempered functions that are dense in $\mathcal{L}^2$. From the commutation of X_ν and P_ν it follows that $[A, B] = -i\hbar$ and that A, B are mean zero for the state $|\psi\rangle$. From the interpretation of $\langle\psi|A\psi\rangle$ as the mean value of the quantity associated with the operator A for normalized states $\langle\psi|\psi\rangle = 1$, identify variances of location and momenta as

$$\sigma_x^2 = \langle\psi|A^2\psi\rangle, \quad \text{and} \qquad \sigma_p^2 = \langle\psi|B^2\psi\rangle.$$

Self-adjointness of A and B follows from the self-adjointness of X_ν and P_ν on $\mathcal{L}^2$. Self-adjointness and the Cauchy-Schwarz-Bunyakovsky inequality then provide that

$$|\langle\psi|AB\psi\rangle|^2 = |\langle A\psi|B\psi\rangle|^2 \leq \langle A\psi|A\psi\rangle \langle B\psi|B\psi\rangle = \langle\psi|A^2\psi\rangle \langle\psi|B^2\psi\rangle = \sigma_x^2\sigma_p^2.$$

The lower bound, equality, is achieved if $|A\psi\rangle = c|B\psi\rangle$ for a complex constant c. From the definition of commutator, sesquilinearity of scalar products, and the property of scalar products that $\langle u|v\rangle = \overline{\langle v|u\rangle}$, identify that

$$\langle\psi|[A, B]\psi\rangle = \langle\psi|AB\psi\rangle - \langle\psi|BA\psi\rangle = \langle\psi|AB\psi\rangle - \langle AB\psi|\psi\rangle = 2i\Im m(\langle\psi|AB\psi\rangle),$$

twice the imaginary part. The imaginary part of z has a magnitude bounded by the magnitude of z. This bound and the commutation relation for A and B result in that

$$|\langle\psi|[A, B]\psi\rangle|^2 = |i\hbar\langle\psi|\psi\rangle|^2 = \hbar^2 = |2i\Im m(\langle\psi|AB\psi\rangle)|^2 \leq 4|\langle\psi|AB\psi\rangle|^2 \leq 4\sigma_x^2\sigma_p^2.$$

This is Heisenberg's uncertainty principle. For states in $\mathcal{L}^2$,

$$\sigma_x\sigma_p \geq \hbar/2,$$

a lower limit on the geometric mean of the variances of location and momentum. This limits the simultaneous accuracy of the knowledge of location and momentum descriptions in each

dimension ν. The lower bound is achieved if $\langle\psi|AB\psi\rangle$ is imaginary and $|A\psi\rangle = c|B\psi\rangle$ for $c \in \mathbb{C}$. The labels $\psi(x)$ with $x \in \mathbb{R}^3$ for the states that meet both lower bounds are Gaussian functions (105),

$$\psi(x) = \frac{e^{-(x-x_o)^2/(4L^2)+ip_o x/\hbar}}{(2\pi L^2)^{\frac{1}{4}}}.$$

These Gaussian functions satisfy

$$\langle\psi|AB\psi\rangle = i\int_{\mathbb{R}^3} dx\ \frac{(x-x_o)^2}{2L^2}|\psi(x)|^2$$

that is imaginary and

$$A\psi(x) = (x-x_o)\psi(x) = cB\psi(x) = c(-i\hbar\frac{d\psi(x)}{dx} - p_o\psi(x)) = c\ \frac{i\hbar(x-x_o)}{2L^2}\psi(x)$$

for real $x_o = \langle\psi|X_\nu\psi\rangle$ and $p_o = \langle\psi|P_\nu\psi\rangle$, $c = 2L^2/(i\hbar)$ and $\langle\psi|\psi\rangle = 1$. The parameter $L = \sigma_x$ and $\sigma_p = \hbar/(2L)$. This is the function shape in each dimension. The spatial function that labels a minimum uncertainty state is the product of three factors, one for each spatial dimension. These Gaussian functions are denoted minimum packets and are the most classical particle-like state describing functions in the sense that the geometric mean of the uncertainties in simultaneous knowledge of the location and momentum of the state is minimized.

The Heisenberg uncertainty principle applies in relativistic physics with some revision. In a relativistic development, the operators X_ν ($|X_\nu\psi\rangle = |x_\nu\psi\rangle$) are not self-adjoint. As a consequence, the development above of the Heisenberg uncertainty relation does not apply for operator pairs X_ν, P_ν. Nevertheless, the Heisenberg uncertainty principle applies when classical approximations to the relativistic physics apply, and with X_ν replaced by the Hermitian $\widehat{X}_\nu = -i\hbar\omega^{1/2}d/dp_\nu\omega^{-1/2}$, the relativistic, single body location operator [44]. Classical particle approximations apply when states are described by functions with isolated (119) concentrations of support well-represented by a single location (116) and momentum (118). The eigenfunctions of $\widehat{X}_\nu$ are the relativistically invariant localized functions. If the $\widehat{X}_\nu$ is used in the development of the Heisenberg uncertainty relation, then the minimum uncertainty packets are not Gaussian functions. The relativistic minimum uncertainty packets are normalized inverse Fourier transforms of

$$\tilde{\psi}(\mathbf{p}) = \omega^{1/2}e^{-\sigma_o^2(\mathbf{p}-\mathbf{w})^2}e^{i\mathbf{p}\cdot\mathbf{x}_o/\hbar}.$$

Gaussian functions provide an explicit example of the linear expansion of a state description as a linear combination of more localized Gaussian elements in the Hilbert space. The identity

$$e^{-\frac{x^2}{4L_0^2}} = \sqrt{\frac{1}{\epsilon^2(1-\epsilon^2)4L_0^2\pi}}\ \int ds\ e^{-\frac{s^2}{4(1-\epsilon^2)L_0^2}}e^{-\frac{(x-s)^2}{4\epsilon^2L_0^2}}$$

with $0 < \epsilon \ll 1$ is an expansion of the more broadly supported Gaussian function as a linear combination of arbitrarily narrower Gaussian functions centered on s. This expansion is in one dimension. The likelihood of a transition from the normalized state characterized by

$$\psi(x) = N_\psi e^{-\frac{x^2}{4L_0^2}}$$

to the more localized, normalized state characterized by

$$g_s(x) = N_s e^{-\frac{(x-s)^2}{4\epsilon^2 L_0^2}}$$

is

$$|\langle g_s|\psi\rangle|^2 = \frac{2\epsilon}{1+\epsilon^2}\, e^{-\frac{s^2}{4(1+\epsilon^2)L_0^2}}$$

with $\langle\psi|\psi\rangle = \langle g_s|g_s\rangle = 1$. As anticipated, the likelihood of transition to a function centered on s declines rapidly the farther s is from the center of support at $x = 0$. And, the likelihood of transition to a function of width $\epsilon\sigma$ declines slowly with reductions in relative function spread.

8.7 Equivalence of the cluster and operator expansions for free field VEV

The free field VEV function sequences $\underline{\mathcal{F}}$ constructed from the cluster expansion (54) equal the VEV function sequences that follow from the algebraic properties of free field creation and annihilation operators. This equivalence is demonstrated in this appendix. Both constructions are based upon the two-point connected function (42).

The properties of the free field VEV suffice to provide that the quantum field (29) is a densely defined Hermitian Hilbert space operator [9, 23, 24, 54, 58, 63]. The free field VEV generate algebraically from the Hermitian field operator. The free field operator has creation and annihilation components

$$\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa + \Phi_o^-(f)_\kappa$$

with commutation relations

$$[\Phi_o^-(f_1)_{\kappa_1}, \Phi_o^+(f_2)_{\kappa_2}]_\pm = W_2(f_1\, f_2)_{\kappa_1\kappa_2}$$

and that commute otherwise. Notation is that Φ_o^+ is the creation and Φ_o^- the annihilation component. Whether the commutator or anti-commutator is used depends on the values of κ_1, κ_2, whether the statistics type (35) of the field components are boson or fermion. The anti-commutator applies if both κ_1, κ_2 are fermion types. The vacuum state $|\Omega_o\rangle$ is cyclic with

$$\langle \ldots \Omega| \ldots \Phi_o^-(f)_\kappa \Omega_o\rangle = \langle \Omega|\Phi_o^+(f)_\kappa \ldots \Omega_o\rangle = 0.$$

For this algebraic evaluation, each $f(x) \in \mathcal{S}(\mathbb{R}^4)$. From (14), the VEV functions and VEV of the field operators are related

$$(D\cdot)_k \mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n} = \langle \Phi_o(x_k)_{\kappa_k} \dots \Phi_o(x_1)_{\kappa_1} \Omega | \Phi_o(x_{k+1})_{\kappa_{k+1}} \dots \Phi_o(x_n)_{\kappa_n} \Omega \rangle$$

with D from (11), and with the matrix notation (12). The only non-zero VEV functions are

$$\mathcal{F}_{n,n}((x)_{2n})_{(\kappa)_{2n}} = \sum_{pairs} \sigma(S, (\kappa)_{2n}) \prod_{j=1}^{n} W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}. \tag{225}$$

$\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ results from use of the commutation relations of the creation and annihilation field components, and the annihilation of the vacuum. The summation is over all $(2n)!/(2^n n!)$ ways of pairing the indices $i_j, \ell_j \in \{1, 2n\}$ without regard to order and the indices are ordered $i_j < \ell_j$ within each factor $W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}$. The sign of each term $\sigma(S, (\kappa)_{2n}) = \pm 1$ is determined by particle statistics from the types of the indices $(\kappa)_n$. The $\sigma(S, (\kappa)_{2n})$ are positive if all indices $(\kappa)_{2n}$ are boson indices; fermions introduce sign changes from the anti-commutations of $\Phi_o(x_j)^+_{\kappa_j}$ with $\Phi_o(x_j)^-_{\kappa_j}$ that achieve the pairing of indices $S = \{(i_1, \ell_1), \dots (i_n, \ell_n)\}$, section 3.3.2. $\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ with an odd number of arguments n are zero.

$$\begin{aligned} \mathcal{F}_{1,1}(\{1,2\}) &= W_2(\{1,2\}) \\ &= {}^C\mathcal{F}_{1,1}(\{1,2\}) \end{aligned} \tag{226}$$

in agreement with the cluster expansion (54) and in the abbreviated notation (43).

If $|f(x)\rangle \in \mathbf{H}_{\mathcal{P}}$, then $f(x)$ is supported only on positive energies for energy-momenta on the mass shells, and $\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa$ and $\Phi_o(f^*)_\kappa = \Phi_o^-(f^*)_\kappa$. As a consequence, if $g_k, f_{n-k} \in \underline{\mathcal{P}}$, the only contributions from $\mathcal{F}_{k,n-k}(g_k^* f_{n-k})$ to scalar products (13) are

$$\mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_{S} \sigma(S, (\kappa)_{2k}) \prod_{j=1}^{k} W_2(x_j, x_{i_j})_{\kappa_j \kappa_{i_j}}. \tag{227}$$

with the summation $\sum_S$ including only the $k!$ distinct pairings j, i_j with $j \in \{1, k\}$ and $i_j \in \{k+1, 2k\}$. Two-point functions with both function or both $*$-dual function arguments are zero (38) due to the support restriction (7) on functions in $\mathbf{H}_{\mathcal{P}}$. The scalar product (6) and definition (225) provide that $\sigma(S_o, (\kappa)_{2k}) = 1$ for the pairing

$$S_o = \{k, k-1, \dots 1, k+1, k+2, \dots 2k\}.$$

For $\underline{f} \in \underline{\mathcal{P}} \subset \underline{\mathcal{S}}$, only $k!$ of the $(2k)!/(2^k k!)$ terms for $n = 2k$ in (225) contribute to the free field VEV. These properties of free field VEV for the basis space of function sequences $\underline{\mathcal{P}}$ provide that $\underline{\mathcal{F}}$ exhibits split signed symmetry (44).

Equality of the cluster expansion (54) for $\underline{\mathcal{F}}$ and the sequence $\underline{\mathcal{F}}$ constructed algebraically from the commutation relations of the creation and annihilation components of the free field in (227) is demonstrated by induction. Since ${}^{C}\mathcal{F}_{1,1}$ is the only non-zero element of ${}^{C}\underline{\mathcal{F}}$, $\mathcal{F}_{k,k}$ are the only nonzero elements of $\underline{\mathcal{F}}$, and

$$\mathcal{F}_{k,k} = \frac{1}{k!}({}^{C}\underline{\mathcal{F}}\circ)^{k}$$

from (56). The induction follows from a demonstration that the only non-zero element of the sequence on the left-hand side equals

$$\frac{1}{(k+1)!}({}^{C}\underline{\mathcal{F}}\circ)^{k+1} = \sum_{S}\sigma(S,(\kappa)_{2k+2})\prod_{j=1}^{k+1}W_2(\{j,i_j\})$$

given the assumption that

$$\frac{1}{k!}({}^{C}\underline{\mathcal{F}}\circ)^{k} = \sum_{S}\sigma(S,(\kappa)_{2k})\prod_{j=1}^{k}W_2(\{j,i_j\}).$$

(226) proves the assumption for $k=1$.

The assumption results in

$$\begin{aligned}\frac{1}{(k+1)!}({}^{C}\underline{\mathcal{F}}\circ)^{k+1} &= \frac{1}{k+1}\left(\frac{1}{k!}({}^{C}\underline{\mathcal{F}}\circ)^{k}\right)\circ {}^{C}\underline{\mathcal{F}}\\ &= \frac{1}{k+1}\left(\sum_{S}\sigma(S,(\kappa)_{2k})\prod_{j=1}^{k}W_2(\{j,i_j\})\right)\circ {}^{C}\underline{\mathcal{F}}.\end{aligned}$$

The $\circ$-product (62) provides

$$\frac{1}{(k+1)!}({}^{C}\underline{\mathcal{F}}\circ)^{k+1} = \frac{1}{k+1}\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}\sigma_1\sigma_2\left(\sum_{S}\sigma(S,(\kappa)_{A_o})\prod_{j=1}^{k}W_2(\{j,i_j{+}1\})\,W_2(\{k{+}1,2k{+}2\})\right)$$

with the sum over S from (227), $(\kappa)_{A_o}$ designating the set of κ_j with $j\in A_o$, the abbreviated notation $\sigma_\nu=\sigma(\pi_\nu,(\kappa)_{2k+2})$ for $\nu=1,2$, and the sets of indices in the $\circ$-product are

$$\begin{aligned}A_o &= \{1,k\}\cup\{i_1{+}1,i_k{+}1\}\\ A_o' &= \{1,2k{+}2\}\setminus A_o = \{k{+}1\}\cup\{2k{+}2\}.\end{aligned}$$

from (61). The summation $\sum_S$ includes the $k!$ distinct pairings j,i_j with $j\in\{1,k\}$ and $i_j\in\{k+1,2k\}$. The sums over selected permutations $\mathfrak{s}_1,\mathfrak{s}_2$ are from (62) with $n=2k+2$.

Simplification results in

$$\frac{1}{(k+1)!}({}^{C}\underline{\mathcal{F}}\circ)^{k+1} = \frac{1}{k+1}\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}\sum_{S}\sigma_1\sigma_2\,\sigma(S,(\kappa)_{A_o})\prod_{j=1}^{k+1}W_2(\{j,i_j{+}1\})$$

with $i_{k+1} = 2k+1$.

Each term in $\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}\sum_S$ has an ordering of indices that results in one of the $(k+1)!$ unique

$$\prod_{j=1}^{k+1}W_2(\{j,i_j{+}1\}).$$

Signs accumulate with transpositions of adjacent arguments, section 3.3.2, and are uniquely associated with an argument ordering. As a consequence, set

$$\sigma(S',(\kappa)_{2k+2}) = \sigma(\pi_1,(\kappa)_{2k+2})\sigma(\pi_2,(\kappa)_{2k+2})\sigma(S,(\kappa)_{2k})$$

as the sign assigned to each of the $(k+1)!$ distinct pairings j, i_j with $j \in \{1, k+1\}$ and $i_j \in \{k+2, 2k+2\}$. Since the permutations included in $\mathfrak{s}_1, \mathfrak{s}_2$ place indices in natural order, the additional terms included append a $k{+}1$ as the first index to one of the k $i_j + 1$, or append $2k{+}2$ as the second index to a j. There is an additional factor of W_2 with indices that are the j displaced by $k{+}1$ and the i_j displaced by $2k{+}2$. And, the sums over $\mathfrak{s}_1, \mathfrak{s}_2$ include unpermuted terms. As a consequence, there are $(k+1)^2$ times the $k!$ terms in $\sum_S$ contributed by the sums over $\mathfrak{s}_1, \mathfrak{s}_2$. This results in $(k+1)$ of each of the $(k+1)!$ unique terms labeled by a factor

$$\prod_{j=1}^{k+1}W_2(\{j,i_j{+}1\}).$$

Finally,

$$\frac{1}{(k+1)!}({}^{C}\underline{\mathcal{F}}\circ)^{k+1} = \sum_{S'}\sigma(S',(\kappa)_{2k+2})\prod_{j=1}^{k+1}W_2(\{j,i_j\})$$

with the summation over the $(k+1)!$ distinct pairings of arguments. This completes demonstration of the equivalence of (54) and (227) by induction.

It is well-established [9, 58] that the sesquilinear form (13) defines a degenerate scalar product for function sequences from $\underline{\mathcal{S}}$ with free field VEV (225). $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$. The methods of this construction provide a brief demonstration of the nonnegativity of the scalar product (13) for function sequences from $\underline{\mathcal{P}}$ with free field VEV (227). The cluster expansion (54) introduced in section 3.3.4 establishes that

$$\begin{aligned}\underline{\mathcal{F}} &= \exp\circ({}^{C}\underline{\mathcal{F}})\\ &= \mathbb{I} + {}^{C}\underline{\mathcal{F}} + \ldots + \frac{1}{n!}{}^{C}\underline{\mathcal{F}}\circ{}^{C}\underline{\mathcal{F}}\circ\ldots\circ{}^{C}\underline{\mathcal{F}} + \ldots\end{aligned}$$

with the sequence

$$^{C}\underline{\mathcal{F}} = (0, 0, W_2, 0, \ldots).$$

Then, identifying that positively weighted linear combinations of VEV function sequences are nonnegative if the terms are individually nonnegative, and that $\circ$-products preserve the nonnegativity of individually nonnegative, split signed symmetric VEV sequences, section 4.2.2, the nonnegativity of $\underline{\mathcal{F}}$ follows if $\sum_{\kappa_1,\kappa_2} W_2(f^*_{\kappa_1}\, f_{\kappa_2}) \geq 0$. This is manifestly nonnegative,

$$\begin{aligned}\sum_{\kappa_1,\kappa_2} W_2(f^*_{\kappa_1}\, f_{\kappa_2}) &= \sum_{\ell} \int dp\; \delta^+(p) \left| \sum_{\kappa} C_{\ell\kappa} \tilde{f}(p)_\kappa \right|^2 \\ &\geq 0\end{aligned}$$

from the Parseval equality definition of the Fourier transform of generalized functions, the two-point function (32), the $*$-dual sequences (11), the nonnegativity (33) of the matrix DM and simplification.

8.8 Realizations of $Q_{2,2}$

This appendix illustrates the arrays $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ in (50) in section 3.3.3. The arrays determine $n \geq 4$, n-argument connected VEV functions that describe interaction in the constructed realizations of RQP. The illustrated example is $n = 4$, $k = 2$, and any number of constituent fields N_c. The matrices $\mathfrak{h}(p)$ in (51) are therefore $N_c \times N_c$ and two factors of $\mathfrak{h}(p)$ appear in each term of the expansion for $Q_{2,2}((p)_4)_{(\kappa)_4}$. The full set of illustrations assumes that the array $\mathfrak{h}(p)$ includes both boson and fermion components.

Keeping only the detail necessary to identify terms, terms are distinguished by transpositions of the arguments p_j, κ_j of the four fields in a 4-point VEV, $j = 1, 2, 3, 4$. Transposition of arguments corresponds with commutation of the field operators within VEV. Selection of $\kappa_j \in \{1, N_c\}$ determines a type, boson or fermion, and an associated mass m_{κ_j} for each j. Argument type of the jth argument follows from the values of κ_j using (35) in section 3.2. There are four energy-momenta $(p)_4$ and N_c^4 possible selections for $(\kappa)_4$.

The constructions begin with expansions in the constituents $\mathfrak{B}(p)$ and $\Upsilon(p)$ of $\mathfrak{h}(p)$. The summation $a, b \in \mathbb{J}_{2,2}$ in (53) with $a \in \{1, 2\}$ and $b \in \{3, 4\}$results in

$$\begin{aligned}\exp\Big(\sum_{a,b\in\mathbb{J}_{2,2}} \rho_a\rho_b\mathfrak{h}_{\kappa_a\kappa_b}\Big) = \exp(\rho_1\rho_2\mathfrak{B}(p_1+\alpha_o p_2)_{\kappa_1\kappa_2})\; \exp(\rho_3\rho_4\mathfrak{B}(\alpha_o p_3+p_4)_{\kappa_3\kappa_4}) \\ \times \exp(\rho_2\rho_3\Upsilon(-p_2+p_3)_{\kappa_2\kappa_3} + \rho_1\rho_4\Upsilon(-p_1+p_4)_{\kappa_1\kappa_4}).\end{aligned} \tag{228}$$

Notation is from section 3.3.3.

In the abbreviated notation (43), $i_1i_2i_3i_4$ designates one of the permutations π_j of the four sets of field arguments $(p, \kappa)_4$ and $\sigma(i_1i_2i_3i_4) = \sigma(\pi_j, (\kappa)_4)$ designates the associated sign. The

associated sign is identified below (44) in section 3.3.2. The reference order A_o is 2134. Similarly to the abbreviated notation (43), designate

$$\begin{aligned} \mathfrak{B}_i(ab) &= \mathfrak{B}(p_a+\alpha_o p_b)_{\kappa_a\kappa_b} \\ \mathfrak{B}_o(ab) &= \mathfrak{B}(\alpha_o p_a+p_b)_{\kappa_a\kappa_b} \\ \Upsilon(ab) &= \Upsilon(-p_a+p_b)_{\kappa_a\kappa_b}. \end{aligned} \tag{229}$$

In this abbreviated notation and neglecting the factors of $\delta(p_j^2-\lambda_{cj}^2)$, the construction of $Q_{2,2}((p)_4)_{(\kappa)_4}$ follows from the split signed symmetrization (45) of $\mathfrak{q}_{2,2}(1234)$. From (50) and (228),

$$\begin{aligned} \mathfrak{q}_{2,2}(1234) &= \mathfrak{q}_{2,2}((p)_4)_{(\kappa)_4} \\ &= \frac{d}{d\rho_1}\frac{d}{d\rho_2}\frac{d}{d\rho_3}\frac{d}{d\rho_4}\ \exp(\sum_{a,b\in\mathbb{J}_{2,2}}\rho_a\rho_b\mathfrak{h}_{\kappa_a\kappa_b}) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34)+\Upsilon(23)\Upsilon(14) \end{aligned} \tag{230}$$

after evaluation with all $\rho_j=0$. The split signed symmetrization (45) produces

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \sum_{\pi_2}\sigma(\pi_2,(\kappa)_n)\left(\sum_{\pi_1}\sigma(\pi_1,(\kappa)_n)\ \mathfrak{q}_{2,2}(\pi_2(\pi_1(\{1,n\})))\right) \\ &= \sum_{\pi_2}\sigma(\pi_2,(\kappa)_n)\Big(\ \sigma(1234)\mathfrak{q}_{2,2}(\pi_2(1234))+\sigma(2134)\mathfrak{q}_{2,2}(\pi_2(2134))\Big) \\ &= \sigma(1234)\Big(\ \sigma(1234)\mathfrak{q}_{2,2}(1234)+\sigma(2134)\mathfrak{q}_{2,2}(2134)\Big) \\ &\quad +\sigma(1243)\Big(\ \sigma(1234)\mathfrak{q}_{2,2}(1243)+\sigma(2134)\mathfrak{q}_{2,2}(2143)\Big). \end{aligned} \tag{231}$$

Five cases of interest follow from the index types of selections for the $(\kappa)_4$. The five cases have species types: 1,2,3,4=boson; 1,2,3,4=fermion; 1,2=boson and 3,4=fermion; 1,2=fermion and 3,4=boson; and 1,3=boson and 2,4=fermion. Other cases are zero or related by split signed symmetry. These cases associate with boson scattering, fermion scattering, fermion pair production from bosons, fermion pair annihilation to bosons, and fermion-boson scattering, respectively.

Boson scattering: 1,2,3,4=boson. All signs $\sigma(i_1i_2i_3i_4)$ are positive.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} = &\ \mathfrak{q}_{2,2}(1234)+\mathfrak{q}_{2,2}(2134)+\mathfrak{q}_{2,2}(1243)+\mathfrak{q}_{2,2}(2143) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34)+\Upsilon(23)\Upsilon(14)+\mathfrak{B}_i(21)\mathfrak{B}_o(34)+\Upsilon(13)\Upsilon(24) \\ &\quad +\mathfrak{B}_i(12)\mathfrak{B}_o(43)+\Upsilon(24)\Upsilon(13)+\mathfrak{B}_i(21)\mathfrak{B}_o(43)+\Upsilon(14)\Upsilon(23) \\ &= (\mathfrak{B}_i(12)+\mathfrak{B}_i(21))(\mathfrak{B}_o(34)+\mathfrak{B}_o(43))+2\Upsilon(13)\Upsilon(24)+2\Upsilon(23)\Upsilon(14). \end{aligned} \tag{232}$$

Fermion scattering: 1,2,3,4=fermion. Each transposition from the reference order results in a negative sign. $\sigma(1234) = \sigma(2143) = -1$.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= -\mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) + \mathfrak{q}_{2,2}(1243) - \mathfrak{q}_{2,2}(2143) \\ &= -\mathfrak{B}_i(12)\mathfrak{B}_o(34) - \Upsilon(23)\Upsilon(14) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) + \Upsilon(13)\Upsilon(24) \\ &\quad +\mathfrak{B}_i(12)\mathfrak{B}_o(43) + \Upsilon(24)\Upsilon(13) - \mathfrak{B}_i(21)\mathfrak{B}_o(43) - \Upsilon(14)\Upsilon(23) \\ &= (\mathfrak{B}_i(21) - \mathfrak{B}_i(12))(\mathfrak{B}_o(34) - \mathfrak{B}_o(43)) - 2\Upsilon(23)\Upsilon(14) + 2\Upsilon(13)\Upsilon(24). \end{aligned} \tag{233}$$

Fermion pair production from bosons: 1,2=boson and 3,4=fermion. Only $\sigma(2143) = -1$ and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) - \mathfrak{q}_{2,2}(1243) - \mathfrak{q}_{2,2}(2143) \\ &= \mathfrak{B}_i(12)\mathfrak{B}_o(34) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) - \mathfrak{B}_i(12)\mathfrak{B}_o(43) - \mathfrak{B}_i(21)\mathfrak{B}_o(43) \\ &= (\mathfrak{B}_i(21) + \mathfrak{B}_i(12))(\mathfrak{B}_o(34) - \mathfrak{B}_o(43)). \end{aligned} \tag{234}$$

Fermion pair annihilation to bosons: 1,2=fermion and 3,4=boson. Only $\sigma(1234) = -1$ and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= -\mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) - \mathfrak{q}_{2,2}(1243) + \mathfrak{q}_{2,2}(2143) \\ &= -\mathfrak{B}_i(12)\mathfrak{B}_o(34) + \mathfrak{B}_i(21)\mathfrak{B}_o(34) - \mathfrak{B}_i(12)\mathfrak{B}_o(43) + \mathfrak{B}_i(21)\mathfrak{B}_o(43) \\ &= (\mathfrak{B}_i(21) - \mathfrak{B}_i(12))(\mathfrak{B}_o(34) + \mathfrak{B}_o(43)). \end{aligned} \tag{235}$$

Boson-fermion scattering: 1,4=boson and 2,3=fermion. All signs are positive and mixed type $\mathfrak{h}(p)$ are zero.

$$\begin{aligned} Q_{2,2}((p)_4)_{(\kappa)_4} &= \mathfrak{q}_{2,2}(1234) + \mathfrak{q}_{2,2}(2134) + \mathfrak{q}_{2,2}(1243) + \mathfrak{q}_{2,2}(2143) \\ &= \Upsilon(23)\Upsilon(14) + \Upsilon(14)\Upsilon(23) \\ &= 2\Upsilon(23)\Upsilon(14). \end{aligned} \tag{236}$$

The expressions for $Q_{2,2}((p)_4)_{(\kappa)_4}$ each include a factor

$$\prod_{j=1}^{4} \delta(p_j^2 - \lambda_{cj}^{-2})$$

that constrains energy-momentum support to mass shells. These factors are neglected in the abbreviated notation.

With a transposition of indices, $\mathfrak{B}_i(ab) = \mathfrak{B}(p_a + \alpha_o p_b)_{\kappa_a \kappa_b}$ becomes

$$\begin{aligned}\mathfrak{B}(\alpha_o p_a + p_b)^T &= \int d\mu_{\mathfrak{B}}(s)\; e^{is(\alpha_o p_a + p_b)} M(s)^T \\ &\neq \mathfrak{B}(p_a + \alpha_o p_b).\end{aligned}$$

if $\alpha_o \neq 1$ despite the symmetry, $M(s) = M(s)^T$ in the example realization of $M(s)$ in section 3.2.2. If the fermion component of $\mathfrak{B}$ were symmetric with transposition of fermion type arguments, $\mathfrak{B}_\nu(ab) = \mathfrak{B}_\nu(ba)$ for $\nu = i, o$, then fermion pair production and fermion pair annihilation amplitudes would vanish. The fermion component of $\mathfrak{B}$ is symmetric with transposition of fermion type arguments if $\alpha_o = 1$ for the example realization of a fermion in section 3.2.2.

8.9 Feynman-Dyson series and the constructions

First order perturbative series for RQFT scattering amplitudes [23, 54, 63] are discussed in this appendix to contrast Feynman series scattering amplitudes with the explicit scattering amplitudes from the constructions in section 3. In this appendix, the comparison of quantum mechanical constructions with Feynman series is illustrated for the example of one neutral scalar field $\Phi(x)$, $N_c = 1$. Feynman series are also referred to as Feynman-Dyson, or Dyson series. Feynman series apply only to scattering [8], to infinite interval transition likelihoods.

From section 4.5, the constructed scattering amplitudes are infinite interval limits of more general state transition amplitudes. Appropriate states are interpretable as classical bodies described by momenta in plane wave limits of localized states. For states (2) described by product functions of point support at times λ_j and with momentum support centered on $\mathbf{q}_j$,

$$f_n((x)_n) = \prod_{j=1}^{n} \ell(x_j; \lambda_j, \mathbf{q}_j)$$

[34, 36, 38], a transition from m to n freely propagating particles is described by the LSZ (Lehmann-Symanzik-Zimmermann) expressions for scattering amplitudes [9],

$$\begin{aligned}S_{n.m} &= \lim_{\lambda\to\infty} \langle U(\lambda)\ell(\lambda, \mathbf{q}_{m+1}) \dots \ell(\lambda, \mathbf{q}_{m+n}) | U(-\lambda)\ell(-\lambda, \mathbf{q}_1) \dots \ell(-\lambda, \mathbf{q}_m)\rangle \\ &= \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+1})) \dots \Phi(\ell(\lambda, \mathbf{q}_{m+n}))\Omega | U(-\lambda)\Phi(\ell(-\lambda, \mathbf{q}_1)) \dots \Phi(\ell(-\lambda, \mathbf{q}_m))\Omega\rangle\end{aligned} \tag{237}$$

and

$$S_{n.m} = \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+n}))U(\lambda)^{-1} \dots U(\lambda)\Omega | U(-\lambda)\Phi(\ell(-\lambda, \mathbf{q}_1))U(-\lambda)^{-1} \dots U(-\lambda)\Omega\rangle$$

with $U(\lambda)$ the unitary time translation operator, and applying the definition of field (2) with the notation of section 3.1.2. The state describing functions $\ell(x_j; \lambda_j, \mathbf{q}_j)$ are described in (104) in section 4.5.

$$\tilde{\ell}(p_j; \lambda_j, \mathbf{q}_j) = e^{ip_{j0}\lambda_j}(\omega_j + p_{j0})\tilde{f}(\mathbf{p}_j - \mathbf{q}_j)$$

with $\lambda_j \in \mathbb{R}$, $\mathbf{q}_j \in \mathbb{R}^3$ and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. $\ell(x_j; \lambda, \mathbf{q}_j)$ labels states within the completion $\mathbf{H}_{\mathcal{P}}$ of the $\underline{\mathcal{P}}$ described in section 3.4, Ω designates the vacuum and the VEV are constructed in section 3. $\mathcal{S}(\mathbb{R}^3)$ includes functions with Fourier transforms that are delta sequences with supports concentrated near the momenta $\mathbf{q}_j$. A convenient choice for function $\tilde{f}(\mathbf{p})$ is the Gaussian function (105), a point-wise nonnegative delta sequence. The cluster decomposition axiom A.6 provides that the Gaussian functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are described at large times as free particles if the functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are translated to center the spatial support on free particle trajectories with distinct momenta $\mathbf{q}_j$.

The LSZ scattering amplitudes (237) are VEV of products of temporal translations of fields $\Phi(\ell(t, \mathbf{q}))$. From section 4.5 and with $\lambda_j = \lambda$, temporal translations of these fields are independent of time.

$$U(t)\Phi(\ell(t, \mathbf{q}))U(t)^{-1} = \Phi(\ell(0, \mathbf{q}))$$

due to the limitation of the spectral support of the constructed VEV to mass shells and with the selection of compensating phases in $\ell(x_j; \lambda_j, \mathbf{q}_j)$. Then, the scattering amplitudes are the plane wave limits of

$$S_{n.m} = \langle \Phi(\ell(0, \mathbf{q}_{m+n})) \dots \Omega | \Phi(\ell(0, \mathbf{q}_1)) \dots \Omega \rangle.$$

The RQFT [7, 23, 54, 63] rule for calculation of plane wave scattering amplitudes is to evaluate

$$S_{n,m}((p)_{n+m}) = \langle \widetilde{\Phi_o^+}(p_n) \dots \widetilde{\Phi_o^+}(p_1)\Omega_o | U_D(t, -t) \widetilde{\Phi_o^+}(p_{n+1}) \dots \widetilde{\Phi_o^+}(p_{n+m})\Omega_o \rangle \tag{238}$$

for $t = \infty$. These scattering amplitudes are expanded in free field VEV. Free field operators are distinguished in this appendix by the notation $\Phi_o(x)$. The Dyson operator $U_D(t_1, t_2)$ is provided below in (243). Creation components $\Phi_o^+(x)$ of free fields are introduced in section 3.2. The scattering amplitudes $S_{n,m}((p)_{n+m})$ apply the Fourier transform of generalized functions [19],

$$\Phi_o^+(\underline{f}) = \widetilde{\Phi_o^+}(\underline{\tilde{f}}). \tag{239}$$

Introduced in section 3.2, the free field

$$\Phi_o = \Phi_o^+ + \Phi_o^-$$

has a cyclic vacuum state Ω_o and an annihilation component $\Phi_o^- \, \Omega_o = 0$. From section 3.1.3, the operator adjoint of Φ_o^+ is

$$\Phi_o^+(\underline{f})^* = \Phi_o^-(\underline{f}^*) \tag{240}$$

using the dual (11) for functions on the right-hand side and the Hilbert space operator adjoint on the left-hand side. The adjoint is

$$\widetilde{\Phi}_o^+(p)^* = \widetilde{\Phi}_o^-(-p)$$

from the Fourier transform (239) and $*$-dual (11). For each $f_\kappa \in \mathcal{P}(\mathbb{R}^4)$, $\Phi_o(\underline{f}) = \Phi_o^+(\underline{f})$ as a consequence of the support constraint on functions in $\underline{\mathcal{P}}$. From section 3.2, the commutators of free field creation Φ_o^+ and annihilation Φ_o^- components provide that

$$[\Phi_o^-(x), \widetilde{\Phi_o^+}(p)] = \frac{\delta^+(p)e^{ipx}}{(2\pi)^2} \tag{241}$$

using the definition of Fourier transforms for generalized functions (239), the Fourier transform (21), and the notation

$$\delta^\pm(p) = \theta(\pm p_0)\delta(p^2 - \lambda_c^{-2})$$

from (32). The adjoint results in

$$[\widetilde{\Phi_o^-}(-p), \Phi_o^+(x)] = \frac{\delta^+(p)e^{-ipx}}{(2\pi)^2}.$$

The commutation relations and annihilation of the vacuum by Φ_o^- provide the convenient result

$$\langle \Omega_o | (\Phi_o^-(x))^k \prod_{j=1}^{n} \widetilde{\Phi_o^+}(p_j)\, \Omega_o \rangle = \frac{n!\, \delta_{k,n}}{(2\pi)^{2n}} \prod_{j=1}^{n} \delta^+(p_j) e^{ip_j x} \tag{242}$$

demonstrated by induction and with the Kronecker delta, $\delta_{k,n}$.

The Feynman series for scattering amplitudes is a Neumann series solution with the Dyson operator substituted into the scattering amplitudes (238). The Dyson operator is

$$\begin{aligned} U_D(\lambda_1, \lambda_2) &= e^{iH_0\lambda_1} e^{-iH(\lambda_1 - \lambda_2)} e^{-iH_0\lambda_2} \\ &= \mathbb{I} - i\int_{\lambda_2}^{\lambda_1} ds\; H_{int}(s)\, U_D(s, \lambda_2) \end{aligned} \tag{243}$$

evaluated with $\lambda_1 = -\lambda_2 \to \infty$. The resulting Volterra equation of the second kind for $U_D(\lambda_1, \lambda_2)$ is satisfied formally by the Dyson operator. In (238), $\Phi_o(x)$ is a neutral scalar free quantum field and H_0 generates time translations of these free fields. The Hamiltonian H is expressed in free fields.

$$H = H_0 + H_{int}$$

with a conjectured interaction Hamiltonian for a self-interacting, neutral scalar field of

$$H_{int}(x_0) = \sum_{\ell \geq 4} a_\ell \int d\mathbf{x}\; :(\Phi_o(x_0, \mathbf{x}))^\ell: . \tag{244}$$

The summation is over all $\mathbf{x} \in \mathbb{R}^3$. The notation $:(\Phi_o)^\ell:$ designates normal ordering of the Hamiltonian (244). Normal ordering designates that the factors of Φ_o^+ and Φ_o^- in the binomial

expansion of $(\Phi_o)^\ell$ are ordered with every Φ_o^- to the right of any Φ_o^+ [7, 23, 54, 63]. Normal ordering sets $\langle\Omega_o|H_{int}\,\Omega_o\rangle = 0$ but the vacuum $|\Omega_o\rangle$ is not in the null space of the Hamiltonian, $|H_{int}\,\Omega_o\rangle \neq 0$. From the Campbell-Baker-Hausdorff expression for this example of a boson field,

$$\begin{aligned} :\exp(\alpha\Phi_o): &= :\exp(\alpha\Phi_o^- + \alpha\Phi_o^+): \\ &= \exp(\alpha\Phi_o^-)\exp(\alpha\Phi_o^+) \\ &= \exp(\alpha\Phi_o^- + \alpha\Phi_o^+)\exp(-\tfrac{\alpha^2}{2}[\Phi_o^-, \Phi_o^+]). \end{aligned}$$

From section 3.2, the commutator is central, commutes with both Φ_o^+ and Φ_o^-, in the algebra generated by Φ_o^+ and Φ_o^-. Note that $[\Phi_o^-(x), \Phi_o^+(y)]$ diverges for $x = y$ and this is one of many divergences encountered in the canonical formalism. Normal ordering is not a linear operation on the algebra generated from Φ_o^+ and Φ_o^-. A contradiction to a general specification for normal order as a linear operation in the algebra of fields is illustrated by the free field commutation relation

$$[\Phi_o^-(f_1), \Phi_o(f_2)] = W_2(f_1\, f_2).$$

Normal ordering produces

$$:[\Phi_o^-(f_1), \Phi_o^+(f_2)]: = 0 \neq :W_2(f_1\, f_2): = W_2(f_1\, f_2).$$

The choice of Hamiltonian (244) associates RQFT with a classical field model. Normal ordering determines an order for the non-commuting operators.

The first contributing order of the Neumann series for the Dyson operator $U_D(\lambda_1, \lambda_2)$ approximates

$$U_D(\lambda_1, \lambda_2) \approx \mathbb{I} - i\int_{\lambda_2}^{\lambda_1} ds\; H_{int}(s).$$

Then, the first contributing order to the Feynman rules scattering amplitude (238) is

$$\begin{aligned} S_{k,n-k}((p)_n) &= \langle\widetilde{\Phi_o^+}(p_k)\ldots\widetilde{\Phi_o^+}(p_1)\Omega_o|U_D(\infty,-\infty)\widetilde{\Phi_o^+}(p_{k+1})\ldots\widetilde{\Phi_o^+}(p_n)\Omega_o\rangle \\ &\approx \langle\Omega_o|\prod_{j=1}^{k}\widetilde{\Phi_o^-}(-p_j)\;(\mathbb{I} - i\int_{-\infty}^{\infty} ds\; H_{int}(s))\prod_{j=k+1}^{n}\widetilde{\Phi_o^+}(p_j)\Omega_o\rangle \\ &= \langle\Omega_o|\prod_{j=1}^{k}\widetilde{\Phi_o^-}(-p_j)\;(\mathbb{I} - i\sum_\ell a_\ell\int dx\sum_\nu\binom{\ell}{\nu}(\Phi_o^+(x))^\nu(\Phi_o^-(x))^{\ell-\nu})\prod_{j=k+1}^{n}\widetilde{\Phi_o^+}(p_j)\Omega_o\rangle. \end{aligned}$$

The normal ordered binomial expansion and commutation relations (241) provide that the only term that does not include forward scattering contributions has $\ell = n$ and $\nu = k$. Other terms include forward contributions and are discarded in this characterization of the connected

function ${}^{C}\mathcal{W}_{k,n-k}$ associated with a non-forward scattering likelihood. An associated connected VEV derives from the non-forward contribution to

$$\begin{aligned}
-ia_n \binom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\, (\Phi_o^+(x))^k (\Phi_o^-(x))^{n-k} \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j) \Omega_o \rangle \\
&= -ia_n \binom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\, (\Phi_o^+(x))^k \Omega_o \rangle \langle \Omega_o (\Phi_o^-(x))^{n-k} \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j) \Omega_o \rangle \\
&= -i \frac{n!\, a_n}{(2\pi)^{2n-4}}\ \delta(-p_1 \ldots - p_k + p_{k+1} \ldots + p_n) \prod_{\ell=1}^{n} \delta^+(p_\ell) \\
&= -ic_n\ \delta(-p_1 \ldots - p_k + p_{k+1} \ldots + p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_c^{-2}) \\
&= -i\ {}^{C}\widetilde{\mathcal{W}}_{k,n-k}((-p)_k, (p)_{k+1,n}).
\end{aligned}$$

Commutations of $\widetilde{\Phi_o^-}(-p_j)$ with $\widetilde{\Phi_o^+}(p_\ell)$ produce forward contributions and are discarded in the second line. Substitution of (242) and the Fourier expansion of the delta function results in the third line. Equality of the third lines and fourth lines applies for the energy support constrained functions from the completion of basis spaces $\underline{\mathcal{P}}$, and not for all tempered functions $\underline{\mathcal{S}}$. Factors $\theta(p_{j0})$ in the $\delta^+(p_j)$ are redundant with the support constraints for either the appropriate function in $\underline{\mathcal{P}}$ or $*$-dual (11) of $\underline{\mathcal{P}}$. The coefficients a_ℓ in the Hamiltonian (244) and the c_n in the description of the VEV (48) are related

$$c_n = \frac{n!\, a_n}{(2\pi)^{2n-4}}.$$

In the final line, a neutral scalar field connected VEV ${}^{C}\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.3 is identified and substituted. For Hamiltonians (244), the first order contribution to the Feynman series scattering amplitude (238) coincides with the Fourier transform of a connected VEV ${}^{C}\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.3. While the Hamiltonian (102) of the construction is distinct from the conjectured canonical formalism Hamiltonian (244), the weak coupling scattering amplitudes of the two developments nearly coincide: the Feynman series continues with renormalized, higher powers of the interaction Hamiltonian. The phase 'i' is irrelevant except for phase differences between forward and non-forward contributions, and the effects of these phases vanish in the scattering limit. In the constructions, a unity phase is required for nonnegativity of the scalar product A.2.

The first contributing order of the scattering amplitudes (238) coincide up to a phase with the constructed scattering amplitudes (106) for a neutral scalar field. Then, the scattering likelihoods of first order expansion in RQFT and the constructions are the same: the Feynman rules

scattering likelihoods asymptotically coincide at weak coupling with the scattering likelihoods from the fully quantum mechanical constructions. This equality persists in multiple constituent field examples if relativistic corrections to the Feynman series are made in appropriate cases [36].

The RQFT rule (238) follows from a conjectured asymptotic equality of free and interacting fields. If the free field were unitary similar to the interacting field at asymptotic times, $\lambda \to \pm\infty$,

$$e^{-iH\lambda}\Phi(f)e^{iH\lambda} = e^{-iH_0\lambda}\Phi_o(f)e^{iH_0\lambda}$$

with

$$\Omega_o = e^{iH\lambda}e^{-iH_0\lambda}\Omega,$$

then the scattering amplitude (238) would follow. The unitary similarity of interacting and free fields, and translation invariance of vacuums would provide that

$$\begin{aligned}\langle\Phi(f_k)\dots\Phi(f_1)\Omega|\Phi(f_{k+1})\dots\Phi(f_n)\Omega\rangle \\ = \langle\Phi_o(f_k)\dots\Phi_o(f_1)\Omega_o|U_D(\lambda_{out},\lambda_{in})\Phi_o(f_{k+1})\dots\Phi_o(f_n)\Omega_o\rangle.\end{aligned} \tag{245}$$

However, neither precedent is true.

1. If a unitary similarity implied (245), that similarity would contradict the Haag (Haag-Hall-Wightman-Greenberg) theorem [9, 58]. Haag's theorem demonstrates that unitary similarity of free and interacting fields is not possible. Unitary similarity applies to fields Φ and Φ_o that are densely defined operators in a common Hilbert space [24].

2. In the Fock space of the free field, translation invariance of the free field VEV (227) provides that the free field vacuum Ω_o is translation invariant. But, this vacuum is not invariant to $e^{iH\lambda}$, $H_{int}\Omega_o \neq 0$. The all creation operator term contributes. For example, from the commutation relation (241) and VEV (242), the Fourier transform of delta functions, and the annihilation of the vacuum by Φ_o^-, it follows that

$$\begin{aligned}\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|H_{int}\Omega_o\rangle &= a_k\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|\int d\mathbf{x}\ (\Phi_o^+(x))^k\Omega_o\rangle \\ &= \frac{k!\,a_k}{(2\pi)^{2k-3}}\ \delta(\mathbf{p}_1+\dots+\mathbf{p}_k)\prod_{j=1}^{k}e^{i\omega_j x_0}\delta^-(p_j) \\ &\neq 0\end{aligned}$$

   for the Hamiltonians (244).

If (243) were used to extrapolate transition likelihoods from (238) to finite time intervals $\lambda_1-\lambda_2$, these transition likelihoods would not be Lorentz invariant. The scattering likelihoods from

Feynman-Dyson series are not relativistically invariant for finite transition times λ but the $\lambda \to \infty$ limits, the scattering likelihoods from S-matrices, are covariant [8, 63]. Selection of a Hamiltonian $H_0 + H_{int}$ distinct from the generator of time translation in the representation of the Poincaré group precludes relativistic covariance.

Scattering amplitudes are infinite interval limits of relativistically covariant transition amplitudes $\langle \underline{f} | \underline{h} \rangle$ specified by state describing function sequences $\underline{f}, \underline{h}$, elements of the constructed rigged Hilbert spaces $\mathbf{H}_{\mathcal{P}}$. Demonstrated above and in [36], the constructed likelihoods $|\langle \underline{f} | \underline{h} \rangle|^2$ approximate Feynman series scattering likelihoods in scattering instances.

$$|\langle \underline{f} | \underline{h} \rangle|^2 \approx \lim_{\lambda \to \infty} |\langle \underline{f} |_o e^{iH_0\lambda} e^{-i(H_0 + gH_{int})2\lambda} e^{iH_0\lambda} \underline{h} \rangle_o|^2 \tag{246}$$

with: particle-like state describing function "in" and "out" sequences $\underline{f}, \underline{h}$; interaction Hamiltonian H_{int} derived from a classical action and expressed in free field operators; free field Hamiltonian H_0; transition interval λ; coupling constant $g^2 \ll 1$ to approximate the Feynman-Dyson series with a small number of terms; free field, Fock space, scalar product $\langle \cdot |_o \cdot \rangle_o$; and a physically nontrivial scalar product $\langle \cdot | \cdot \rangle$ constructed to approximate H_{int} in the plane wave scattering limit. Particle-like state describing function sequences and cluster decomposition provide that $\langle \underline{f} | \underline{h} \rangle \approx \langle \underline{f} |_o \underline{h} \rangle_o$, section 4.3. State describing functions are classical particle-like if the spatial support of each argument is isolated and well-represented by a single location, the support of the Fourier transform of each argument is well-represented by a single momentum, and supports are not entangled.

8.10 The generalized function $\delta(H_B(\mathbf{q}_S, \mathbf{q}_D))$

A convenient evaluation of the generalized function $\delta(H_B)$ is developed in this appendix. In (193), the sum and difference momentum arguments of H_B designated $\mathbf{q}_S, \mathbf{q}_D$ follow from momenta $\mathbf{p}_1, \mathbf{p}_2$ or $\mathbf{p}'_3, \mathbf{p}'_4$. H_B includes a binding energy parameter E.

The generalized function $\delta(H_B)$ is defined if H_B has zeros with finite derivatives at the zeros [19]. First, zeros of H_B are evaluated. From (181) with (8), (182) and (184) expressed in sum and difference coordinates (193), the function H_B is

$$\begin{aligned} H_B &= \omega_1 + \omega_2 - \omega_b - E \\ &= \sqrt{\lambda_c^{-2} + (\tfrac{1}{2}\mathbf{q}_S + \mathbf{q}_D)^2} + \sqrt{\lambda_c^{-2} + (\tfrac{1}{2}\mathbf{q}_S - \mathbf{q}_D)^2} - \sqrt{\lambda_b^{-2} + \mathbf{q}_S^2} - E \\ &= \sqrt{\lambda_c^{-2} + \tfrac{1}{4}\rho_S^2 + \rho_D^2 + \rho_S\rho_D \cos\phi} + \sqrt{\lambda_c^{-2} + \tfrac{1}{4}\rho_S^2 + \rho_D^2 - \rho_S\rho_D \cos\phi} - \omega_b(\rho_S) - E \end{aligned} \tag{247}$$

with designations

$$\rho_S = \|\mathbf{q}_S\|, \qquad \rho_D = \|\mathbf{q}_D\|, \qquad \mathbf{q}_S \cdot \mathbf{q}_D = \rho_S\rho_D \cos\phi.$$

If $H_B = 0$, then the positive terms must be the same magnitude as the negative terms. Squaring,

$$2\lambda_c^{-2}+\tfrac{1}{2}\rho_S^2+2\rho_D^2+2\sqrt{\left(\lambda_c^{-2}+\tfrac{1}{4}\rho_S^2+\rho_D^2\right)^2-\rho_S^2\rho_D^2\cos^2\phi} \;= \left(\omega_b(\rho_S)+E\right)^2.$$

Reorganization, substitution of (8) and squaring again results in a linear equation for ρ_D^2.

$$\begin{aligned} 4\left(\omega(\tfrac{1}{2}\rho_S)^2+\rho_D^2\right)^2-4\rho_S^2\rho_D^2\cos^2\phi &= \left(\left(\omega_b(\rho_S)+E\right)^2-2\omega(\tfrac{1}{2}\rho_S)^2-2\rho_D^2\right)^2 \\ 4\omega(\tfrac{1}{2}\rho_S)^4+8\rho_D^2\omega(\tfrac{1}{2}\rho_S)^2-4\rho_S^2\rho_D^2\cos^2\phi &= \left(\left(\omega_b(\rho_S)+E\right)^2-2\omega(\tfrac{1}{2}\rho_S)^2\right)^2 \\ &\quad -4\rho_D^2\left(\left(\omega_b(\rho_S)+E\right)^2-2\omega(\tfrac{1}{2}\rho_S)^2\right). \end{aligned}$$

The $4\rho_D^4$ terms add to zero.

Organizing terms, it follows that

$$\rho_D^2\left(4\left(\omega_b(\rho_S)+E\right)^2-4\rho_S^2\cos^2\phi\right) = \left(\left(\omega_b(\rho_S)+E\right)^2-2\omega(\tfrac{1}{2}\rho_S)^2\right)^2-4\omega(\tfrac{1}{2}\rho_S)^4. \tag{248}$$

Substitution of (182) with a trigonometric identity and binomial expansion results in a demonstration of the positive definiteness of the coefficient of ρ_D^2 in (248).

$$\begin{aligned} 4\left(\omega_b(\rho_S)+E\right)^2-4\rho_S^2\cos^2\phi &= 4\lambda_b^{-2}+4\rho_S^2\sin^2\phi+8E\,\omega_b(\rho_S)+4E^2 \\ &> 0. \end{aligned}$$

Binomial expansion of the constant coefficient from (248) results in

$$\left(\left(\omega_b(\rho_S)+E\right)^2-2\omega(\tfrac{1}{2}\rho_S)^2\right)^2-4\omega(\tfrac{1}{2}\rho_S)^4 = \left(\omega_b(\rho_S)+E\right)^2\left(\left(\omega_b(\rho_S)+E\right)^2-4\omega(\tfrac{1}{2}\rho_S)^2\right).$$

The constant coefficient is nonnegative if

$$\begin{aligned} 0 &\le \left(\omega_b(\rho_S)+E\right)^2-4\omega(\tfrac{1}{2}\rho_S)^2 \\ &= \lambda_b^{-2}+\rho_S^2+2E\,\omega_b(\rho_S)+4\lambda_c^{-2}-4\lambda_c\lambda_b^{-1}+\lambda_b^{-2}-4\lambda_c^{-2}-\rho_S^2 \\ &= 2\lambda_b^{-2}+2E\,\omega_b(\rho_S)-4\lambda_c\lambda_b^{-1}. \end{aligned}$$

from substitution of (8), (182) and (184) and simplification. The constant coefficient has a minimum for $\rho_S^2 = 0$.

$$\begin{aligned} 2\lambda_b^{-2}+2E\,\omega_b(\rho_S)-4\lambda_c\lambda_b^{-1} &\ge 2\lambda_b^{-2}+2(2\lambda_c^{-1}-\lambda_b^{-1})\lambda_b^{-1}-4\lambda_c\lambda_b^{-1} \\ &= 0 \end{aligned}$$

from $\omega_b(\rho_S) \geq \lambda_b^{-1}$. The constant coefficient is positive definite for $\rho_S^2 > 0$ and is zero when $\rho_S^2 = 0$.

The nonnegative ρ_D that sets $H_B = 0$ is designated ρ_o. Then,

$$\begin{aligned} \rho_o &= \sqrt{\frac{\beta}{\alpha}} \\ \alpha &= 4\left(\omega_b(\rho_S) + E\right)^2 - 4\rho_S^2 \cos^2\phi \\ \beta &= \left(\omega_b(\rho_S) + E\right)^2 \left(\left(\omega_b(\rho_S) + E\right)^2 - 4\omega(\tfrac{1}{2}\rho_S)^2\right). \end{aligned}$$

β is nonnegative and α is positive definite. Numerical evaluation verifies that the root ρ_o is not introduced with the squares.

From the unique solution ρ_o to (248) for the zero of H_B in the domain $\rho_D \in \{0, \infty\}$,

$$\delta(H_B) = \frac{\delta(\rho_D - \rho_o)}{\left|\frac{\partial H_B}{\partial \rho_D}\right|}.$$

The derivative,

$$\frac{\partial H_B}{\partial \rho_D} = \frac{\rho_D + \frac{1}{2}\rho_S \cos\phi}{\sqrt{\lambda_c^{-2} + \frac{1}{4}\rho_S^2 + \rho_D^2 + \rho_S\rho_D \cos\phi}} + \frac{\rho_D - \frac{1}{2}\rho_S \cos\phi}{\sqrt{\lambda_c^{-2} + \frac{1}{4}\rho_S^2 + \rho_D^2 - \rho_S\rho_D \cos\phi}}. \tag{249}$$

Two terms of opposite sign and equal magnitude add to zero. The squares of the two terms must be equal if a zero occurs.

$$\frac{(\rho_D + \frac{1}{2}\rho_S \cos\phi)^2}{\lambda_c^{-2} + \frac{1}{4}\rho_S^2 + \rho_D^2 + \rho_S\rho_D \cos\phi} = \frac{(\rho_D - \frac{1}{2}\rho_S \cos\phi)^2}{\lambda_c^{-2} + \frac{1}{4}\rho_S^2 + \rho_D^2 - \rho_S\rho_D \cos\phi}.$$

Multiplying by non-zero factors to place the two terms over a common denominator and simplification from terms that add to zero results in a numerator

$$2\rho_D\rho_S \cos\phi\,(\lambda_c^{-2} + \frac{1}{4}\rho_S^2 \sin^2\phi = 0$$

and a finite denominator. For a zero of $\partial H_B/\partial\rho_D$ to occur, either $\rho_D = 0$ or $\rho_S \cos\phi = 0$.

It follows from inspection of the derivative (249) that $\rho_S \cos\phi = 0$ does not imply that $\partial H_B/\partial\rho_D = 0$ if $\rho_D > 0$. $\rho_D = 0$ with $\rho_S = 0$ or $\cos\phi = 0$ is the only zero of $\partial H_B/\partial\rho_D$. For $\rho_D \ll sqrt\lambda_c^{-2} + \frac{1}{4}\rho_S^2$ and $\rho_S \cos\phi = 0$,

$$\frac{\partial H_B}{\partial \rho_D} \approx \frac{2\rho_D}{\sqrt{\lambda_c^{-2} + \frac{1}{4}\rho_S^2}} + \mathcal{O}(\rho_d^3).$$

]

$\partial H_B/\partial\rho_D$ does not diverge for any real ρ_D, ρ_S or $\cos\phi$ and therefore

$$\frac{\rho_D}{\left|\frac{\partial H_B}{\partial \rho_D}\right|} = \frac{2}{\sqrt{\lambda_c^{-2} + \frac{1}{4}\rho_S^2}}$$

at the only zero of $\partial H_B/\partial\rho_D$. From (249), $\partial H_B/\partial\rho_D$ is infinitely differentiable then the quotient rule provides that

$$\frac{\partial^k}{\partial \rho_D^k}\left(\frac{1}{\rho_D^{-1}\left|\frac{\partial H_B}{\partial \rho_D}\right|}\right)$$

does not diverge.

$$\frac{\partial}{\partial \rho_D}\,\frac{u}{v} = \frac{u'v - uv'}{v^2}$$

and the denominator function

$$v = \rho_D^{-1}\left|\frac{\partial H_B}{\partial \rho_D}\right| = \frac{2}{\sqrt{\lambda_c^{-2} + \frac{1}{4}\rho_S^2}} + \mathcal{O}(\rho_d^2) \tag{250}$$

lacks zeros.

8.11 Center of momentum frames and energy-momentum conservation

For any time-like energy-momentum Lorentz four-vector q, with momenta $\mathbf{q}$, there is a Lorentz boost to the rest frame with $\mathbf{q}' = 0$. Time-like is that $q^2 > 0$. The transformation from the original energy-momentum vector to the primed reference frame is

$$q' = \Lambda q$$

with Λ a Lorentz boost. From that any time-like energy momentum vector p with $p^2 = \mu^2 > 0$ is equivalent to $\sqrt{\mu^2 + \mathbf{p}^2}, \mathbf{p}$, it is readily demonstrated that any n time-like energy-momenta p_j define a time-like $q = p_1 + p_2 \ldots p_n$ and hence a center of momentum reference frame with $\mathbf{q}' = \mathbf{p}_1' + \mathbf{p}_2' \ldots \mathbf{p}_n' = 0$.

The velocity

$$\mathbf{v} = \frac{\mathbf{q}}{\mathbf{q}_0} \tag{251}$$

determines the Lorentz boost and satisfies $\mathbf{v}^2 < 1$. With $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$, the proper orthochronous Lorentz transformation Λ that transforms $\mathbf{q}$ to the center of momentum frame is

the boost

$$\begin{bmatrix} \frac{q_0}{\gamma} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1+\frac{\gamma^2}{1+\gamma}v_x^2 & \frac{\gamma^2}{1+\gamma}v_x v_y & \frac{\gamma^2}{1+\gamma}v_x v_z \\ -\gamma v_y & \frac{\gamma^2}{1+\gamma}v_x v_y & 1+\frac{\gamma^2}{1+\gamma}v_y^2 & \frac{\gamma^2}{1+\gamma}v_y v_z \\ -\gamma v_z & \frac{\gamma^2}{1+\gamma}v_x v_z & \frac{\gamma^2}{1+\gamma}v_y v_z & 1+\frac{\gamma^2}{1+\gamma}v_z^2 \end{pmatrix} \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z. \end{bmatrix}$$

The determinant of Λ is $+1$ and the Trace is $2(1+\gamma)$.

The constructed four-argument connected VEV include both energy-momentum conservation and on mass shell m Dirac delta functions. The summations that evaluate scalar products (185) are constrained to surfaces within $(p)_4 \in \mathbb{R}^{16}$ in the support of

$$\delta(p_1 + p_2 - p_3 - p_4) \prod_{j=1}^{4} \delta(p_j^2 - \lambda_c^{-2}).$$

Within this surface, the energy and momenta of four on mass shell m Lorentz vectors p_j add to zero. The joint support of the constructed VEV and the functions in the Hilbert space $\mathbf{H}_{\mathcal{P}}$ provides that each energy for a state describing function is on a positive energy mass shell m, $p_{j0} = \omega(\mathbf{p}_j)$ with the definition (8). A convenient expression for the energy support condition for four on mass shell m energy-momentum vectors is evaluated in this appendix.

For any two on mass shell m energy-momentum vectors $(\omega_1, \mathbf{p}_1), (\omega_2, \mathbf{p}_2)$, a Lorentz transformation Λ follows that translates the momenta $\mathbf{p}_1, \mathbf{p}_2$ to the center of momentum reference frame. With

$$p'_j = \Lambda p_j,$$

in the center of momentum reference frame $\mathbf{p}'_2 = -\mathbf{p}'_1$ and as a consequence, $\omega'_1 = \omega'_2$. Then, energy and momentum are conserved if

$$\begin{aligned} \omega'_3 &= \omega'_1 \\ \mathbf{p}'_4 &= -\mathbf{p}'_3. \end{aligned}$$

Conservation of energy constrains $\rho'_3 = \|\mathbf{p}'_3\|$ to equal $\rho'_1 = \|\widehat{\mathbf{p}}_1\|$ in the center of momentum reference frame of $\mathbf{p}_1, \mathbf{p}_2$.

In the primed, center of momentum reference frame, the energy-momentum conservation

delta function equals

$$\begin{aligned}\delta(p'_1+p'_2-p'_3-p'_4) &= \delta(\mathbf{p}'_3+\mathbf{p}'_4)\,\delta(2\omega'_3-2\omega'_1)\\ &= \delta(\mathbf{p}'_3+\mathbf{p}'_4)\,\frac{\delta(\rho'_3-\rho'_1)}{2\left|\frac{d\omega(\rho'_3)}{d\rho'_3}\right|}\\ &= \delta(\mathbf{p}'_3+\mathbf{p}'_4)\,\frac{\omega(\rho'_3)}{2\rho'_3}\,\delta(\rho'_3-\rho'_1)\end{aligned} \tag{252}$$

with the notation that $\rho'_j = \|\mathbf{p}'_j\|$.

8.12 Similarity transforms and compositions of $M(p), D, S(A)$ triples

A common similarity transform of a triple of arrays $M(p), D, S(A)$ that satisfy (39) also satisfy (39) possibly with exception of the direct sum decomposition of $M(p)$ into boson $B(p)$ and fermion $F(p)$ components. The similarity transform applies a real ($\overline{V} = V$) or pure imaginary ($\overline{V} = -V$) orthogonal matrix ($V^T = V^{-1}$). To preserve the decomposition $M(p) = B(p) \oplus F(p)$ with nonzero $B(p)$ and $F(p)$, the similarity transform is limited to apply individually to $B(p)$ and $F(p)$, $V = V_B \oplus V_F$ with V_B of the same dimension as $B(p)$ followed by a second similarity transformation by a permutation matrix [26]. Then the triple

$$VM(p)V^T, VDV^T, VS(A)V^T \tag{253}$$

satisfies (39) if $M(p), D, S(A)$ satisfy (39). Demonstration reveals the sufficiency of a real or pure imaginary orthogonal transform and $V = V_B \oplus V_F$,

$$\begin{aligned}\overline{VDV^T}VDV^T &= V\overline{D}V^TVDV^T = V\overline{D}DV^T = \mathbb{I}_{N_c}\\ VDV^TVM(p)V^T &= VDM(p)V^T = (VC(p)V^T)^*VC(p)V^T\\ VM(\Lambda^{-1}p)V^T &= VS(A)M(p)S(A)^TV^T = (VS(A)V^T)VM(p)V^T(VS(A)V^T)^T\\ VDV^TVS(A)V^T &= \overline{VS(A)V^T}VDV^T\\ VM(p)V^T &= (V_B \oplus V_F)B(p) \oplus F(p)(V_B^T \oplus V_F^T) = (V_BB(p)V_B^T) \oplus (V_FF(p)V_F^T)\\ (V_BB(p)V_B^T)^T &= V_BB(-p)V_B^T\\ (V_FF(p)V_F^T)^T &= -V_FF(-p)V_F^T.\end{aligned}$$

A common similarity transform also preserves satisfaction of (74).

$$VS(A)V^T = VS(A_1)S(A_2)V^T = VS(A_1)V^TVS(A_2)V^T$$

from the invertability of V. A final, common permutation matrix similarity transform O permutes the indices κ_j but applies individually to the boson and fermion components, and effectively preserves a decomposition of $M(p)$ into boson and fermion constituents.

Given two solutions $M_1(p), D_1, S_1(A)$ and $M_2(p), D_2, S_2(A)$ of dimensions N_1 and N_2 respectively that satisfy the conditions to construct a physically nontrivial realization of relativistic quantum physics (39) and (74), their direct sums

$$M(p) = M_1(p) \oplus M_2(p), D = D_1 \oplus D_2, S(A) = S_1(A) \oplus S_2(A) \tag{254}$$

and Kronecker products

$$M(p) = M_1(p) \otimes M_2(p), D = D_1 \otimes D_2, S(A) = S_1(A) \otimes S_2(A) \tag{255}$$

also satisfy (39). The direct sum results in arrays

$$U \oplus V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

of dimension $N_c = N_1 + N_2$. $U, V = M_j(p), D_j, S_j(A)$ and $j = 1, 2$. The components $B_j(p)$ of dimension N_{B_j} and $F_j(p)$ of dimension N_{F_j} are the boson and fermion components of $M_j(p)$. The Kronecker product is

$$U \otimes V = \begin{pmatrix} U_{11}V & U_{12}V & \cdots & U_{1N_1}V \\ U_{21}V & U_{22}V & \cdots & U_{2N_1}V \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_11}V & U_{N_12}V & \cdots & U_{N_1N_1}V \end{pmatrix}$$

with U_{ij} the elements of the $N_1 \times N_1$ matrix array U and V is the $N_2 \times N_2$ matrix. The result is an $(N_1N_2) \times (N_1N_2)$ array. That both the direct sum and Kronecker product of solutions also satisfy the required conditions (39) and (74) follows from the mixed product properties of direct sums of matrices

$$(U \oplus V)(C \oplus D) = UC \oplus VD$$

and Kronecker products of matrices,

$$(U \otimes V)(C \otimes D) = UC \otimes VD,$$

the linearity of the direct sum

$$(U + C) \oplus (V + D) = U \oplus V + U \oplus D + C \oplus V + C \oplus D$$

and Kronecker product

$$(U + C) \otimes (V + D) = U \otimes V + U \otimes D + C \otimes V + C \otimes D,$$

and the transpose properties

$$U^T \oplus V^T = (U \oplus V)^T$$

and

$$U^T \otimes V^T = (U \otimes V)^T.$$

To satisfy these properties, U and C have the same dimension N_1, and V and D have the same dimension N_2.

The demonstrations are for the Kronecker product but the common properties provide that the demonstrations also apply for the direct sum with the evident modifications.

If $\overline{D}_j D_j = \mathbb{I}_{N_j}$ for $j = 1, 2$, then

$$\begin{aligned} \overline{D}D &= (\overline{D}_1 \otimes \overline{D}_2)(D_1 \otimes D_2) \\ &= (\overline{D}_1 D_1) \otimes (\overline{D}_2 D_2) \\ &= \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2} \\ &= \mathbb{I}_{N_c} \end{aligned}$$

from the mixed product property and the Kronecker product of identity matrices.

If $\overline{D}_j M_j = C_j^* C_j$ for $j = 1, 2$, then

$$\begin{aligned} \overline{D}M &= (\overline{D}_1 \otimes \overline{D}_2)(M_1 \otimes M_2) \\ &= (D_1 M_1) \otimes (D_2 M_2) \\ &= (C_1^* C_1) \otimes (C_2^* C_2) \\ &= (C_1^* \otimes C_2^*)(C_1 \otimes C_2) \\ &= C^* C \end{aligned}$$

with $C = C_1 \otimes C_2$ from the mixed product and transpose properties.

If $S_j M_j S_j^T = M_j'$ with $M_j = M_j(p)$, $M_j' = M_j(\Lambda^{-1} p)$ and $j = 1, 2$, then

$$\begin{aligned} SMS^T &= (S_1 \otimes S_2)(M_1 \otimes M_2)(S_1 \otimes S_2)^T \\ &= ((S_1 M_1) \otimes (S_2 M_2))(S_1^T \otimes S_2^T) \\ &= (S_1 M_1 S_1^T) \otimes (S_2 M_2 S_2^T) \\ &= M_1' \otimes M_2' \\ &= M' \end{aligned}$$

from the mixed product and transpose properties.

If $D_j S_j = \overline{S}_j D_j$ for $j = 1, 2$, then

$$\begin{aligned} DS &= (D_1 \otimes D_2)(S_1 \otimes S_2) \\ &= (D_1 S_1) \otimes (D_2 S_2) \\ &= (\overline{S}_1 D_1) \otimes (\overline{S}_2 D_2) \\ &= (\overline{S}_1 \otimes \overline{S}_2)(D_1 \otimes D_2) \\ &= \overline{(S_1 \otimes S_2)}(D_1 \otimes D_2) \\ &= \overline{S} D \end{aligned}$$

from the mixed product property and complex conjugation.

If both $M_1(p)$ and $M_2(p)$ satisfy (35) and (36), then a permutation matrix O similarity transform produces

$$M(p) = M_1(p) \oplus M_2(p), \quad \text{or} \quad M(p) = M_1(p) \otimes M_2(p)$$

of the form

$$O M(p) O^T = B(p) \oplus F(p).$$

For the block diagonal direct sum it follows that $B(p) = B_1(p) \oplus B_2(p)$ and $F(p) = F_1(p) \oplus F_2(p)$. For the Kronecker product, define $N_j \times N_j$ matrices that fill out the arrays B_j and F_j with zeros such that

$$M_j = \begin{pmatrix} B_j & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_j \end{pmatrix}.$$

Then

$$M = M_1 \otimes M_2$$

equals

$$M_1 \otimes M_2 = \begin{pmatrix} B_1 \otimes B_2 & 0 & 0 & 0 \\ 0 & B_1 \otimes F_2 & 0 & 0 \\ 0 & 0 & F_1 \otimes B_2 & 0 \\ 0 & 0 & 0 & F_1 \otimes F_2 \end{pmatrix}$$

from linearity. $B_1 \otimes B_2$ is a square matrix of dimension $N_{B_1} N_{B_2}$ and $F_1 \otimes F_2$ is a square matrix of dimension $N_{F_1} N_{F_2}$. The transpose property provides that both transform with a positive sign. $B_1 \otimes F_2$ is a square matrix of dimension $N_{B_1} N_{F_2}$ and $F_1 \otimes B_2$ is a square matrix of dimension $N_{B_1} N_{F_2}$. The transpose property provides that both transform with a negative sign. A permutation matrix O places M into the direct sum form

$$O(M_1 \otimes M_2) O^T = \begin{pmatrix} B(p) & 0 \\ 0 & F(p) \end{pmatrix}$$

with

$$B(p) = \begin{pmatrix} B_1 \otimes B_2 & 0 \\ 0 & F_1 \otimes F_2 \end{pmatrix}$$

and

$$F(p) = \begin{pmatrix} B_1 \otimes F_2 & 0 \\ 0 & F_1 \otimes B_2 \end{pmatrix}.$$

The dimension of the direct sum of the four block diagonal matrix components is

$$N_c = N_{B_1} N_{B_2} + N_{B_1} N_{F_2} + N_{F_1} N_{B_2} + N_{F_1} N_{F_2}$$

from $N_c = N_1 N_2$ and $N_j = N_{B_j} + N_{F_j}$.

If $S_j(A) = S_j(A_1) S_j(A_2)$ for $j = 1, 2$, then

$$\begin{aligned} S(A) &= S_1(A) \otimes S_2(A) \\ &= (S_1(A_1) S_1(A_2)) \otimes (S_2(A_1) S_2(A_2)) \\ &= (S_1(A_1) \otimes S_2(A_1))(S_1(A_2) \otimes S_2(A_2)) \\ &= S(A_1) S(A_2) \end{aligned}$$

from the mixed product property.

A triple $M(p), D, S(A)$ is designated as decomposable if there is a real ($V = \overline{V}$) or imaginary ($V = -\overline{V}$) orthogonal matrix V ($V^{-1} = V^T$) such that

$$\begin{aligned} V M(p) V^T &= M_1(p) \oplus M_2(p) \\ V D V^T &= D_1 \oplus D_2 \\ V S(A) V^T &= S_1(A) \oplus S_2(A) \end{aligned} \tag{256}$$

with $N_1 \times N_1$ dimensional triple $M_1(p), D_1, S_1(A)$ and $(N_c - N_1) \times (N_c - N_1)$ dimensional triple $M_2(p), D_2, S_2(A)$ that individually satisfy (39) and (74). To preserve satisfaction of axioms A.1-7, and physically, to preserve superselection, V can not connect boson and fermion indices κ. Then, $OV = V_B \oplus V_F$ with V_B of the same dimension as $B(p)$, and O is the permutation matrix [26] that sets

$$\begin{aligned} V M(p) V^T &= M_1(p) \oplus M_2(p) \\ &= (B_1(p) \oplus F_1(p)) \oplus (B_2(p) \oplus F_2(p)) \\ &= O^T (B_1(p) \oplus B_2(p)) \oplus (F_1(p) \oplus F_2(p)) O. \end{aligned}$$

Then the additional real orthogonal similarity transform O follows V and transforms the decomposable triple $M(p), D, S(A)$ into the required form (35). Permutation matrices are real and orthogonal [26].

A low dimension example of a decomposable (256) Kronecker product of triples is the product of the triples $M(p), D, S(A)$ for two charged scalar boson representations. The Kronecker products are

$$M(p) \otimes M(p) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, $D \otimes D = M(p) \otimes M(p)$ and $S(A) \otimes S(A) = \mathbb{I}_4$. $N_c = 2^2 = 4$. The permutation matrix

$$O = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

produces

$$O(M(p) \otimes M(p))O^T = O \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} O^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the Kronecker product composed triple decomposes as the direct sums

$$\begin{aligned} O(M(p) \otimes M(p))O^T &= M(p) \oplus M(p) \\ O(D \otimes D)O^T &= D \oplus D \\ O(S(A) \otimes S(A))O^T &= S(A) \oplus S(A). \end{aligned}$$

The triple from the Kronecker products of two charged scalar field representations is permutation matrix similar to the direct sum of two, two component scalar field representations. A charge operator is the generator of the symmetry $S_\phi \otimes S_\phi$,

$$Q = -i \, \frac{d}{d\phi}(S_\phi \otimes S_\phi)\Big|_{\phi=0}$$

with S_ϕ the charged scalar boson charge symmetry operator (40).

$$O(S_\phi \otimes S_\phi)O^T = O \begin{pmatrix} e^{2i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i\phi} \end{pmatrix} O^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2i\phi} & 0 \\ 0 & 0 & 0 & e^{-2i\phi} \end{pmatrix}$$

and the charges are $0, 0, 2, -2$ for the four component functions $(O\cdot)_1 f_1(x)_\kappa$, $1 \leq \kappa \leq 4$ in the one-argument subspace, with the shorthand notation (12). The Kronecker product of two charged scalar field representations includes an uncharged particle-antiparticle pair and a charge two particle-antiparticle pair.

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

8.13 Two body classical trajectories

In this appendix, notation is established and results for two body trajectories from classical mechanics are collected.

Non-relativistic two body problems provide explicit examples of classical trajectories $\mathbf{u}_j(\lambda)$. The motion of two classical bodies interacting by a scalar pair potential is executed within a plane. A scalar pair potential depends solely on the body separation and properties of the bodies such as mass or charge. The location of bodies in a plane at each time λ is specified by Cartesian spatial coordinates,

$$\mathbf{u}_j = \begin{pmatrix} r_j \cos\theta_j \\ r_j \sin\theta_j \\ 0 \end{pmatrix} \tag{257}$$

with $r_j = \|\mathbf{u}_j\|$ the Euclidean length. Notation is abbreviated, $\mathbf{u}_j = \mathbf{u}_j(\lambda)$ and similarly for r_j, θ_j.

The Lagrangian for a two body problem in the notation of this note is

$$\mathcal{L} = \frac{1}{2} m_1 c^2 \dot{\mathbf{u}}_1^2 + \frac{1}{2} m_2 c^2 \dot{\mathbf{u}}_2^2 - V(\|\mathbf{u}_1 - \mathbf{u}_2\|)$$

with V the pair potential. The separation of the two bodies is

$$\mathbf{u}_s = \mathbf{u}_1 - \mathbf{u}_2.$$

In a reference frame with the center of mass coincident with the origin of coordinates,

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = 0.$$

The solutions for $\mathbf{u}_i$ given the separation $\mathbf{u}_s$ in the center of mass coordinate frame are

$$\mathbf{u}_1 = \frac{m_2 \mathbf{u}_s}{m_1 + m_2}, \qquad \mathbf{u}_2 = -\frac{m_1 \mathbf{u}_s}{m_1 + m_2}.$$

Substitution results in a single body Lagrangian for $\mathbf{u}_s$,

$$\mathcal{L} = \frac{1}{2} \mu c^2 \dot{\mathbf{u}}_s^2 - V(\|\mathbf{u}_s\|),$$

with μ denoted the reduced mass.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This results in Newton's equation of motion for the separation,

$$\mu c^2 \ddot{\mathbf{u}}_s = -\frac{\partial V}{\partial \mathbf{u}_s} \tag{258}$$

with

$$\frac{\partial V}{\partial \mathbf{u}_s} = \nabla_{\mathbf{u}_s} V$$

a gradient vector. For equal masses in the center of mass reference frame,

$$\mu = \frac{m}{2}, \qquad \mathbf{u}_2 = -\mathbf{u}_1$$

and with a $-g/r$ pair potential, $V = -gmc^2/\|\mathbf{u}_s\|$, substitution of $2\|\mathbf{u}_1\| = \|\mathbf{u}_s\|$ for the separation results in equation of motion

$$\ddot{\mathbf{u}}_1 = -\frac{g}{4}\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|^3}. \tag{259}$$

The interaction is characterized by the length g. For gravity, $g = Gm^2/mc^2$ and for electrostatics $g = Ke^2/mc^2$. For a mass of one a.m.u. and an elementary charge of one, $g = 1.23 \times 10^{-53}$ m for gravity and $g = 1.54 \times 10^{-18}$ m for electrostatics.

With

$$L = \mathbf{u}_1^2 \dot{\theta}_1,$$

the relationships of Cartesian and radial coordinates for two equal mass bodies $m = m_1 = m_2$ in the center of mass reference frame ($\mathbf{u}_2 = -\mathbf{u}_1$) include

$$\begin{aligned} \mathbf{u}_j^2 &= r_j^2 \\ \dot{\mathbf{u}}_j \cdot \mathbf{u}_j &= \dot{r}_j r_j \\ \dot{\mathbf{u}}_j^2 &= \dot{r}_j^2 + \frac{L^2}{r_j^2}. \end{aligned} \tag{260}$$

The trajectory $\mathbf{u}_1$ in the center of mass reference frame for the pair of equal mass bodies interacting via a $-g/r$ pair potential are conveniently parameterized by $\theta = \theta_1$ from (257). A solution to Kepler's problem is

$$r_1 = \frac{L^2/4g}{1 - \epsilon_r \cos\theta}$$

and

$$\lambda(\theta) = \frac{1}{L}\int_{\theta_0}^{\theta} d\phi \; r_1(\phi)^2$$

with $\lambda(\theta_0) = 0$ and

$$\epsilon_r = \sqrt{1 + 32e_1L^2/g^2}.$$

For $e_1 > 0$, the solution is unbound and diverges when $\epsilon_r \cos\theta = 1$. For these unbound trajectories, θ is constrained to the interval $(\beta, 2\pi - \beta)$ with $\beta = \cos^{-1}(\epsilon_r^{-1})$. Bound states have $e_1 \leq 0$ and

$$e_1 \geq -\frac{g^2}{32L^2}.$$

$e_1 = -g^2/32L^2$ are the circular orbits. For the circular orbits of $-g/r$ pair potentials,

$$4L^2 = gr_1.$$

8.14 Non-relativistic energy approximations

In this appendix, non-relativistic approximations to the Hamiltonian (102) are developed.

A non-relativistic approximation of the energies ω_k from (8) derives from the Taylor theorem polynomial approximation for momenta $\mathbf{p}_k$ near $\widehat{\mathbf{p}}_k$. To second order in $\mathbf{p}_k - \widehat{\mathbf{p}}_k$,

$$\omega_k \approx \omega(\widehat{\mathbf{p}}_k) + \frac{\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k)}{\omega(\widehat{\mathbf{p}}_k)} + \frac{(\mathbf{p}_k - \widehat{\mathbf{p}}_k)^2}{2\omega(\widehat{\mathbf{p}}_k)} - \frac{(\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k))^2}{2\omega(\widehat{\mathbf{p}}_k)^3}$$

and the Cauchy-Schwarz-Bunyakovski inequality provides that

$$(\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k))^2 \leq \widehat{\mathbf{p}}_k^2 \, (\mathbf{p}_k - \widehat{\mathbf{p}}_k)^2.$$

For non-relativistic velocities

$$\widehat{\mathbf{p}}_k^2 \ll \omega(\widehat{\mathbf{p}}_k)^2$$

and then

$$\frac{(\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k))^2}{2\omega(\widehat{\mathbf{p}}_k)^3} \ll \frac{(\mathbf{p}_k - \widehat{\mathbf{p}}_k)^2}{2\omega(\widehat{\mathbf{p}}_k)}.$$

This justifies neglect of the last term from the Taylor series in (261) if the support of state describing functions excludes relativistic momenta, if, for example, (156) of section 5.4 applies. The resulting non-relativistic approximation for ω_k is

$$\omega_k \approx \omega(\widehat{\mathbf{p}}_k) + \frac{\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k)}{\omega(\widehat{\mathbf{p}}_k)} + \frac{(\mathbf{p}_k - \widehat{\mathbf{p}}_k)^2}{2\omega(\widehat{\mathbf{p}}_k)}. \tag{261}$$

The approximation applies for non-relativistically supported state describing functions. If $\widehat{\mathbf{p}}_k$ is a representative for the support of a state describing function and $\mathbf{p}$ is any point from the non-relativistic dominant support, then $\|\mathbf{p}_k - \widehat{\mathbf{p}}_k\| \ll \lambda_c^{-1} \leq \omega(\widehat{\mathbf{p}}_k)$. (261) provides a convenient approximation for the Hamiltonian (102) applied in each n-argument subspace,

$$H = \sum_k \omega_k.$$

If $\widehat{\mathbf{p}}_k$ and $\mathbf{p}_k$ are both non-relativistic, $\|\widehat{\mathbf{p}}_k\|, \|\mathbf{p}_k\| \ll \lambda_c^{-1}$, then both correction terms to $\omega(\widehat{\mathbf{p}}_k)$ in (261) are second order in small quantities. Neglect of the term proportional to $(\mathbf{p}_2 - \widehat{\mathbf{p}}_k)^2$ in (157) results in a convenient linear in $\mathbf{p}_2$ approximation for the energy. However, with non-relativistic momenta, this approximation is also second order in small quantities and neglect of the quadratic term is not justified. Neglect of the second correction requires that $(\mathbf{p}_k - \widehat{\mathbf{p}}_k)^2 \ll |\widehat{\mathbf{p}}_k \cdot (\mathbf{p}_k - \widehat{\mathbf{p}}_k)|$ in addition to the non-relativistic momenta.

8.15 Gaussian quadratures

In this section, the non-relativistic, brief interval, limited acceleration approximation of the scalar products $Q(1)$ from (169) is evaluated as elementary functions for trajectories determined by initial conditions $\mathbf{u}(0), \dot{\mathbf{u}}(0)$.

The contribution of the free field VEV $Q_F(1)$ to $Q(1)$ is distinguished from the contribution of the four-argument connected function$Q_C(1)$ in (170). Evaluation of $Q_F(1)$ and $Q_C(1)$ follows from Gaussian quadrature

$$\sqrt{\alpha} \int_{-\infty}^{\infty} ds\, e^{-\alpha s^2 + \beta s} = \sqrt{\pi}\, e^{\beta^2/(4\alpha)} \tag{262}$$

for $\alpha, \beta \in \mathbb{C}$ with $\mathbb{R}e(\alpha) > 0$.

From the expression (170) for the free field VEV contribution $Q_F(1)$ to the scalar product (169), evaluation of the momentum conservation delta function, $e^{a+b} = e^a e^b$, translation of the summation variables, and substitution of the Gaussian summation (262) result in

$$\begin{aligned} Q_F(1) &= \frac{4A_g}{\lambda_c^2} \int d\mathbf{q}_2\, e^{-2L(0)^2(\mathbf{q}_2 - \mathbf{w})^2} \\ &= \frac{4A_g}{\lambda_c^2} \int d\mathbf{q}_2\, e^{-2L(0)^2 \mathbf{q}_2^2} \\ &= \frac{4A_g}{\lambda_c^2} \left(\frac{\pi}{2L(0)^2} \right)^{\frac{3}{2}} \end{aligned} \tag{263}$$

for the free field VEV contribution to $Q(1)$.

$$L(0)^2 \in \mathbb{R}$$

is selected to set the initial state describing function as a most classical-like state description. From (263), $Q_F(1) > 0$.

The contribution of the four-argument connected VEV $Q_C(1)$ to the scalar product (169),

$$Q_C(1) = \frac{c_4 A_g}{\lambda_c} \int d\mathbf{q}_2 d\mathbf{q}_4\, \delta(\mathbf{q}_2^2 - \mathbf{q}_4^2)\, e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}} e^{-L(0)^2(\mathbf{q}_2 - \mathbf{w})^2}\, e^{-L(0)^2(\mathbf{q}_4 - \mathbf{w})^2},$$

is approximated. If the separation of the Gaussian envelope from the origin is very small compared to its width,

$$L(0)^2\mathbf{w}^2 \ll 1, \tag{264}$$

then

$$\begin{aligned} L(0)^2(\mathbf{q}_2-\mathbf{w})^2+L(0)^2(\mathbf{q}_4-\mathbf{w})^2 &= L(0)^2(\mathbf{q}_2^2+\mathbf{q}_4^2) - 2L(0)^2(\mathbf{q}_2+\mathbf{q}_4)\cdot\mathbf{w} + 2L(0)^2\mathbf{w}_2^2 \\ &\approx L(0)^2(\mathbf{q}_2^2+\mathbf{q}_4^2) \end{aligned}$$

follows. Within the dominant contributions to the summation that evaluates $Q_C(1)$, both $L(0)^2\mathbf{q}_4^2$ and $L(0)^2\mathbf{q}_2^2$ are $\mathcal{O}(1) \gg L(0)^2\mathbf{w}^2$. The Cauchy-Schwarz-Bunyakovski inequality provides that

$$(L(0)^2(\mathbf{q}_2+\mathbf{q}_4)\cdot\mathbf{w})^2 \ll L(0)^2(\mathbf{q}_2+\mathbf{q}_4)^2\ L(0)^2\mathbf{w}^2.$$

With the approximation,

$$Q_C(1) \approx \frac{c_4 A_g}{\lambda_c}\int d\mathbf{q}_2 d\mathbf{q}_4\ \delta(\mathbf{q}_2^2-\mathbf{q}_4^2)\ e^{-2i(\mathbf{q}_2-\mathbf{q}_4)\cdot\mathbf{u}} e^{-L(0)^2(\mathbf{q}_2^2+\mathbf{q}_4^2)}.$$

This is designated the limited momentum, $|L(0)^2|\mathbf{w}^2 \ll 1$, approximation, and is in addition to the non-relativistic, brief interval, and limited acceleration approximations.

Expressing the summation for $Q_C(1)$ in spherical coordinates is convenient to evaluate the energy conservation delta function. With

$$\mathbf{q}_j = (\rho_j \cos\theta_j \sin\phi_j, \rho_j \sin\theta_j \sin\phi_j, \rho_j \cos\phi_j)$$

and the z_j-axes selected to align with the real vector $\mathbf{u}$. The dot products are

$$\begin{aligned} \mathbf{q}_2\cdot\mathbf{u} &= \rho_2 r\cos\phi_2 \\ \mathbf{q}_4\cdot\mathbf{u} &= \rho_4 r\cos\phi_4. \end{aligned}$$

The real vector magnitudes are designated

$$r = \|\mathbf{u}\| \tag{265}$$

and $\rho_j = \|\mathbf{q}_j\|$. The rotational invariance of the Gaussian state describing function (155) justifies the selection of axes.

With the change to spherical coordinates,

$$\begin{aligned} Q_C(1) &= \frac{c_4 A_g}{\lambda_c}\int_0^\infty \rho_2^2 d\rho_2 \int_0^\infty \rho_4^2 d\rho_4\ \delta(\rho_2^2-\rho_4^2)\ e^{-L(0)^2(\mathbf{q}_2^2+\mathbf{q}_4^2)} \\ &\quad\times \int_0^{2\pi} d\theta_2 \int_0^\pi \sin\phi_2 d\phi_2\ e^{-2i\rho_2 r\cos\phi_2} \int_0^{2\pi} d\theta_4 \int_0^\pi \sin\phi_4 d\phi_4\ e^{2i\rho_4 r\cos\phi_4}. \end{aligned}$$

The θ_j and ϕ_j summations are elementary.

$$\begin{aligned}\int_0^{2\pi} d\theta_2 \int_0^{\pi} \sin\phi_2\, d\phi_2\; e^{-2i\rho_2 r\cos\phi_2} &= 2\pi\,\frac{e^{-2i\rho_2 r}-e^{2i\rho_2 r}}{-2i\rho_2 r}\\ \int_0^{2\pi} d\theta_4 \int_0^{\pi} \sin\phi_4\, d\phi_4\; e^{2i\rho_4 r\cos\phi_4} &= 2\pi\,\frac{e^{2i\rho_4 r}-e^{-2i\rho_4 r}}{2i\rho_4 r}.\end{aligned}$$

Only the first term of the delta function [19]

$$\delta(\rho_2^2-\rho_4^2) = \frac{\delta(\rho_2-\rho_4)}{2|\rho_2|} + \frac{\delta(\rho_2+\rho_4)}{2|\rho_2|}$$

contributes within the region of summation and then substitution of the θ_j and ϕ_j summations results in

$$\begin{aligned}Q_C(1) &= -\frac{\pi^2}{r^2}\frac{c_4 A_g}{2\lambda_c}\int_0^{\infty}\rho_2 d\rho_2\; e^{-2L(0)^2\rho_2^2}\;\left(e^{-2i\rho_2 r}-e^{2i\rho_2 r}\right)^2\\ &= \frac{\pi^2}{r^2}\frac{c_4 A_g}{2\lambda_c}\int_0^{\infty}\rho_2 d\rho_2\; e^{-2L(0)^2\rho_2^2}\;\left(2-e^{-4i\rho_2 r}-e^{4i\rho_2 r}\right).\end{aligned} \tag{266}$$

The integration is three terms of the form

$$\int_0^{\infty}\rho_2 d\rho_2\; e^{-\alpha\rho_2^2+i\beta\rho_2} = \frac{1}{2\alpha} + \frac{\beta\sqrt{\pi}}{4\alpha^{\frac{3}{2}}}e^{-\frac{\beta^2}{4\alpha}}\left(i+\ \mathrm{erfi}(\frac{\beta}{2\sqrt{\alpha}})\right) \tag{267}$$

for real $\alpha > 0$, real β, and with the imaginary error function $\mathrm{erfi}(x)$.

$$\mathrm{erfi}(x) = -i\mathrm{erf}(ix).$$

For real arguments, the imaginary error function $\mathrm{erfi}(x)$ is an odd, real function [1]. The summation (267) is evaluated by completing the square in the exponent, translating the summation interval by a complex value, and decomposing the integrand into an elementary integrable term and a term expressed in error functions.

The sum of the three contributions to the integration is

$$\begin{aligned}&\int_0^{\infty}\rho_2 d\rho_2\; e^{-2L(0)^2\rho_2^2}\;\left(2-e^{-4i\rho_2 r}-e^{4i\rho_2 r}\right)\\ &\quad= \frac{r\sqrt{\pi}}{(2L(0)^2)^{\frac{3}{2}}}e^{-\frac{r^2}{2L(0)^2}}\left(i+\ \mathrm{erfi}(\frac{r}{\sqrt{2L(0)^2}})\right) - \frac{r\sqrt{\pi}}{(2L(0)^2)^{\frac{3}{2}}}e^{-\frac{r^2}{2L(0)^2}}\left(i+\ \mathrm{erfi}(\frac{-r}{\sqrt{2L(0)^2}})\right)\\ &\quad= \frac{2r\sqrt{\pi}}{(2L(0)^2)^{\frac{3}{2}}}e^{-\frac{r^2}{2L(0)^2}}\ \mathrm{erfi}(\frac{r}{\sqrt{2L(0)^2}}).\end{aligned}$$

Collecting results, the value of $Q_C(1)$ is

$$Q_C(1) = \frac{\pi^2}{r}\frac{c_4 A_g}{\lambda_c}\frac{\sqrt{\pi}}{(2L(0)^2)^{\frac{3}{2}}}\exp\left(-\frac{r^2}{2L(0)^2}\right)\mathrm{erfi}\left(\frac{r}{\sqrt{2L(0)^2}}\right). \tag{268}$$

From (268) and $r > 0$, $Q_C(1) \geq 0$ and then with (263), $Q(1) > 0$.

8.16 Expected value of the energy of $\varphi_2(0)$

In this appendix, the expected value of the energy

$$\frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\|\varphi_2(0)\|^2}$$

is evaluated in elementary functions. H is the Hamiltonian (102) in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$ and the state describing functions $\varphi_2(0)$ are of the form (141) with (146) and (155). With appropriate selections of the parameters $\mathbf{u}(0)$, $\dot{\mathbf{u}}(0)$ and $L(0)^2$, these state describing functions exhibit quantum-classical correspondences.

The Hamiltonian (102) in the two-argument subspace is $\omega_1 + \omega_2$. From (143) and (157) in section 5.4.1, for non-relativistic momenta $\lambda_c^2\mathbf{p}_1^2 \ll 1$ and $\lambda_c^2\mathbf{p}_2^2 \ll 1$,

$$\omega_1 + \omega_2 \approx 2\lambda_c^{-1} + \lambda_c\mathbf{q}_2^2 + \frac{1}{4}\lambda_c\mathbf{q}_1^2$$

in the Jacobi coordinates (142). The contribution $\frac{\lambda_c}{4}\mathbf{q}_1^2$ of the center of mass to the energy is negligible if either the representative momentum condition (118) applies or the description of the center of momentum $\tilde{f}_M$ in (141) has a near zero expectation for $\mathbf{q}_1^2$, consistent with the non-relativistic approximation. Then, in the brief interval, non-relativistic, limited acceleration approximation,

$$\begin{aligned}\langle H\varphi_2(0)|\varphi_2(0)\rangle &\approx \langle(2\lambda_c^{-1} + \lambda_c\mathbf{q}_2^2)\varphi_2(0)|\varphi_2(0)\rangle \\ &= 2\lambda_c^{-1}Q(1) + \lambda_c Q(\mathbf{q}_2^2)\end{aligned}$$

from (169) in section 5.4.2.

$Q(1)$ is the sum of free field and connected VEV contributions (170) evaluated in (263) and (268) in appendix 8.15. The non-relativistic, brief interval, and limited acceleration approximations apply in the evaluation of both $Q_F(1)$ and $Q_C(1)$. $r^2 = \mathbf{u}^2$, and $\mathbf{w} = \mathbf{w}(0)$ and $\mathbf{u} = \mathbf{u}(0)$ are notation for initial momenta and locations. Inspection of the realizations of (263) and (266), and the product rule applied to (263) and (268) provide that

$$\begin{aligned}Q_F(\mathbf{q}_2^2) &= \frac{4A_g}{\lambda_c^2}\int d\mathbf{q}_2\ \mathbf{q}_2^2\ e^{-2L(0)^2(\mathbf{q}_2-\mathbf{w})^2} \\ &= \frac{4A_g}{\lambda_c^2}\int d\mathbf{q}_2\ (\mathbf{q}_2+\mathbf{w})^2\ e^{-2L(0)^2\mathbf{q}_2^2} \\ &= \mathbf{w}^2 Q_F(1) - \frac{\partial}{\partial(2L(0)^2)}\ Q_F(1) \\ &= \mathbf{w}^2 Q_F(1) - \frac{\partial}{\partial(2L(0)^2)}\left(\frac{4A_g}{\lambda_c^2}\left(\frac{\pi}{2L(0)^2}\right)^{\frac{3}{2}}\right) \\ &= \left(\mathbf{w}^2 + \frac{3}{4L(0)^2}\right)\ Q_F(1)\end{aligned}$$

from the zero summation of $\mathbf{q}_2 \cdot \mathbf{w}$. The interchange of summation and differentiation is justified by dominated convergence.

The contribution $Q_C(\mathbf{q}_2^2)$ includes the same approximation (264) as the evaluation (268) of $Q_C(1)$ in appendix 8.15. From (169) and (170),

$$Q_C(\mathbf{q}_2^2) = \frac{c_4 A_g}{\lambda_c} \int d\mathbf{q}_2 d\mathbf{q}_4 \; \mathbf{q}_2^2 \, \delta(\mathbf{q}_2^2 - \mathbf{q}_4^2) \; e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}} e^{-\overline{L(0)^2}(\mathbf{q}_2 - \mathbf{w})^2} e^{-L(0)^2(\mathbf{q}_4 - \mathbf{w})^2}.$$

With a convenient approximation to $\mathbf{q}_2^2$ consistent with the limited momentum approximation (264), the limited momentum approximation produces

$$Q_C(\mathbf{q}_2^2) \approx \frac{c_4 A_g}{\lambda_c} \int d\mathbf{q}_2 d\mathbf{q}_4 \; (\mathbf{q}_2 + \mathbf{w})^2 \delta(\mathbf{q}_2^2 - \mathbf{q}_4^2) \; e^{-2i(\mathbf{q}_2 - \mathbf{q}_4)\cdot \mathbf{u}} e^{-\overline{L(0)^2}(\mathbf{q}_2^2 + \mathbf{q}_4^2)}.$$

From the definition (170) of $Q_F(1)$ and the energy conservation delta function,

$$Q_C(\mathbf{q}_2^2) = \mathbf{w}^2 Q_F(1) - \frac{\partial}{\partial(2L(0)^2)} \, Q_C(1)$$

with the zero contribution from the summation of $\mathbf{q}_2 \cdot \mathbf{w}$.

From the evaluation (268) of $Q_C(1)$,

$$\begin{aligned}
&-\frac{\partial}{\partial(2L(0)^2)} \, Q_C(1)) \\
&\quad = -\frac{\partial}{\partial(2L(0)^2)} \left(\frac{\pi^2}{r} \frac{c_4 A_g}{\lambda_c} \frac{\sqrt{\pi}}{(2L(0)^2)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{2L(0)^2} \right) \operatorname{erfi}\left(\frac{r}{\sqrt{2L(0)^2}} \right) \right) \\
&\quad = \left(\frac{3}{4L(0)^2} - \frac{r^2}{4L(0)^4} + \frac{r}{2(2L(0)^2)^{\frac{3}{2}}} \frac{d \, \ln(\operatorname{erfi}(x))}{dx} \right) Q_C(1)
\end{aligned}$$

evaluated for

$$x = \frac{r}{\sqrt{2L(0)^2}}.$$

Collecting results,

$$\begin{aligned}
\frac{\langle H\varphi_2(0) | \varphi_2(0) \rangle}{\|\varphi_2(0)\|^2} &= \frac{2}{\lambda_c} + \lambda_c \frac{Q(\mathbf{q}_2^2)}{Q(1)} \\
&= \frac{2}{\lambda_c} + \left(\dot{\mathbf{u}}^2 + \frac{3\lambda_c^2}{4L(0)^2} \right) \frac{Q_F(1)}{\lambda_c Q(1)} + \left(\dot{\mathbf{u}}^2 + \frac{3\lambda_c^2}{4L(0)^2} - \frac{\lambda_c^2 r^2}{4L(0)^4} \right. \\
&\qquad \left. + \frac{\lambda_c^2 r}{2(2L(0)^2)^{\frac{3}{2}}} \frac{d \, \ln(\operatorname{erfi}(x))}{dx} \right) \frac{Q_C(1)}{\lambda_c Q(1)} \\
&= \frac{2}{\lambda_c} + \frac{3\lambda_c}{4L(0)^2} + \frac{\dot{\mathbf{u}}^2}{\lambda_c} - \left(\frac{\lambda_c^2 r^2}{4L(0)^4} - \frac{\lambda_c^2 r}{2(2L(0)^2)^{\frac{3}{2}}} \frac{d \, \ln(\operatorname{erfi}(x))}{dx} \right) \frac{Q_C(1)}{\lambda_c Q(1)}
\end{aligned} \tag{269}$$

with the non-relativistic approximation $\lambda_c \mathbf{w} = \dot{\mathbf{u}}$ and

$$x = \frac{r}{\sqrt{2L(0)^2}}.$$

The contribution of the four-argument connected functions, the contribution proportional to $Q_C(1)$, is negative for large $x = r/\sqrt{2L(0)^2}$. From section 5, $\mathbf{u}^2 \gg L(0)^2$ for a quantum classical correspondence to apply. From $Q(1) \geq Q_C(1) > 0$, the last term in in the last line of the evaluation of (269) is the negative of a positive constant times

$$\begin{aligned}
\frac{\lambda_c^2}{2L(0)^2}\left(x^2 - \frac{x}{2}\,\frac{d\ln(\mathrm{erfi}(x))}{dx}\right) &= \frac{\lambda_c^2}{2L(0)^2}\left(x^2 - \frac{x}{\sqrt{\pi}}\,\frac{e^{x^2}}{\mathrm{erfi}(x)}\right) \\
&\approx \frac{\lambda_c^2}{2L(0)^2}\left(x^2 - \frac{x^2}{1+\dfrac{1}{2x^2}}\right) \\
&> \frac{\lambda_c^2}{4L(0)^2} \\
&> 0
\end{aligned}$$

from the derivative of $\mathrm{erfi}(x)$

$$\frac{d\mathrm{erfi}(x)}{dx} = \frac{2}{\sqrt{\pi}}\,e^{x^2}$$

and the leading terms in the asymptotic expansion for $x \gg 1$,

$$\mathrm{erfi}(x) \approx \frac{e^{x^2}}{x\sqrt{\pi}}\left(1 + \frac{1}{2x^2}\right). \tag{270}$$

Finally, for instances with quantum-classical correspondences,

$$\frac{\langle H\varphi_2(0)|\varphi_2(0)\rangle}{\|\varphi_2(0)\|^2} = \frac{2}{\lambda_c} + \frac{\dot{\mathbf{u}}^2}{\lambda_c} + \frac{3\lambda_c}{4L(0)^2} - \frac{\lambda_c^2}{4L(0)^2}\frac{Q_C(1)}{\lambda_c Q(1)}. \tag{271}$$

8.17 Essentially local functions

There are functions arbitrarily dominantly supported within finite volumes among the anti-local functions in $\underline{\mathcal{P}}$. These functions are essentially localized. There are no strictly localized functions in $\underline{\mathcal{P}}$, [34].

The operation $(a^2 - \Delta)^{\frac{1}{2}}$ over $\mathbb{R}^3$ has the anti-local property [55]: if both f and $(a^2 - \Delta)^{\frac{1}{2}} f$ vanish within some finite volume of $\mathbb{R}^3$, then $f \in \mathcal{L}^2(\mathbb{R}^3)$ is identically zero. Δ is the Laplacian for $\mathbf{x} \in \mathbb{R}^3$. If $\varphi \in \underline{\mathcal{P}}$ has a Fourier transform of the form

$$\tilde{\varphi}(p) = (p_0 + \omega)\,\tilde{g}(p)$$

with $g \in \mathcal{S}(\mathbb{R}^4)$ and $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$ from (8), then

$$\varphi(x) = -i\frac{dg(x)}{dx_0} + (\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g(x) \tag{272}$$

and $\varphi \in \mathcal{S}(\mathbb{R}^4)$ vanishing in a finite volume provides that both $g \in \mathcal{S}(\mathbb{R}^4)$ and $(\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g$ vanish in the volume [34]. There are no solutions $g(x)$ to the wave equation (272) within the tempered functions. Anti-locality can be motivated by the observation that if $\varphi(x)$ was supported solely within a finite volume of $\mathbb{R}^3$, then the Fourier transform would be an entire analytic function of $\mathbf{p} \in \mathbb{C}^3$, but both $p_0\, \tilde{g}(p)$ and $\omega\, \tilde{g}(p)$ can not both be entire due to a branch cut of the multi-valued $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$.

One example suffices to demonstrate the existence of essentially localized functions within $\underline{\mathcal{P}}$. The selected example φ has a Fourier transform

$$\tilde{\varphi}(p) = (p_0 + \omega)\tilde{g}(p_0) \exp(-\alpha \mathbf{p}^2)$$

with $g \in \mathcal{S}(\mathbb{R})$. Then

$$\varphi(x) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \left(-ig'(x_0) + \omega g(x_0)\right) \exp(-\alpha \mathbf{p}^2 - i\mathbf{p} \cdot \mathbf{x}) \tag{273}$$

from the properties (23) of the Fourier transform (21).

From the Gaussian summation (262) in appendix 8.15,

$$\begin{aligned}
\int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \exp(-\alpha \mathbf{p}^2 - i\mathbf{p} \cdot \mathbf{x}) &= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} d\theta \int_0^\pi \sin\phi d\phi \, \exp(-\alpha\rho^2 - i\rho r \sin\phi) \\
&= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2) \frac{\left(e^{-i\rho r} - e^{i\rho r}\right)}{-i\rho r} \\
&= \frac{i}{r} \int_{-\infty}^\infty \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr} \int_{-\infty}^\infty \frac{d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr} \frac{1}{\sqrt{2\alpha}} \exp\left(-\frac{r^2}{4\alpha}\right) \\
&= (2\alpha)^{-\frac{3}{2}} \exp\left(-\frac{r^2}{4\alpha}\right)
\end{aligned}$$

using the spherical symmetry, change to polar coordinates with the p_z-axis aligned with $\mathbf{x}$,

$$\mathbf{p} = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

and ρ, r are the Euclidean lengths, $\rho^2 = \mathbf{p}^2$ and $r^2 = \mathbf{x}^2$, respectively. Similarly,

$$\begin{aligned}\left| \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, \omega \exp(-\alpha \mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x}) \right| &= \left| \frac{i}{r} \int_{-\infty}^{\infty} \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}}\, \omega \exp(-\alpha\rho^2 - i\rho r) \right| \\ &= \left| \frac{i}{r} \int_{-\infty}^{\infty} \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}}\, \omega \exp(-\alpha\rho^2) \left(\frac{1}{(-ir)^k} \frac{d^k}{d\rho^k} \right) \exp(-i\rho r) \right| \\ &< \frac{1}{r^{k+1}} \int_{-\infty}^{\infty} d\rho\, G(|\rho|)\, exp(-\alpha\rho^2) \\ &= \frac{a}{r^{k+1}}\end{aligned}$$

with a finite constant a independent of r. Multiple integrations by parts, the infinite differentiability of ω, and bounding the absolute value of the sum with the sum of absolute values result in a $G(|\rho|)$ of polynomial growth and the indicated bound a on the integration is finite. This bound suffices to demonstrate that the dominant support of $\varphi(x)$ from (273) lies within finite spheres: the likelihood per unit volume decreases as $1/r^{k+1}$ for arbitrarily large k in this instance of a Gaussian function. $\varphi(0)$ is finite and

$$\varphi(x) < \frac{|g'(x_0)|}{(2\alpha)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4\alpha}\right) + |g(x_0)|\, \frac{a}{r^{k+1}}$$

for every $g \in \mathcal{S}(\mathbb{R})$ with g from (273).

8.18 Illustrative relative states

A measurement process is illustrated in this appendix. The example illustrates interpretations of projection operators, unitary time translations, and relative states discussed in [11] and appendix 8.2.8. The illustration is not derived from the realizations constructed in section 3, and is meant to provide expedient illustrations of general concepts.

Consider an observer with three orthogonal states of interest interacting with an observed characterized by two orthogonal states. The three states of the observer are designated "no observation," "observed state 1" and "observed state 2." Designate the two observed states as "up" and "down." The two states could describe any observed quantity characterized by two possibilities, for example: located within detector A or B; Schrödinger's live or dead cat; spin up or down. Then, there are six states of interest within the Hilbert space if the observer is described independently of the observed, for example, when the observer is distantly space-like separated from the observed and not entangled. Designate these states using "no" for "no observation," "sup" for "observed state 1" and "sdn" for "observed state 2" with "up" and "dn" for the two observed states. For an ideal measurement, $|\text{sdn}, \text{up}\rangle$ and $|\text{sup}, \text{dn}\rangle$ never appear in

nature and these states are not coupled to the states of interest. These two states need not be considered further. This leaves four states of interest.

$$|\text{no, up}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\text{no, dn}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\text{sup, up}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\text{sdn, dn}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{274}$$

Presumably, these states represent orthogonal subspaces of states $|\underline{f}\rangle$ labeled by particular function sequences $\underline{f}$, and their interaction is described by one of the VEV constructed in section 3.3. The observation is conceived as a scattering event: initially independently described observed and observer are spatially distant, they approach and interact, and then entangled observed with observer states propagate away and cease to interact. An effective Hamiltonian is described below. The Hamiltonian is selected for expedience.

The evolution of an initial state described by the four orthogonal states (274) is described

$$|v(\lambda)\rangle = U(\lambda)|v_o\rangle$$

with a unitary, 4×4 block diagonal

$$U(\lambda) = \begin{pmatrix} U_2(\lambda) & 0 \\ 0 & U_2(\lambda) \end{pmatrix}$$

using 2×2 unitary

$$U_2(\lambda) = \begin{pmatrix} e^{i\eta}\cos\theta & -e^{i\phi - i\rho}\sin\theta \\ e^{i\rho}\sin\theta & e^{i\phi - i\eta}\cos\theta \end{pmatrix}.$$

The four parameters, θ, ϕ, η, ρ, are real. To simplify this example, the two unitary block submatrices of $U(\lambda)$ are set equal with $\eta = \rho = \phi = 0$ and θ is

$$\theta(\lambda) = \frac{\pi}{2}\frac{1}{I_\infty}\int_{-\infty}^{\lambda} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}}.$$

The normalization I_∞ is a beta function valid if $\epsilon > 0$.

$$I_\infty = \int_{-\infty}^{\infty} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}} = \frac{\Gamma(\frac{1}{2})\Gamma(\epsilon)}{\Gamma(\frac{1}{2}+\epsilon)}.$$

$\theta(\lambda)$ is absolutely continuous with $\theta(-\infty) = 0$ and $\theta(\infty) = \pi/2$.

The Hamiltonian generates time translation.

$$\begin{aligned} H(\lambda)v(\lambda) &= i\frac{dv(\lambda)}{d\lambda} \\ &= i\dot{U}(\lambda)v_o \\ &= i\dot{U}(\lambda)U^{-1}(\lambda)v(\lambda) \\ &= i\dot{U}(\lambda)U^{*}(\lambda)v(\lambda) \end{aligned}$$

in units of inverse length. For the 2×2 blocks of the Hamiltonian,

$$\begin{aligned} H_2(\lambda) &= i\dot{U}_2(\lambda)U_2^*(\lambda) \\ &= \begin{pmatrix} -i\dot{\theta}\sin\theta & -i\dot{\theta}\cos\theta \\ i\dot{\theta}\cos\theta & -i\dot{\theta}\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i\dot{\theta}(\lambda) \\ i\dot{\theta}(\lambda) & 0 \end{pmatrix} \end{aligned}$$

with

$$\dot{\theta}(\lambda) = \frac{\pi}{2}\frac{1}{I_\infty}\frac{a^{2\epsilon}}{(a^2+\lambda^2)^{\frac{1}{2}+\epsilon}}.$$

The Hamiltonian is Hermitian as a consequence of the unitarity of $U_2(\lambda)$. The evolution of the initial state v_o occurs most rapidly near $\lambda = 0$. The parameter a is a length characterizing the effective approach of observed to observer.

An initial state

$$|v_o\rangle = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and normalization $|\alpha|^2 + |\beta|^2 = 1$ evolves to

$$|v(\lambda)\rangle = U(\lambda)|v_o\rangle = \begin{pmatrix} \alpha\cos\theta(\lambda) \\ \alpha\sin\theta(\lambda) \\ \beta\cos\theta(\lambda) \\ \beta\sin\theta(\lambda) \end{pmatrix} \tag{275}$$

with $\cos\theta(-\infty) = 1$, $\sin\theta(-\infty) = 0$ and $\cos\theta(\infty) = 0$, $\sin\theta(\infty) = 1$. Initially, no observation has occurred and as time evolves, the observed and observer approach and the likelihood of state transitions increases. Eventually, the observer and observed separate and likelihoods stabilize with the relevant observer states entangled with observed states.

The projections onto the perceived observed states are

$$P_{\text{up}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{\text{dn}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the projections onto the possible states of the observer are

$$E_{\text{no}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\text{sup}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\text{sdn}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These projections commute, $[E_{\rm xx}, P_{\rm yy}] = 0$. This commutation implies the decomposition into relative states discussed in [11] and appendix 8.2.8. The projections onto perceived observed states, $P_{\rm yy}$, commute with time translation, $[U(\lambda), P_{\rm yy}] = 0$, providing that the likelihoods of the two observed states do not vary with time. The projections onto the possibilities for observer state, $E_{\rm xx}$, do not commute with time translation and state of the observer evolves with time. Likelihoods are

$$\text{up likelihood } = E[P_{\rm up}] = \langle v(\lambda)|P_{\rm up}v(\lambda)\rangle = |\alpha|^2$$

and

$$\text{dn likelihood } = E[P_{\rm dn}] = \langle v(\lambda)|P_{\rm dn}v(\lambda)\rangle = |\beta|^2$$

both independently of λ in the first example. The likelihoods of the observer having the indicated perceived history of observations are

$$\text{likelihood of no observation} = E[E_{\rm no}] = \langle v(\lambda)|E_{\rm no}v(\lambda)\rangle = \cos^2\theta(\lambda) \xrightarrow{\lambda\to\infty} 0$$

$$\text{likelihood that up observed} = E[E_{\rm sup}] = \langle v(\lambda)|E_{\rm sup}v(\lambda)\rangle = |\alpha|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\alpha|^2$$

$$\text{likelihood that dn observed} = E[E_{\rm sdn}] = \langle v(\lambda)|E_{\rm sdn}v(\lambda)\rangle = |\beta|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\beta|^2.$$

From (207) and for the state density operator $\rho = |v(\lambda)\rangle\langle v(\lambda)|$ for the pure state $|v(\lambda)\rangle$, the relative state density operators,

$$\rho^{xx} = \frac{E_{xx}\rho E_{xx}}{E[E_{xx}]},$$

are

$$\rho^{\rm no} = \begin{pmatrix} |\alpha|^2 & 0 & \alpha\overline{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ \overline{\alpha}\beta & 0 & |\beta|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\rm sup} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\rm sdn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These are unit trace matrices operating in orthogonal subspaces, that is, with ranges in the null spaces of the other state density operators. $\rho^{xx}\rho^{xx'} = 0$ unless xx'=xx.

The selected form for $U(\lambda)$ couples the observer with the observed and lacks any evolution of "up" and "dn" observed states, e.g., evolution of live into dead cats. A more general unitary transformation implements changes to the likelihoods of live or dead with time, for example, the composition of a rotation of $v_o = (1, 0, 0, 0)$ to $(\cos\vartheta, 0, \sin\vartheta, 0)$ with $U(\lambda)$. If the transition is internal to the observed, more general effective interactions apply than if the transition is due to interaction with the observer. In the case with the transition due to interaction with the observer, $\vartheta(\lambda)$ would have similar properties to $\theta(\lambda)$ except the transition need not be from one certain extreme to the other. $\vartheta(\lambda)$ is constrained by physical considerations, e.g., dead cats do not become live again. For this example composition, the Hamiltonian is time-dependent,

$$H(\lambda) = \begin{pmatrix} 0 & -i\dot{\theta} & -i\dot{\vartheta}\cos^2\theta & -i\dot{\vartheta}\cos\theta\sin\theta \\ i\dot{\theta} & 0 & -i\dot{\vartheta}\cos\theta\sin\theta & -i\dot{\vartheta}\sin^2\theta \\ i\dot{\vartheta}\cos^2\theta & i\dot{\vartheta}\cos\theta\sin\theta & 0 & -i\dot{\theta} \\ i\dot{\vartheta}\cos\theta\sin\theta & i\dot{\vartheta}\sin^2\theta & i\dot{\theta} & 0 \end{pmatrix}$$

and reproduces the result (275) but with $\alpha = \cos\vartheta(\lambda)$ and $\beta = \sin\vartheta(\lambda)$.

References

1. *Handbook of Mathematical Functions*, ed. M. Abramowitz, and I.A. Stegun, National Bureau of Standards, Applied Mathematics Series - 55, 1970.

2. S. Albeverio, H. Gottschalk, and J.-L. Wu, "Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions," *Rev. Math. Phys.*, Vol. 8, 1996, pg. 763-817.

3. N.I. Akhiezer and, I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, Vol. 1, trans. M. Nestell, New York, NY: Frederick Ungar Publishing Co., 1966.

4. H. Baumgärtel, and M. Wollenberg, "A Class of Nontrivial Weakly Local Massive Wightman Fields with Interpolating Properties," *Commun. Math. Phys.*, Vol. 94, 1984, p. 331.

5. J.S. Bell, "On the Einstein-Podolsky-Rosen Paradox," *Physics*, Vol. 1, 1964, p. 195.

6. G. Birchoff and J. von Neumann, "The Logic of Quantum Mechanics," *The Annals Of Mathematics*, Vol. 37, No. 4, 1936, p. 823.

7. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, New York, NY: McGraw-Hill, 1965.

8. A.S. Blum, "The state is not abolished, it withers away: How quantum field theory became a theory of scattering," *Studies in History and Philosophy of Modern Physics*, Vol. 60, 2017, p. 46.

9. N.N. Bogolubov, A.A. Logunov, and I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, trans. by Stephen Fulling and Ludmilla Popova, Reading, MA: W.A. Benjamin, 1975.

10. H.J. Borchers, "On the structure of the algebra of field operators," *Nuovo Cimento*, Vol. 24, 1962, p. 214.

11. B.S. DeWitt, H. Everett III, N. Graham, J.A. Wheeler in *The Many-worlds Interpretation of Quantum Mechanics*, ed. B.S. DeWitt, N. Graham, Princeton, NJ: Princeton University Press, 1973.

12. J. Dieudonné, *Treatise on Analysis, Volume II*, trans. by I.G. MacDonald, Vol. 10-II in Pure and Applied Mathematics, A Series of Monographs and Textbooks, New York, NY: Academic Press, 1970.

13. P.A.M. Dirac, *The Principles of Quantum Mechanics, Fourth Edition*, Oxford: Clarendon Press, 1958.

14. P. Ehrenfest, "Bemerkung über die angenäherte Gültigkeit der klassichen Machanik innerhalb der Quanatenmechanik," *Z. Physik*, Vol. 45, 1927, p. 455.

15. A. Einstein, "Zur Elektrodynamik bewegter Kórper," *Annalen der Physik* Vol. 322(10), 1905, p. 891.

16. A. Einstein, B. Podolsky and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?," *Physical Review*, Vol. 47(10), 1935, p. 777.

17. Richard Feynman, "Simulating Physics with Computers," *International Journal of Theoretical Physics*, vol. 21, 1982, p. 467.

18. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Volume III*, Reading, MA: Addison-Wesley Publishing Co., 1965.

19. I.M. Gel'fand, and G.E. Shilov, *Generalized Functions, Vol. 1*, trans. E. Salatan, New York, NY: Academic Press, 1964.

20. I.M. Gel'fand, and G.E. Shilov, *Generalized Functions, Vol. 2*, trans. by M.D. Friedman, A. Feinstein, and C.P. Peltzer, New York, NY: Academic Press, 1968.

21. I.M. Gel'fand, and N.Ya. Vilenkin, *Generalized Functions*, Vol. 4, trans. A. Feinstein, New York, NY: Academic Press, 1964.

22. I.M. Gel'fand, M.I. Graev, and N.Ya. Vilenkin, *Generalized Functions*, Vol. 5, trans. E. Saletan, New York, NY: Academic Press, 1966.

23. F. Gross, *Relativistic Quantum Mechanics and Field Theory*, New York, NY: Wiley-Interscience Publication, 1993.

24. B. C. Hall, *Quantum Theory for Mathematicians*, New York: Springer, 2013.

25. G.C. Hegerfeldt, "Prime Field Decompositions and Infinitely Divisible States on Borchers' Tensor Algebra," *Commun. Math. Phys.*, Vol. 45, 1975, p. 137.

26. R.A. Horn, and C.R. Johnson, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.

27. R.A. Horn, and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge: Cambridge University Press, 1991.

28. J.D. Jackson, *Classical Electrodynamics*, New York, NY: John Wiley and Sons, 1962.

29. A. Jaffe,A, "High-Energy Behavior in Quantum Field Theory. I. Strictly Localizable Fields," *Phys. Rev.*, Vol. 158(5), 1967, p. 1454.

30. A. Jaffe, "Constructive Quantum Field Theory," in *Mathematical Physics 2000*, ed. A. S. Fokas, London: Imperial College Press, 2000.

31. G.E. Johnson, "Relativistic propagation of isolated states," August 29, 2026, *www.-relquantphys.org*.

32. G.E. Johnson, "Consistent Descriptions of Quantum Fields," *Reports on Mathematical Physics*, Vol. 80(2), 2017, p. 193.

33. G.E. Johnson, "Classical approximations of relativistic quantum physics," April 2016, arXiv:quant-ph/1604.07654.

34. G.E. Johnson, "Introduction to quantum field theory exhibiting interaction," Feb. 2015, arXiv:math-ph/1502.07727.

35. G.E. Johnson, "Measurement and self-adjoint operators," May 2014, arXiv:quant-ph/1405.7224.

36. G.E. Johnson, "Fields and Quantum Mechanics," Dec. 2013, arXiv:math-ph/1312.2608.

37. G.E. Johnson, "Massless Particles in QFT from Algebras without Involution," May 2012, arXiv:math-ph/1205.4323.

38. G.E. Johnson, "Algebras without Involution and Quantum Field Theories," March 2012, arXiv:math-ph/1203.2705.

39. G.E. Johnson, "Are strings the aether of our time?" June 2015, arXiv:quant-ph/1506-.05361v1.

40. G.E. Johnson, "Interacting quantum fields," *Rev. Math. Phys.*, Vol. 11, 1999, p. 881 and errata, *Rev. Math. Phys.*, Vol. 12, 2000, p. 687.

41. R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Volume I: Elementary Theory*, and *Volume II: Advanced Theory*, Graduate Studies in Mathematics, Vol. 15: American Mathematical Society, 1997.

42. G. Lechner, "Deformations of quantum field theories and integrable models," *Commun. Math. Phys.* Vol. 312, 2012, p. 265.

43. A. Messiah, *Quantum Mechanics, Vol. I*, New York, NY: John Wiley and Sons, 1968.

44. T.D. Newton and E.P. Wigner, "Localized States for Elementary Systems," *Rev. Modern Phys.*, Vol. 21, 1949, p. 400.

45. M.F. Pusey, J. Barrett, and T. Rudolph, "On the reality of the quantum state," *Nature Physics*, Vol. 8, 2012, p. 475.

46. H. Reeh and S. Schlieder, "Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern", *Nuovo Cimento*, Vol. 22, 1961, p. 1051.

47. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, New York, NY: Dover Publications, 1990.

48. H. Rinke, "A remark on the asymptotic completeness of local fields," *Commun. Math. Phys.*, Vol. 12, 1969, p. 324.

49. Walter Rudin, *Real & Complex Analysis*, New York, NY: McGraw-Hill Book Co., 1966.

50. D. Ruelle, "On the Asymptotic Condition in Quantum Field Theory," *Helv. Phys. Acta.*, Vol. 35, 1962, p. 147.

51. E. Schrödinger, "Der stetige Übergang von der Mikro-zur Makromechanik," *Die Naturwissenschaften*, Vol. 14, Issue 28, 1926, p. 664.

52. E. Schrödinger, "Discussion of probability relations between separated systems," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 31 (4), 1935, p. 555.

53. E. Schrödinger, "Die gegenwärtige Situation in der Quantenmechanik," *Die Naturwissenschaften*, Vol. 23, Issue 48, 1935, p. 807.

54. F. Schwabl, *Advanced Quantum Mechanics*, trans. R. Hilton and A. Lahee, Berlin: Springer-Verlag, 1999.

55. I.E. Segal and R.W. Goodman, "Anti-locality of certain Lorentz-invariant operators," *Journal of Mathematics and Mechanics*, Vol. 14, 1965, p. 629.

56. O. Steinmann, "Structure of the Two-Point Function," *J. Math. Phys.*, Vol. 4, 1963, p. 583.

57. A. Douglas Stone, *Einstein and the Quantum*, Princeton, NJ: Princeton University Press, 2013.

58. R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That*, Reading, MA: W.A. Benjamin, 1964.

59. B. Simon, *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory*, Princeton, NJ: Princeton University Press, 1974.

60. F. Strocchi, "The Physical Principles of Quantum Mechanics. A critical review," Jan. 2012, arXiv:quant-ph/1112.1507.

61. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford: The Oxford University Press, 1948.

62. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton, NJ: Princeton University Press, 1955.

63. S. Weinberg, *The Quantum Theory of Fields, Volume I, Foundations*, New York, NY: Cambridge University Press, 1995.

64. A.S. Wightman, "Quantum Field Theory in Terms of Vacuum Expectation Values," *Phys. Rev.*, vol. 101, 1956, p. 860.

65. A.S. Wightman and L. Gårding, "Fields as operator-valued distributions in relativistic quantum theory," *Arkiv för Fysik*, Vol. 28, 1965, p. 129.

66. E.P. Wigner, "On unitary representation of the inhomogenous Lorentz group," *Annals of Mathematics*, Vol. 40, No. 1, 1939, pg. 149.

67. E.P. Wigner, "Remarks on the mind-body question," in I.J. Good, *The Scientist Speculates*, Heinemann, London, 1961.

68. J. Yngvason, "Localization and Entanglement in Relativistic Quantum Physics," Jan. 2014, arXiv:quant-ph/1401.2652.

69. J. Yngvason, "The Role of Type III Factors on Quantum Field Theory," *Reports on Mathematical Physics*, Vol. 55(1), 2005, p. 135.